TOPICS IN ASTROPHYSICS

Much astrophysical research is interdisciplinary, requiring a mix of methods, e.g. observational and theoretical, to solve questions about the Universe

Often need to think “simply” to know how to approach a problem, using order of magnitude estimates of different processes to identify the relevant physics

This course aims to provide a training in how to approach astrophysical problems by exploring two complementary aspects

Hence it is a course of two halves

Lectures 1 - 12 : Timescales, Distributions and Tides (Mark Wyatt)

Shows how specific physical concepts can be applied to a wide diversity of astrophysical phenomena

You will learn about the physics of tides, and how simple concepts can be applied to quasars, black holes, stellar clusters, planets and moons

You will also learn how to identify the relevant physics in a problem through timescales, and the importance of considering populations of astrophysical objects as distributions

Lectures 13 - 24 : Planet Formation and Evolution (Oli Shorttle)

Shows how to apply diverse physical concepts to the specific research theme of planet formation

You will learn about some key results from planetary and exoplanet are science and outstanding challenges

You will also learn about the process of planet accretion within protoplanetary disks

The Point of the Lectures

The lectures will convey information, but will focus on how to approach problems, and will contain many worked examples

Guest lectures will build on the content to show how it is used in cutting-edge research (not examinable)

When Solving Problems

Remember: think “simply”, make order of magnitude estimates, and be scrupulous about units and dimensions

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1.1 Numbers Sheet

$m = -2.5 \log(L/D^2) + \text{const} = -2.5 \log F/F_0$

In addition to the formula booklet, here are some useful scales and scalings to have at your fingertips

Angular distances:

1' = 1/60 degrees

1" = $(1/60)^2$ degrees

1 AU subtends an angle of 1" at a distance of 1 parsec

Formulae involving velocities/time:

Doppler shift (for $v << c$):

$\Delta\nu/\nu = v/c$

An object traveling at 1 km/s covers 1 parsec in 1 Myr.

Number of seconds in a day $\sim 10^5$

Number of seconds in a year $\sim 3 \times 10^7$

Age of Universe $\sim 1.5 \times 10^{10}$ years.

Age of Sun 4.5×10^9 years

For problems involving circular motion it can be useful to scale period of orbit and orbital velocity to the orbital properties of the earth, i.e.

$T = 1$ year $(R_{AU}^3/M_1)^{0.5}$

$v = 30\text{km/s}\ (M_1/R_{AU})^{0.5}$

Formulae involving radiation:

Bolometric flux from black body of temperature T = σT^4 (Wm^{-2})

Location of Wien peak for black body spectrum

$\lambda \sim 3\mu\text{m}\ (T/1000\text{K})^{-1}$

Definition of apparent and absolute magnitude (m and M):

m is measure of flux at earth (W m^{-2}); M is measure of intrinsic luminosity of source (W)

$m - M = 5\log_{10}D - 5$ where D is measured in parsecs

(i.e. absolute magnitude = apparent magnitude if object at distance of 10 pc)

$M = -2.5\log_{10}L+$ constant (where L is luminosity) such that absolute visual magnitude of Sun is 4.83.

Colour is defined by ratio of fluxes at different wavelengths, i.e. in terms of differences in magnitude e.g. B-V. Redder colours have larger colour indices.

Indicative size and mass scales:

Distances between galaxies $\sim$ Mpc
Sizes of galaxies $\sim$ 10s of kpc

Sizes of clusters $\sim$ pc

Sizes of (extra-) solar systems $\sim$ AU (for planets) to 10^4 AU (comet cloud)

Size of stars: $10^8 - 10^{12}$ m

Masses of galaxies: $10^9 - 10^{12} M_\odot$

Masses of central black holes: $10^{6-8} M_\odot$

Masses of globular clusters: $10^6 M_\odot$

Masses of other clusters: $10^{2-3} M_\odot$

Typical mass of a star: $1 M_\odot$

Mass of a brown dwarf: $10^{-2} - 10^{-1} M_\odot$

Mass of a giant planet: $10^{-3} - 10^{-2} M_\odot$

Mass of a terrestrial planet: $10^{-5} M_\odot$

Typical densities:

Number density of stars in solar neighbourhood $\sim 0.1\text{pc}^{-3}$

Mean number density of interstellar medium $\sim 10^6\text{m}^{-3}$(1 per cm^3, but very large dynamic range)

Mean density of Sun, density of rock and of water are all of similar order of magnitude (1000s kg/m^3).

1.2 Timescales and Length-scales

For any quantity Q we can define:

a timescale $\tau = Q \Big/ \left|\frac{dQ}{dt}\right|$

determines which processes dominate

which are in equilibrium?

If t is timescale of interest, $\tau \ll t \rightarrow$ equilibrium

$\tau \gg t \rightarrow$ hardly evolving

and a length-scale $L = Q \Big/ \left|\frac{dQ}{dx}\right|$

used to assess "box size" for simulations

and resolution requirements

1.3 Exponentials vs Power Laws

Many astrophysical variables have an exponential or power law dependence on time or length

Consider exponentials, such as

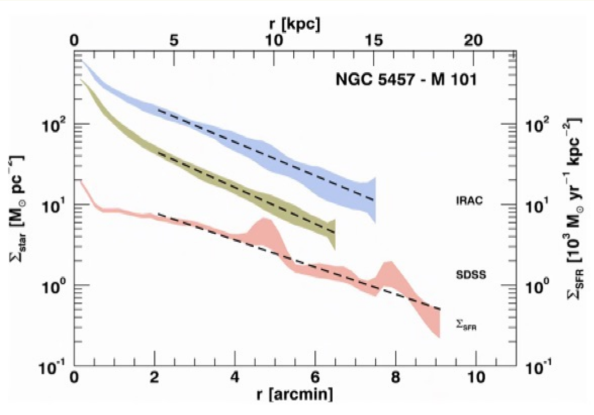
$$Q = Q_0 e^{-t/T}$$

$\therefore dQ/dt = -Q/T$

$\therefore \tau = T$

→ there is an intrinsic timescale

eg radioactive decay



$$Q = Q_0 e^{-x/L}$$ → intrinsic length-scale

eg stellar density in many galaxies

Note that exponentials are straight lines on log-linear plots, e.g., ln(Q) vs x

what about galaxy formation allows every point to know about 3.5kpc e-folding length?

Consider power laws

$$Q = Q_0 \left(\tau / t\right)^n$$

$$\therefore dQ/dt = nQ/t$$

$\therefore \tau = t/n \rightarrow$ characteristic timescale of change $\sim$ age (and $L \sim x$)

$\rightarrow$ no intrinsic time- or length-scale

$\rightarrow$ "scale-free" or "self-similar"

E.g., supernova blast waves, star clusters and accretion disks all evolve in a self-similar fashion

$\therefore$ physical variables change in such a way that the instantaneous growth time is $\sim$ the present time

Note that power laws are straight lines on log-log plots, e.g., ln(Q) vs ln(t)

1.4 Some Important Timescales and How to Calculate Them

1.4.1 Dynamical Timescale, t_{dyn}

Consider an object a distance R from a mass M

There are several ways to get dynamical times:

(i) Radial infall from stationary onto point mass

$$\ddot{R} = -GM/R^2$$

$$\therefore \dot{R}\ddot{R} = -GM\dot{R}/R^2$$

$$\therefore \frac{d}{dt}\left[\frac{1}{2}\dot{R}^2 - GM/R\right] = 0$$

$$\therefore \dot{R} = -\sqrt{GM}\sqrt{R^{-1} - R_0^{-1}}$$

Solve by substituting $R = R_0 \sin\theta$

$$\therefore \tau = \frac{\pi}{2\sqrt{2}}\sqrt{R_0^3/GM}$$

(ii) Radial infall assuming mass uniformly distributed inside R

$$M_{enc}(R) = M(R/R_0)^3$$

$$\therefore \ddot{R} = -\left(\frac{GM}{R_0^3}\right)R$$

$\rightarrow$ SHM with period $2\pi\sqrt{R_0^3/GM}$

(iii) Escape velocity

$$\frac{1}{2}V_{esc}^2 = GM/R_0$$

$$\therefore V_{esc} = \sqrt{2GM/R_0}$$

$$\therefore \tau \simeq R_0/V_{esc} = \frac{1}{\sqrt{2}}\sqrt{R_0^3/GM}$$

(iv) Dimensional analysis

Variables: $G\ [L^3 M^{-1} T^{-2}]$, $M\ [M]$, $R_0\ [L]$

$\rightarrow \tau \sim \sqrt{R_0^3/GM}$

(v) Circular orbit

To maintain $\quad \omega^2 R_0 = GM/R_0^2$

$\therefore \tau = 2\pi/\omega = 2\pi\sqrt{R_0^3/GM}$

To order unity all methods give $\underline{\tau_{dyn} \sim \sqrt{R_0^3/GM}}$

1.4.2 Sound Crossing Time , τ_{cs}

This is the timescale on which pressure disturbances are conveyed (in the absence of supersonic bulk transport), and is the communication time for gaseous systems

For a region of size D $\quad \tau_{cs} \simeq D / c_s \quad$ where $c_s = \sqrt{P/\rho}$

number of moles

For an ideal gas $\quad PV = nRT \quad$ 8.3J/mol/K

$\therefore \; P = \rho R^* T / \mu \quad$ where $R^* = 1000\,R = 8300$ J/K/kg

μ = relative molecular weight

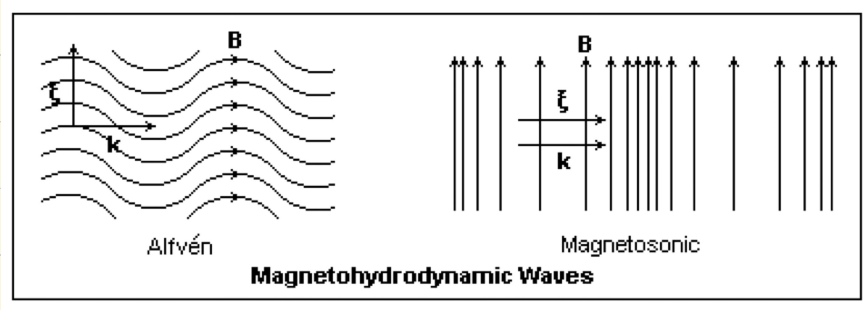
= 2.35 for ISM

$\therefore \; \tau_{cs} = D \sqrt{\mu / R^* T}$

1.4.3 Alfven Wave Crossing Time, τ_A

Magnetic fields resist being bent or squashed

and so have an associated pressure and tension



This results in waves analogous to sound

waves in magnetised media

The effective magnetic pressure is of magnitude $\frac{1}{2} B^2/\mu_0$

$$\therefore V_A \simeq \sqrt{P/\rho} = \sqrt{B^2/\rho\mu_0}$$

$$\therefore \tau_A \simeq D/V_A = D\sqrt{\rho\mu_0}/B$$

Can propagate information faster than sound waves

1.4.4 Light Crossing Time, τ_{lc}

$$\tau_{lc} \sim R/c$$

= absolute minimum communication time

= timescale for energy transfer in optically thin media

1.4.4.1 Example 1 - Proof that Quasars Host Black Holes

Quasars outshine entire galaxies, yet they are variable on timescales $\Delta t \sim$ hours

Thus maximum size is $\sim c\Delta t \sim 10\,\mathrm{au}$

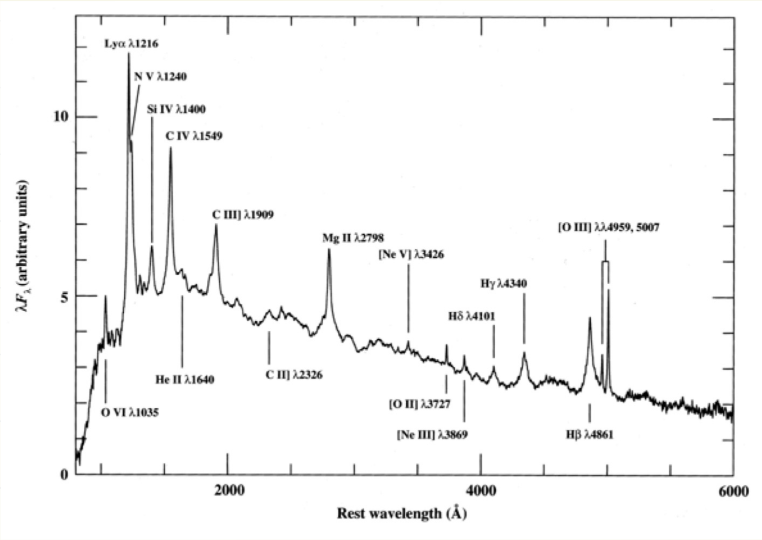
But a galaxy with all stars touching has a size $\sim N^{1/3} R_* \sim 10-100\,\mathrm{au}$ as $N \sim 10^9 - 10^{12}$, $R_* \sim 7\times10^8\,\mathrm{m}$

$\rightarrow$ quasars involve black holes

Since event horizon-crossing time $(2GM_{BH}/c^2)/c < \Delta t$ NB set $V_{esc} = c$ to get R_s

$$\therefore M_{BH} < 10^9 M_\odot$$

1.4.4.2 Example 2 - Weighing Quasar Black Holes Using the "Light Echo" Technique

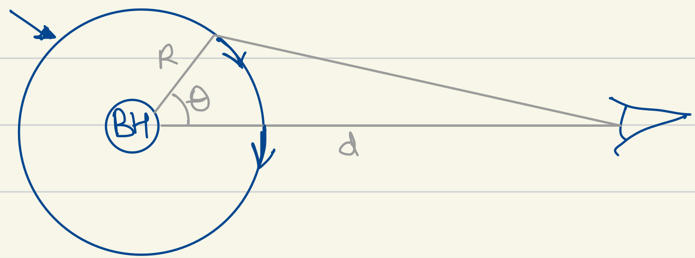


Emission lines in quasar spectra are from circum-black hole material, and are broadened due to orbital velocity

$$v(R) = \sqrt{GM_{BH}/R}$$

While the emission is spatially unresolved, different wavelengths (i.e., different projected velocities V_{los}) probe different bits of circum-black hole material and so have a different time-lag τ

Consider a ring of line-emitting circum-black hole material

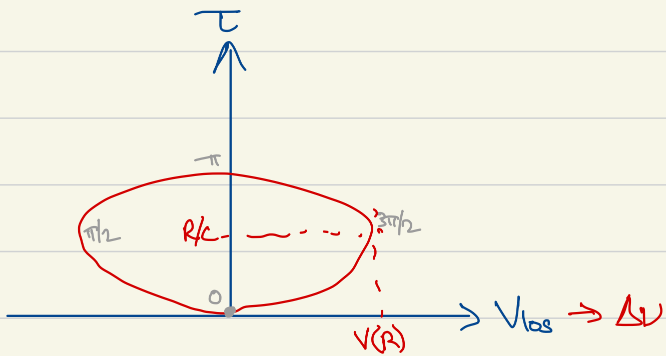


$$V_{los} = V(R)\sin\theta$$

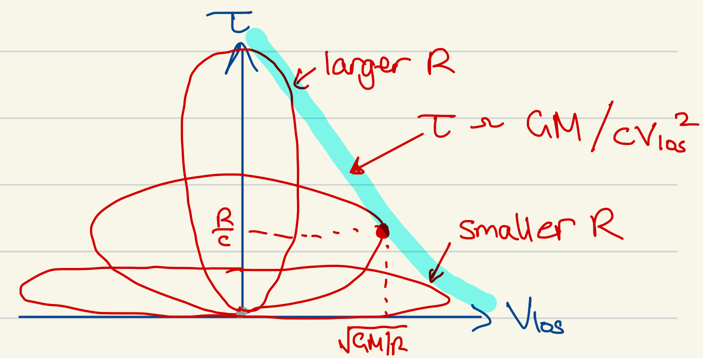
Light travel distance $\simeq d + R(1-\cos\theta)$

$$\therefore\ \tau \simeq \frac{R}{c}(1-\cos\theta)$$

Time lag vs line-of-sight velocity for a ring:

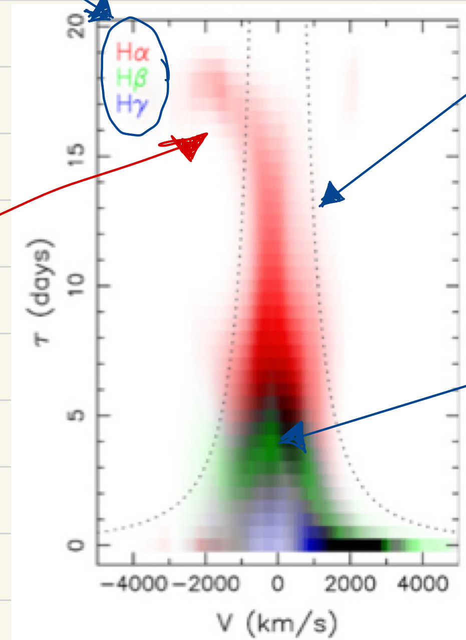


And for a disk:



Different lines probe different parts of the disk

Upper envelope constrains M_{BH}



Spherical infall

Circum-black hole disk

At R: $\tau = \frac{R}{c}(1 - \cos\theta)$ as before

If radial infall at $V(R)$

$$V_{los} = V(R)\cos\theta$$

$$\therefore \tau = \frac{R}{c}\left(1 - V_{los}/V(R)\right)$$

1.4.5 Thermal Timescale, τ_{th}

Thermal equilibrium means that heating rate is equal to the cooling rate

But this doesn't mean that $\tau_{th} = \infty$

Rather $\tau_{th} = Q/|\dot{Q}_{heat}| = Q/|\dot{Q}_{cool}|$ ie rate in one turned off

where Q = thermal content / unit mass

$= C_V T$

depends on d.o.f.

For an ideal gas, thermal energy / mass $\sim R^* T/\mu$

and thermal energy / volume $\sim P$

For a photon gas, thermal energy / volume $\sim a T^4$

radiation constant

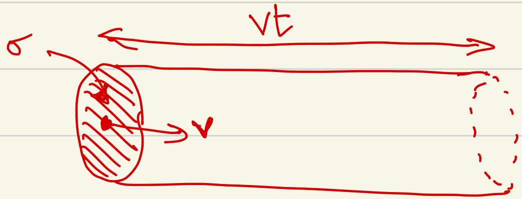
1.4.6 Collision Timescale, τ_{coll}

"Particle in a box" collision rate:

Consider an object moving at velocity v

through a sea of n impactors per unit volume

with an impact cross-section of σ



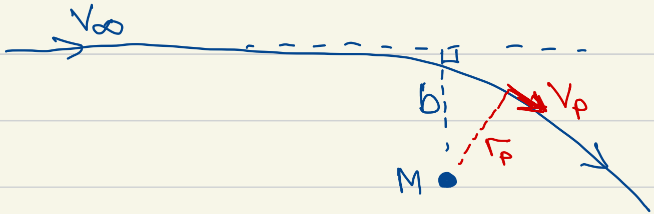
Volume swept out / time $= \sigma v$

Collision rate $= n\sigma v$

Collision time $\tau_{coll} = 1/(n\sigma v)$

Gravitational focussing means that the collision cross-section may be larger than the physical size

Consider an object approaching another of mass M and radius R at velocity V_∞ with impact parameter b



Angular momentum: $bV_\infty = r_p V_p$

Energy: $\frac{1}{2}V_\infty^2 = \frac{1}{2}V_p^2 - GM/r_p$

Eliminate V_p $\rightarrow b^2 = r_p^2\left[1 + \frac{2GM}{r_p V_\infty^2}\right]$

Ratio of PE @ peri to KE at ∞

For a collision: $r_p = R$ $\therefore$ $\sigma = \pi b^2 = \pi R^2\left[1 + \frac{2GM}{RV_\infty^2}\right]$

$$= \pi R^2\left[1 + \left(\frac{V_{esc}}{V_\infty}\right)^2\right]$$

1.4.7 Diffusion Timescale, τ_{diff}

Objects undergoing a random walk with step length λ would have travelled a mean distance after N steps of $\sqrt{N}\,\lambda$

Thus, the number of steps required to traverse a distance R is: $(R/\lambda)^2$

And the time required to traverse this distance at speed v is:

$$\tau_{diff} = (R/\lambda)^2 \cdot (\lambda/v)$$
$$= R^2/(\lambda v)$$

NB diffusive processes have a quadratic dependence of time on distance

1.4.7.1 Example 1 - Diffusion of Photons out of the Sun

Mean free path
$$\lambda = \frac{1}{n\sigma} = \frac{m}{\rho\sigma} = \frac{1}{\kappa\rho}$$

where m = mass of particle

κ = opacity $\equiv$ cross-section / mass

So the diffusion time to reach the surface at a radius $R_\odot$ (remembering that $\tau_{diff} = R^2/(\lambda v)$) is

$$\tau_{diff} = R_\odot^2 \kappa\rho / c$$

And for $\rho \sim M_\odot / R_\odot^3$:

$$\tau_{diff} = \left(\frac{M_\odot}{R_\odot}\right)\left(\frac{\kappa}{c}\right)$$

has units of $\frac{M}{L} \cdot \frac{L^2/M}{L/T} = T$ ✓

If free electrons are scattering (valid at high temperatures) then $\kappa = \kappa_{es} \rightarrow \tau_{diff} \sim 10^4\,yr$

At lower temperatures where electrons are bound to protons opacity ~ 100 times higher

Use $\tau_{diff} = R_\odot^2 \kappa\rho / c$ to get the Sun's luminosity from first principles:

Energy/volume

Energy in the Sun's radiation field $\sim aT^4 R_\odot^3$

$\therefore L_\odot \simeq aT^4 R_\odot^3 / \tau_{diff} \simeq acT^4 R_\odot / (\kappa\rho)$

Compare with the radiative diffusion of energy through a sphere of radius R;

$\frac{dT}{dR} = 3\kappa\rho L / [16\pi acR^2T^3] \sim T/R \rightarrow$ same as same Physics

Compare with the timescale for the Sun to cool in the absence of nuclear reactions

τ_{cool} = total energy in Sun / $L_\odot$ = total energy in Sun / (energy in radiation / τ_{diff})

As $P_{gas} \gg P_{radiation}$, gas thermal energy $\gg$ radiation energy

$\rightarrow \tau_{cool} \gg \tau_{diff}$

1.4.7.2 Example 2 - Two Body Relaxation in Stellar Clusters (see eg. EG 1.5)

Consider a star of mass M orbiting at speed V in a cluster

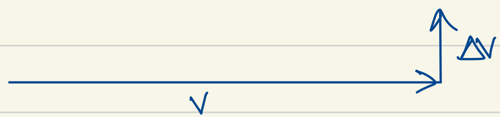
Encounters with other stars (gravitational scattering) impart kicks to the velocity

This acquisition of ΔV is very important for cluster evolution as it provides a mechanism for transferring energy between stellar orbits

This is called **two body relaxation**

What is the two body relaxation timescale τ_{2br} ?

Assume the average kick perpendicular to a star's orbit is Δv per encounter



Direction of Δv is randomly oriented in the plane perpendicular to its velocity $\underline{v}$

→ random walk in velocity space

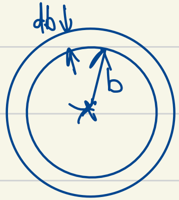
Define τ_{2br} as the timescale for a star to acquire a cumulative $\Delta v_{\perp} \sim v$?

Random walk means this requires a number of encounters $N \sim (v/\Delta v)^2$ ie $\sqrt{N}\,\Delta v = v$

which takes a time of N / rate of encounters

For an accurate calculation you would consider impact parameters in the range $b \to b+db$

which would all result in the same $\Delta v_{\perp}$



For a quick estimate of the rate of encounters: assume all impacts $< b$ occur at b

$\therefore$ rate of encounters $\sim n b^2 v$

For the magnitude of the kick: $\Delta v \simeq$ acceleration at b × time spent at b

$\simeq (Gm/b^2) \cdot (b/v)$

good if perturbation is small

So, what is the timescale for a star to acquire a cumulative $\Delta v_b \sim V$?

Remember: number of encounters required $N \sim (v/\Delta v)^2$

time to achieve this N / rate of encounters

rate of encounters $n b^2 V$

kick per encounter $\Delta v \sim (Gm/b^2) \cdot (b/v)$

$$\therefore T_{2br} \sim \left(v \Big/ \frac{Gm}{bv} \right)^2 \Big/ (n b^2 v)$$
$$= v^3 / (G^2 m^2 n)$$

not dominated by close or distant encounters

Note that "b" disappears → all impact parameters contribute equally

Doing the integral → get a log b dependence.

Lots of implications for phenomena in stellar clusters that we'll get to

1.5 Distributions

A probability density function $p(q)$ is defined such that the probability that a system has a value in the range $q \to q + dq$ is $p(q)\,dq$

Sometimes we know $p(q)$ but want the probability in terms of a different parameter $x(q)$

Defining $p(x)\,dx$ as the probability that a system has a value in the range $x \to x + dx$

$$p(x)\,dx = p(q)\,dq$$

$$\therefore\ p(x) = p(q)\,dq/dx$$

Often we want to know which part of the distribution dominates for which it is helpful to consider $q\,p(q)$

This is because $p(q)\,dq = q\,p(q)\,d\ln q$

$\therefore$ $q\,p(q)$ is the probability per log interval of q

whereas $p(q)$ " linear "

For example, consider a power law distribution $p(q) \propto q^{-\alpha}$ that holds between q_{min} and q_{max}

The probability of being in the range $q_1 \to q_2$ is $\int_{q_1}^{q_2} p(q)\,dq \propto \int_{q_1}^{q_2} q^{-\alpha}\,dq$

$$\propto \left[q^{1-\alpha} \right]_{q_1}^{q_2}$$

And the total number of instances is $\propto \left[q_{max}^{1-\alpha} - q_{min}^{1-\alpha} \right]$

$\therefore$ q_{max} dominates if $\alpha < 1$

q_{min} dominates if $\alpha > 1$

1.5.1 Probability Distribution Example 1 - Initial Mass Function (IMF)

IMF = Initial Mass Function

= probability density function for stars at birth, $f(m)$

$\neq$ PDMF

PDMF = Present Day Mass Function

= distribution of stellar masses in a given sample

$\neq$ IMF as stellar lifetime, t_{MS}, is a function of mass

OMF = Observed Mass Function

$\neq$ PDMF as sample selection effects need to be considered eg. magnitude-limited sample

The IMF is defined by $f(m)\,dm$, which is the fraction of stars formed in the mass range $m \rightarrow m + dm$

Number of stars in that mass range $\propto m f(m)\, d\ln m \qquad \propto m^{1-\alpha}\, d\ln m$

Total mass of stars in that mass range $\propto m f(m)\, dm \qquad \propto m^{2-\alpha}\, d\ln m$

Total luminosity of stars in that mass range $\propto L(m) f(m)\, dm \qquad \propto m^{\beta+1-\alpha}\, d\ln m$

So, if $f(m) \propto m^{-\alpha}$ and $L(m) \propto m^{\beta}$

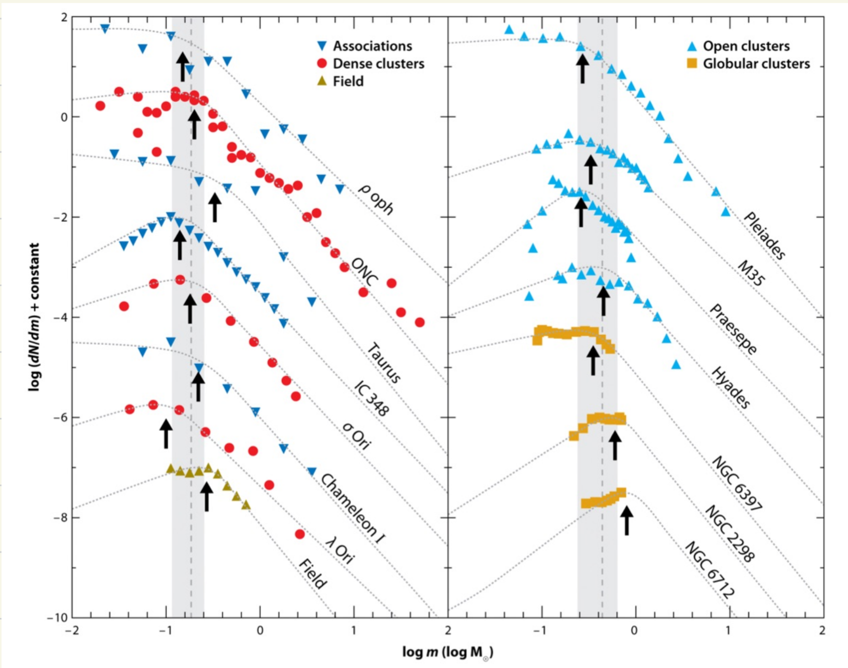
Thus, total mass of stars is dominated by least massive stars if $\alpha > 2$

most $\alpha < 2$

And the total luminosity of stars is dominated by least massive stars $\alpha > \beta + 1$

most $\alpha < \beta + 1$

1.5.1.1 Observations of the IMF in Co-Eval Populations of Nearby Stars



IMF is very similar across a range of cluster densities, ages and metallicities

Why is OMF $\neq$ IMF at high M_* ?

short t_{MS}

Why is OMF $\neq$ IMF at low M_* ?

H-burning and so low L_*
(brighter when young)

Note that distributions can be quoted in different ways

Let $f(>m)$ be the fraction of stars with masses larger than m

For a power law distribution $f(m) \propto m^{-\alpha}$

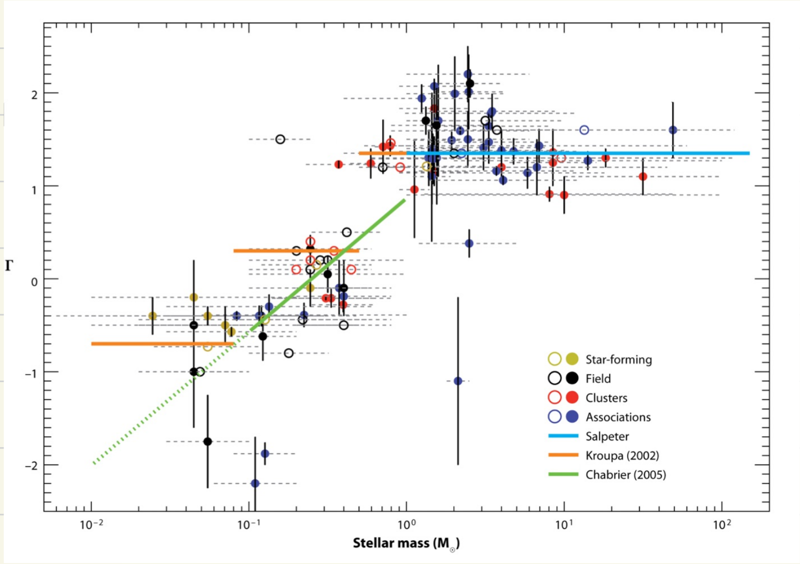
Then $f(>m) \propto \int_m^{m_{max}} m^{-\alpha}\, dm$

what happens if $\alpha < 1$? ($f(>m) = 1$ as M_{max} dominates)

And if most stars are low mass $(\alpha > 1)$ then $f(>m) \propto m^{1-\alpha}$

Thus $f(>m) \propto m^{-\Gamma}$ is equivalent to $\alpha = 1 + \Gamma$

Replotting the observed IMF as the power law index $\Gamma = \alpha - 1$ in the distribution



Hence the IMF is usually parameterised

as a piece wise power law distribution:

$\alpha \sim 2.35$ for $> 1 M_\odot$ (Salpeter)

$\alpha < 2$ for $\ll 1 M_\odot$

$\therefore$ total mass is dominated by $\sim 1 M_\odot$ stars – why is this the characteristic mass scale for fragmentation?

good for life? high M_* too short-lived, low M_* have high E emission from magnetospheres, but high N

1.5.1.2 The IMF in Distant Galaxies

We can't resolve individual stars, so need to probe this using integrated luminosity or spectroscopy

Remember: Total luminosity of stars $\propto L(m) f(m) dm \propto m^{\beta+1-\alpha} \, d\ln m$

For $< 30 M_\odot$: $L \propto m^{3.5}$ $\rightarrow L_{tot} \propto m^{3.5+1-2.35 = 2.15} \, d\ln m$ for $1-30 M_\odot$

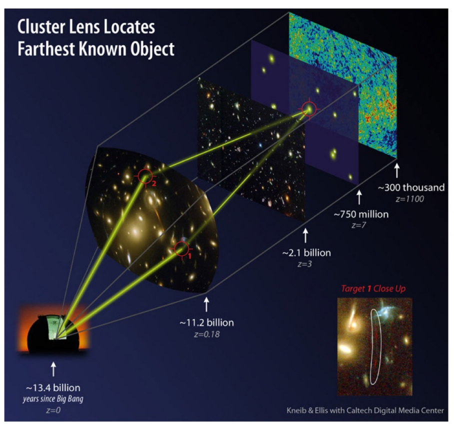
$\propto m^{3.5+1-(<2.35) > 2.15} \, d\ln m$ for $< 1 M_\odot$

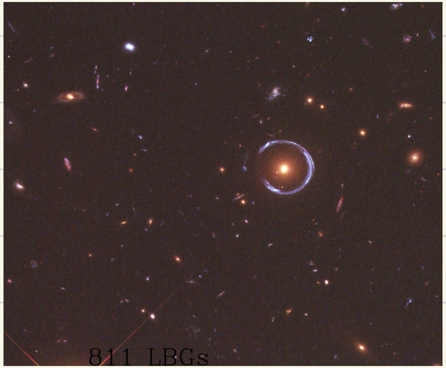
$> 30 M_\odot$: $L \propto m$ $\qquad \propto m^{1+1-2.35 = -0.35} \, d\ln m$ for $> 30 M_\odot$

So a distant galaxy's luminosity is dominated by $\sim 30 M_\odot$ $\rightarrow$ only observe top of IMF

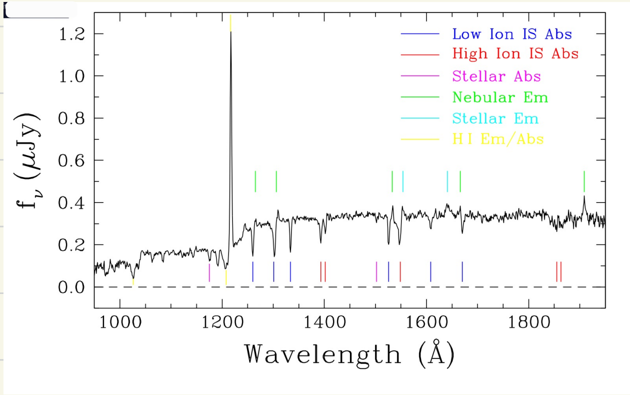
$\rightarrow$ don't know the formation rate of lower mass stars that dominate mass

Can use gravitational lending to detect high redshift galaxies



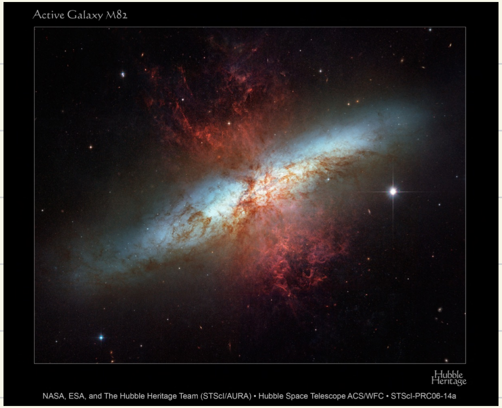


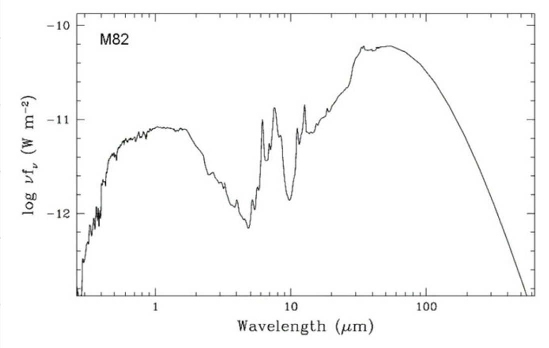
Then take a spectrum to identify stellar populations, which are dominated by the high mass stars



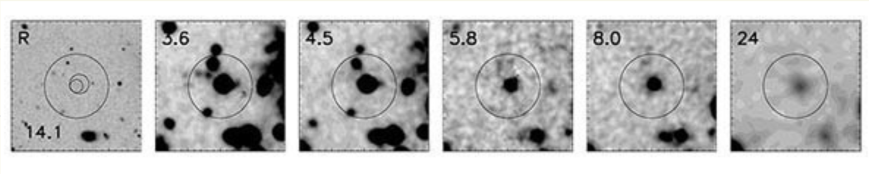
eg. Max Pettini

Note a galaxy's stellar luminosity being dominated by high mass stars doesn't mean that the observed luminosity is





Dust is a further complicating factor which means that energy output may not be at expected wavelengths, i.e., far-IR



Some distant galaxies only become detectable at long wavelengths

eg Richard McMahon

1.5.2 Probability Distribution Example 2 - Stellar Feedback to the ISM

How do previous generations of stars affect their environment?

- energetic radiation
- stellar winds
- supernovae

Is this constructive for star formation?

- encourage collapse of gas

Or destructive?

- sweeps up and removes gas

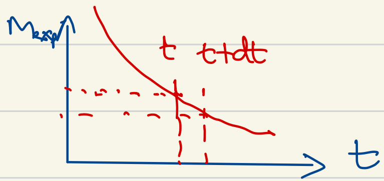
Consider energy input from supernovae in a cluster of stars that all formed together

Make some assumptions: IMF $f(m) \propto m^{-\alpha}$ where $\alpha = 2.35$

Time from star birth to SN explosion for $> 8 M_\odot$: $t_{exp} \propto m^{-b}$ where $b = 2.5$

Energy release per SN is independent of m at $E_{SN} \sim 10^{44}$ J

The mass of stars that are exploding at time t: $m_{exp} \propto t^{-1/b}$



$\therefore$ In $t \to t+dt$ stars exploding are those in $M_{exp} \to M_{exp}+dM$

where $dm = (dm_{exp}/dt)\,dt$

So the number of stars exploding is $\propto f(m)\,dm \propto f(m)(dm_{exp}/dt)\,dt$

$\propto m_{exp}^{-\alpha}\, t^{-1/b-1}\, dt \propto t^{\frac{\alpha-1}{b}-1}\, dt$

And the energy input rate is $L \propto t^{\frac{\alpha-1}{b}-1} \propto t^{-0.46}$

Simulations of energy input to the ISM following a burst of massive star formation

1erg = 10^{-7} J

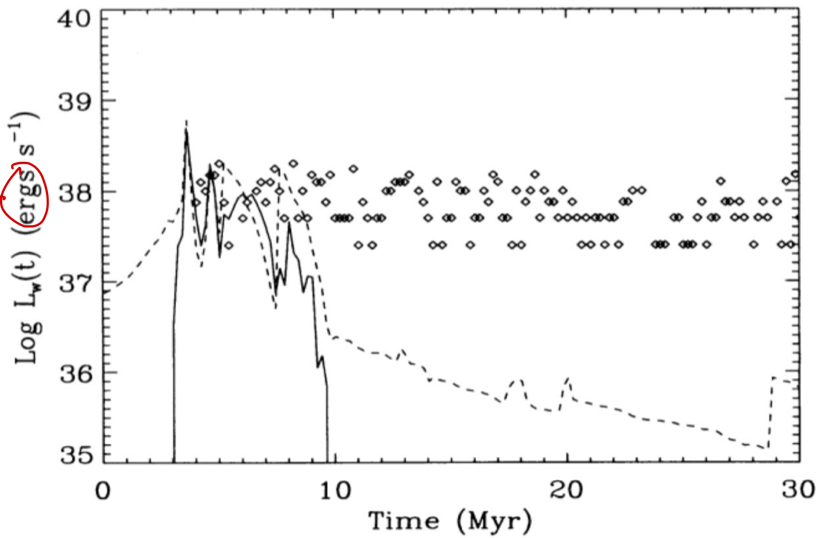
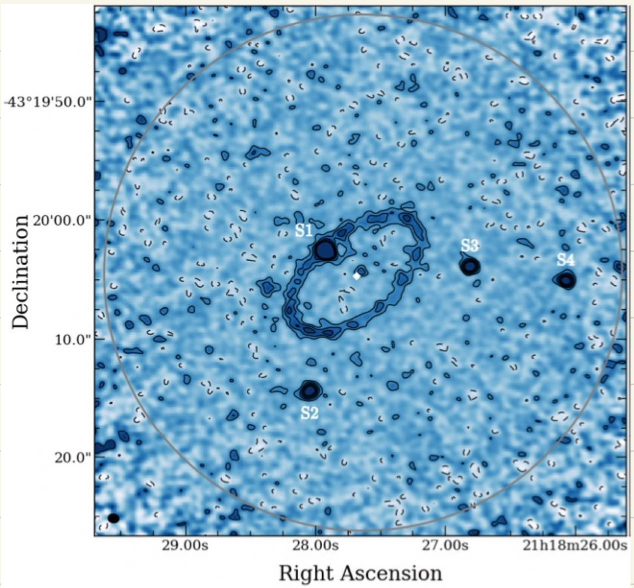


FIG. 3.—Rate of mechaical energy input, $L_w(t)$, for OB stars (*dashed line*), Wolf-Rayet stars (*solid line*) and supernovae (*diamonds*) in a coeval model of 100 stars (8–80 $M_\odot$).

1.5.3 Probability Distribution Example 3 - Sub-mm Galaxy Counts



Background galaxies photobomb images of Kuiper belts of nearby stars

Important to understand how common these are, to assess if these are galaxies or features of the planetary system

There is a distribution of galaxy brightness, with more faint galaxies than bright ones, the number detected depending on how deep you look

The sub-mm galaxy distribution is measured by surveys of large areas

Exam question from 2022:

(i) A survey at a wavelength $\lambda = 1.1\,\mathrm{mm}$ covering an area of $A = 10$ square arcmin has detected a number of galaxies and measured their flux densities S (in mJy) down to a limit of $S_{\mathrm{lim}} = 100\,\mu\mathrm{Jy}$. These detections have been used to determine the number of galaxies per square degree with flux densities in the range S to $S + \mathrm{d}S$ to be

$$n(S)\mathrm{d}S = N_0(S/S_0)^{-\alpha}\mathrm{d}(S/S_0),$$

where $N_0 = 2700\,\mathrm{deg}^{-2}$, $S_0 = 2.6\,\mathrm{mJy}$, and $\alpha = 1.81$. Estimate how many galaxies were detected.

Estimate the flux density of the brightest galaxy detected in the survey.

Comment on whether it is the brightest or faintest galaxies that contribute most to the cosmic infrared background at this wavelength.

Number expected to detect brighter than S_{lim} is $n_{det}(> S_{lim})$

$$= \int_{S_{lim}}^{\infty} n(S)\,dS \,.\, A$$

$$= A \,.\, N_0 S_0^{\alpha-1} \int_{S_{lim}}^{\infty} S^{-\alpha}\,dS$$

$$= A \,.\, N_0 S_0^{\alpha-1} \frac{1}{1-\alpha}\left[S^{1-\alpha}\right]_{S_{lim}}^{\infty}$$

As $\alpha > 1$

$$\sim A \,.\, N_0 \frac{1}{\alpha-1}\left(S_{lim}/S_0\right)^{1-\alpha} \simeq 130$$

Exam question from 2022:

$$n(S)\mathrm{d}S = N_0(S/S_0)^{-\alpha}\mathrm{d}(S/S_0),$$

where $N_0 = 2700\,\mathrm{deg}^{-2}$, $S_0 = 2.6\,\mathrm{mJy}$, and $\alpha = 1.81$. Estimate how many galaxies were detected.

Estimate the flux density of the brightest galaxy detected in the survey.

Comment on whether it is the brightest or faintest galaxies that contribute most to the cosmic infrared background at this wavelength.

Number expected to detect brighter than S_{lim} is $n_{det}(>S_{lim}) = A.N_0 \frac{1}{\alpha-1}(S_{lim}/S_0)^{1-\alpha}$

Estimate brightest galaxy detected in survey by setting $n_{det}(>S_b) \sim 1 \rightarrow S_b = 41$ mJy

Why? "Number expected to detect" is the mean in a Poisson distribution of the number actually observed

For a Poisson distribution with mean λ, a number of occurrences k occurs with probability $\lambda^k e^{-\lambda}/k!$

$\rightarrow$ if $\lambda=1$ we expect 0 : 36.8%, 1 : 36.8%, ≥ 2 : 26.4%

Total flux per square degree $= \int_{S_{min}}^{S_{max}} S\,n(S)\,dS \propto [S_{max}^{0.19} - S_{min}^{0.19}]$

$\rightarrow$ brightest but survey only constrained distribution in range $S_{lim} \rightarrow S_b$

How to tell if galaxies? Proper motion

1.5.4 Probability Distribution Example 4 - Collisional Cascade

Can we predict the size distribution in the asteroid belt, $n(a)$, where a is the size of the asteroid?

Assume a power law: $n(a) \propto a^{-b}$ → what is b?

Consider a logarithmic size bin: number of asteroids in bin $\propto a\, n(a) \propto a^{1-b}$

mass of asteroids in bin $\propto a^3 \cdot a^{1-b} \propto a^{4-b}$

as smaller collisions cause slow erosion and larger ones are rare

Assume that collisions with other asteroids in the bin dominate the mass loss and that collision velocity is independent of a

Rate of mass losing collisions $\propto n \sigma v \propto a^{1-b} a^2 \propto a^{3-b}$

Mass loss rate from bin $\propto a^{3-b} \cdot a^{4-b} \propto a^{7-2b}$

Remember: mass loss from a logarithmic size bin is $\propto a^{7-2b}$

Given that in steady state mass loss from logarithmic size bins is independent of size:

$\therefore b = 7/2$, known as MRN and seen in asteroid belt and ISM

Proof: Consider bin k in the distribution

Assume the fraction of mass going into bin i is scale independent and so can be written $F(k-i)$

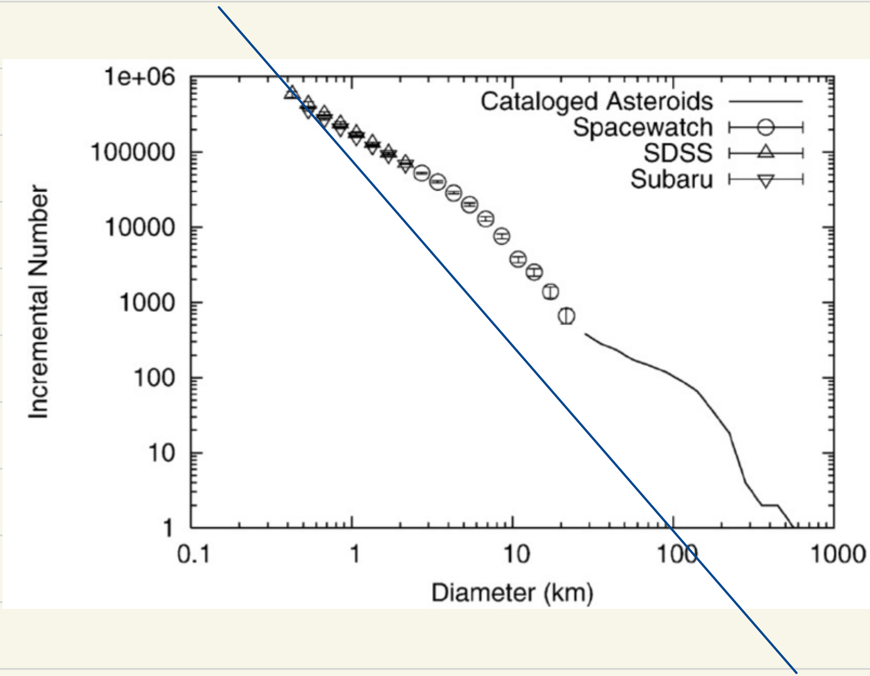
This means that the rate of mass gain in bin i from collisions in other bins is $\dot{m}_i^+ = \sum_k \dot{m}_k^- F(k-i)$

In steady state this is equal to the mass loss from bin $= \dot{m}_i^-$

All mass must go somewhere and so we know $\sum_k F(k-i) = 1$

Thus one solution must be that $\dot{m}_i^- = \dot{m}_k^-$

Observed asteroid belt size distribution $n(>a) \propto a^{1-b} \propto a^{-5/2}$



True distribution is slightly shallower than MRN because larger asteroids are harder to disrupt due to their self gravity

Can now do to QS on EG1