7 Secular Perturbations

- Terms independent of λ and λ' → long term effect of perturber's gravity
- Gauss' averaging method: sec perts are equivalent to perts from potential obtained by spreading the perturber's mass around its orbit w line density determined by velocity
  → think of pl system as interacting wires.

Secular disturbing function from MD99 Appendix B

All direct terms, zeroth order, from Table B.1, ϖ $j=0$; so to 2^{nd} order:

$$R_D^{sec} = [f_1 + f_2(e^2+e'^2) + f_3(s^2+s'^2)]\cos(0) + f_{10}\, ee'\cos(\varpi'-\varpi) + f_{14}\, ss'\cos(\Omega'-\Omega) + 4^{th} \text{ order terms}$$

where $s = \sin I/2 \approx I/2$

NOTE: terms involving eccentricity and incl^n are decoupled to 2^{nd} order (as $\Phi = O(s^2, s'^2, ss')$)

Getting constants from Table B.3:

$$R_D^{sec} = \tfrac{1}{2}b^0_{1/2}(\alpha) + \tfrac{1}{8}(2\alpha D + \alpha^2 D^2)b^0_{1/2}(\alpha)(e^2+e'^2) - \tfrac{1}{4}\alpha\left(b^{-1}_{3/2}(\alpha) + b^1_{3/2}(\alpha)\right)\left[(I/2)^2 + (I'/2)^2\right] + \dots$$
$$+ \tfrac{1}{4}[2 - 2\alpha D - \alpha^2 D^2]\, b^1_{1/2}(\alpha)\, ee'\cos(\varpi'-\varpi) + \alpha\, b^1_{3/2}(\alpha)(I/2)(I'/2)\cos(\Omega'-\Omega)$$

where $\alpha = a/a'$

Simplifying Laplace coefficients $\boxed{6.23}$:

$$b^j_s(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \cos j\psi\,[1 - 2\alpha\cos\psi + \alpha^2]^{-s}\, d\psi$$

$$\therefore\ Db^j_s(\alpha) = \partial/\partial\alpha\,(b^j_s(\alpha)) = \frac{1}{\pi}\int_0^{2\pi}\cos j\psi\,(2s\cos\psi - 2\alpha s)[1 - 2\alpha\cos\psi + \alpha^2]^{-s-1}\, d\psi$$

Using $2\cos j\psi\cos\psi = \cos(j-1)\psi + \cos(j+1)\psi$

$$\therefore\ Db^j_s(\alpha) = \frac{1}{\pi}\int_0^{2\pi}[s\cos(j-1)\psi + s\cos(j+1)\psi - 2\alpha s\cos j\psi][1 - 2\alpha\cos\psi + \alpha^2]^{-s-1}\, d\psi$$
$$= s\,b^{j-1}_{s+1}(\alpha) + s\,b^{j+1}_{s+1}(\alpha) - 2\alpha s\, b^j_{s+1}(\alpha)$$

etc

$$(2\alpha D + \alpha^2 D^2)b^0_{1/2}(\alpha) = \alpha b^1_{3/2}(\alpha)$$
$$(2 - 2\alpha D - \alpha^2 D^2)b^1_{1/2}(\alpha) = -\alpha b^2_{3/2}(\alpha)$$

$\boxed{7.1}$

So $R_D^{sec} = \tfrac{1}{2}b^0_{1/2}(\alpha) + \tfrac{1}{8}\alpha b^1_{3/2}(\alpha)[e^2+e'^2] - \tfrac{1}{8}\alpha b^1_{3/2}(\alpha)[I^2+I'^2] - \tfrac{1}{4}\alpha b^2_{3/2}(\alpha)\, ee'\cos(\varpi'-\varpi) + \tfrac{1}{4}\alpha b^1_{3/2}(\alpha)\, II'\cos(\Omega'-\Omega)$

AND $R = (\mu'/a')R_D^{sec}$
$\quad\ \ R' = (\mu/a')R_D^{sec}$ $\boxed{7.2}$

Rewrite using new variables: $k = e\cos\varpi$, $h = e\sin\varpi$; $q = I\cos\Omega$, $p = I\sin\Omega$ $\Big\}$ $\boxed{7.3}$

$\therefore\ e^2 = k^2 + h^2$

and $ee'\cos(\varpi'-\varpi) = ee'[\cos\varpi\cos\varpi' + \sin\varpi\sin\varpi'] = kk' + hh'$ etc.

Ignoring $\tfrac{1}{2}b^0_{1/2}(\alpha)$ term:

$$R_D^{sec} = \tfrac{1}{8}\alpha b^1_{3/2}(\alpha)[h^2+k^2+h'^2+k'^2-p^2-q^2-p'^2-q'^2] - \tfrac{1}{4}\alpha b^2_{3/2}(\alpha)[kk'+hh'] + \tfrac{1}{4}\alpha b^1_{3/2}(\alpha)[qq'+pp'] \qquad \boxed{7.4}$$

Consider a system of N_{pl} secularly interacting planets

Potential is additive, so for the j-th planet, noting that only terms involving elements of that planet are important:

$$R_j = \sum_{i=1,\, i\neq j}^{N_{pl}} \left(\frac{Gm_i}{a_{out}}\right)\left[\frac{1}{8}\left(\frac{a_{in}}{a_{out}}\right)b^1_{3/2}\left(\frac{a_{in}}{a_{out}}\right)[k_j^2+h_j^2-q_j^2-p_j^2] - \frac{1}{4}\left(\frac{a_{in}}{a_{out}}\right)b^2_{3/2}\left(\frac{a_{in}}{a_{out}}\right)[k_jk_i+h_jh_i] + \frac{1}{4}\left(\frac{a_{in}}{a_{out}}\right)b^1_{3/2}\left(\frac{a_{in}}{a_{out}}\right)[q_jq_i+p_jp_i]\right]$$

Rewrite more succinctly:

$$R_j = n_j a_j^2\left[\tfrac{1}{2}A_{jj}(k_j^2+h_j^2) + \tfrac{1}{2}B_{jj}(q_j^2+p_j^2) + \sum_{i=1,\, i\neq j}^{N_{pl}}\left[A_{ji}(k_jk_i+h_jh_i) + B_{ji}(q_jq_i+p_jp_i)\right]\right]$$

where $A_{ji} = -\tfrac{1}{4}n_j\left(\frac{m_i}{M_*}\right)\alpha_{ji}\bar{\alpha}_{ji}\, b^2_{3/2}(\alpha_{ji})$ $\quad = \left(\frac{Gm_i}{a_{out}}\right)\left(-\frac{1}{4}\frac{a_{in}}{a_{out}}b^2_{3/2}\left(\frac{a_{in}}{a_{out}}\right)\right)/n_ja_j^2$

$\quad B_{ji} = \tfrac{1}{4}n_j\left(\frac{m_i}{M_*}\right)\alpha_{ji}\bar{\alpha}_{ji}\, b^1_{3/2}(\alpha_{ji})$

$\quad A_{jj} = -B_{jj} = \sum_{i=1,\, i\neq j}^{N_{pl}} B_{ji}$ $\qquad \boxed{7.5}$

$\quad \alpha_{ji} = a_i/a_j$, $\bar{\alpha}_{ji} = 1$ if $a_j > a_i$
$\quad \alpha_{ji} = a_j/a_i$, $\bar{\alpha}_{ji} = a_j/a_i$ $\quad < a_i$ $\Big\}$ ie $\alpha_{ji} = a_{in}/a_{out}$, $\bar{\alpha}_{ji} = a_j/a_{out}$

NB $Gm_i/a_{out} = GM_*\left(\frac{M_i}{M_*}\right)/a_{out} = n_j^2 a_j^2\left(\frac{M_i}{M_*}\right)\bar{\alpha}_{ji}$

Get eqns of motion from Lagrange's Planetary Equations

To lowest order, from $\boxed{6.20}$:

$$\dot{e} = -(na^2e)^{-1}\,\partial R/\partial\varpi$$
$$\dot{\varpi} = (na^2e)^{-1}\,\partial R/\partial e$$
$$\dot{I} = -(na^2I)^{-1}\,\partial R/\partial\Omega$$
$$\dot{\Omega} = (na^2I)^{-1}\,\partial R/\partial I$$

Thus from $\boxed{7.3}$:

$$\dot{h} = (\partial h/\partial e)\dot{e} + (\partial h/\partial\varpi)\dot{\varpi} = (na^2e)^{-1}\Big[-(\partial h/\partial e)\big[(\partial R/\partial k)(\partial k/\partial\varpi) + \underbrace{(\partial R/\partial h)(\partial h/\partial\varpi)}_{\text{cancels}}\big] + (\partial h/\partial\varpi)\big[(\partial R/\partial k)(\partial k/\partial e) + \underbrace{(\partial R/\partial h)(\partial h/\partial e)}_{\text{cancels}}\big]\Big]$$
$$= (na^2e)^{-1}\left[-\sin\varpi(-e\sin\varpi) + e\cos\varpi\cos\varpi\right]\partial R/\partial k$$
$$= (na^2)^{-1}\,\partial R/\partial k$$

And similarly

$$\dot{k} = -(na^2)^{-1}\,\partial R/\partial h$$
$$\dot{q} = -(na^2)^{-1}\,\partial R/\partial p$$
$$\dot{p} = (na^2)^{-1}\,\partial R/\partial q$$

$\boxed{7.6}$

Applying this to $\boxed{7.5}$:

$$\therefore\ \dot{h}_j = A_{jj}k_j + \textstyle\sum_{i=1,i\neq j}^{N_{pl}} A_{ji}k_i \quad = \textstyle\sum_{i=1}^{N_{pl}} A_{ji}k_i$$
$$\dot{k}_j = -A_{jj}h_j - \textstyle\sum_{i=1,i\neq j}^{N_{pl}} A_{ji}h_i$$
$$\dot{p}_j = B_{jj}q_j + \textstyle\sum_{i=1,i\neq j}^{N_{pl}} B_{ji}q_i$$
$$\dot{q}_j = -B_{jj}p_j - \textstyle\sum_{i=1,i\neq j}^{N_{pl}} B_{ji}p_i$$

$\boxed{7.7}$

Can write even more succinctly using

$$\underline{z} = \begin{pmatrix} e_1e^{i\varpi_1} \\ \vdots \\ e_{N_{pl}}e^{i\varpi_{N_{pl}}}\end{pmatrix} = \begin{pmatrix} k_1 + ih_1 \\ \vdots \\ k_{N_{pl}} + ih_{N_{pl}}\end{pmatrix}$$

$$\underline{y} = \begin{pmatrix} I_1e^{i\Omega_1} \\ \vdots \\ I_{N_{pl}}e^{i\Omega_{N_{pl}}}\end{pmatrix} = \begin{pmatrix} q_1 + ip_1 \\ \vdots \\ q_{N_{pl}} + ip_{N_{pl}}\end{pmatrix}$$

$\boxed{7.8}$

As $\dot{z}_j = \dot{k}_j + i\dot{h}_j = \sum_{i=1}^{N_{pl}} A_{ji}(-h_i + ik_i) = i\sum_{i=1}^{N_{pl}} A_{ji}z_i$

$$\therefore\ \underline{\dot{z}} = iA\underline{z}$$
$$\underline{\dot{y}} = iB\underline{y}$$

where $A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & \ddots & \\ \vdots & & A_{N_{pl}N_{pl}}\end{pmatrix}$ $\quad B = \begin{pmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & \ddots & \\ \vdots & & B_{N_{pl}N_{pl}}\end{pmatrix}$

$\boxed{7.9}$

Solve as an eigenvalue problem

From $\boxed{3.21}$*: $\underline{z} = W_Ae^{i\Lambda_At}W_A^{-1}\underline{z}_0$, $\quad\underline{y} = W_Be^{i\Lambda_Bt}W_B^{-1}\underline{y}_0$

OR

$$\underline{z} = \begin{pmatrix} e_{11} & e_{12} & \cdots \\ e_{21} & \ddots & \\ \vdots & & e_{N_{pl}N_{pl}}\end{pmatrix}\begin{pmatrix} e^{i[g_1t+\beta_1]} \\ \vdots \\ e^{i[g_{N_{pl}}t+\beta_{N_{pl}}]}\end{pmatrix} \quad \text{and } z_j = \textstyle\sum_{i=1}^{N_{pl}} e_{ji}e^{i[g_it+\beta_i]}$$

$$\underline{y} = \begin{pmatrix} I_{11} & I_{12} & \cdots \\ I_{21} & \ddots & \\ \vdots & & I_{N_{pl}N_{pl}}\end{pmatrix}\begin{pmatrix} e^{i[f_1t+\gamma_1]} \\ \vdots \\ e^{i[f_{N_{pl}}t+\gamma_{N_{pl}}]}\end{pmatrix}$$

$\boxed{7.10}$

- This is Laplace-Lagrange secular solution showing that orbital evol is sum of N_{pl} sinusoidal oscillns w periods given by eigenfrequencies (that are real)
- Good for $e, I \ll 1$ (no orbits overlapping or near resonance); at higher order e and I no longer decoupled
- One of incln eigenvalues is 0 because choice of ref plane is arbitrary and only mutual inclns is important (EG 4.1); not so for $\underline{z}$ as pericentre introduces a preferred orientation

* Note that $\boxed{3.21}$ was solution to $\underline{\dot{X}} = A\underline{X} \Rightarrow \underline{X} = W_Ae^{\Lambda_At}W_A^{-1}\underline{X}_0$; working through the derivation w extra "i" leads to the given equation

Test Particles in a Planetary System (EG 4.2)

Are subject to same disturbing fn $\boxed{7.4}$, which for the N_{pl} system before can be described by $\boxed{7.5}$, but dropping the subscript "j", also dropping $i \neq j$ condition in the sum (NOTE I swapped $i \to j$ to allow i to be used later):

$$\therefore\ R = na^2\left[\tfrac{1}{2}A(k^2+h^2) + \tfrac{1}{2}B(q^2+p^2) + \sum_{j=1}^{N_{pl}}\left[A_j(kk_j+hh_j) + B_j(qq_j+pp_j)\right]\right]$$

$$\left.\begin{aligned} \text{where}\quad A_j &= -\tfrac{1}{4}n\left(\tfrac{m_j}{M_\odot}\right)\alpha_j\bar{\alpha}_j b^{(2)}_{3/2}(\alpha_j)\\ B_j &= \tfrac{1}{4}n\left(\tfrac{m_j}{M_\odot}\right)\alpha_j\bar{\alpha}_j b^{(1)}_{3/2}(\alpha_j)\\ A &= -B = \sum_{j=1}^{N_{pl}} B_j\\ \alpha_j &= a_j/a,\ \bar{\alpha}_j = 1 \text{ if } a > a_j\\ \alpha_j &= a/a_j,\ \bar{\alpha}_j = a/a_j \text{ if } a < a_j \end{aligned}\right\}\quad \boxed{7.11}$$

Can use Lagrange's planetary eqns $\boxed{7.6}$, which gives an analogous set of eqns $\boxed{7.7}$ with dropped subscripts "j" etc:

$$\therefore\ \dot{h} = Ak + \sum_{j=1}^{N_{pl}} A_j k_j$$
$$\dot{k} = -Ah - \sum_{j=1}^{N_{pl}} A_j h_j$$

that can be combined using $z = e\, e^{i\varpi} = k + ih$

$$\therefore\ \dot{z} = iAz + i\sum_{j=1}^{N_{pl}} A_j z_j \qquad \boxed{7.12}$$

where $z_j = \sum_{i=1}^{N_{pl}} e_{ji}\, e^{i[g_i t + \beta_i]}$ from L-L sol^n $\boxed{7.10}$

Solve this using integrating factor

$$d/dt\left[z e^{\int -iA\,dt'}\right] = i\sum_{j=1}^{N_{pl}} A_j z_j e^{\int -iA\,dt'}$$
$$= i\sum_{j=1}^{N_{pl}} A_j \sum_{i=1}^{N_{pl}} e_{ji} e^{i[g_i t + \beta_i - \int A\,dt']}$$

If $A \neq A(t)$ then $\int A\,dt' = At + \beta$

$$z = e^{i(At+\beta)}\int \sum_{i=1}^{N_{pl}}\left(\sum_{j=1}^{N_{pl}} A_j e_{ji}\right) i e^{i[(g_i - A)t + \beta_i - \beta]}\,dt$$

Let $\nu_i = \sum_{j=1}^{N_{pl}} A_j e_{ji}$

$$\therefore\ z = e_p e^{i(At+\beta)} + \sum_{i=1}^{N_{pl}}\left[\frac{\nu_i}{g_i - A}\right] e^{i(g_i t + \beta_i)} \qquad \boxed{7.13}$$

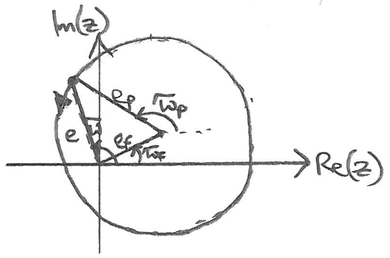
where e_p and β are constants of integration set by initial conditions

Write this as:

$$z = z_p + z_f = e_p e^{i\varpi_p} + e_f e^{i\varpi_f} \qquad \boxed{7.14}$$

where e_f, ϖ_f : are forced eccentricity and forced longitude of pericentre that are known functions of $(m_i, a_i, e_i, \varpi_i)$ that vary slowly in sec. evol^n of planet orbits.

e_p, ϖ_p : are proper (or free) eccentricity and proper longitude of pericentre, set by initial cond^{ts} (ie are intrinsic to particle), though ϖ_p increases linearly in time at rate A

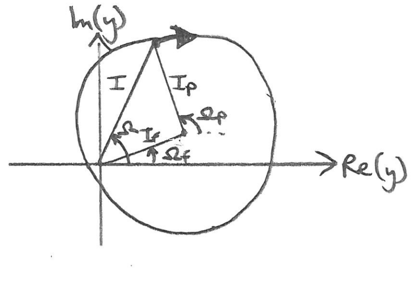


So particle precesses anticlockwise around a circle of radius e_p at a rate A, where the centre of the circle also moves in time

Inclination evolution

Similar: $\dot{y} = -iAy + i\sum_{j=1}^{N_{pl}} B_j y_j$

$$\left.\begin{aligned} y &= I_p e^{i(-At+\gamma)} - \sum_{i=1}^{N_{pl}} \frac{\mu_i}{-A - f_i} e^{i(f_i t + \gamma_i)} \qquad \text{where } \mu_i = \sum_{j=1}^{N_{pl}} B_j I_{ji}\\ &= y_p + y_f = I_p e^{i\Omega_p} + I_f e^{i\Omega_f} \end{aligned}\right\}\quad \boxed{7.15}$$



So particle precesses clockwise ...

Secular resonances

Locations where precession rate equals one of eigenfrequencies of planetary system

$\underline{A = g_i}$ or $\underline{-A = f_i}$. $\boxed{7.16}$

at which point $z_f \to \infty$ or $y_f \to \infty$

Since g_i, f_i are independent of time, and A only depends on a (and m_i, a_i that are fixed), secular resonances are at fixed locations in a

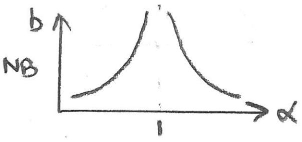
$\boxed{eg}$ • Asteroid belt is bounded by secular resonances (but are surfaces in a, e, I due to higher order terms)
• Secular resonances may move in time eg if a_{pl}, M_{pl} are varying due to non-secular processes
$\to$ secular resonance sweeping

Forced elements near a planet L

Now $z_f = \sum_{i=1}^{N_{pl}} \left[\frac{\sum_{j=1}^{N_{pl}} A_j e_{ji}}{g_i - A}\right] e^{i(g_i t + \beta_i)}$

As $\alpha_L = a/a_L \to 1$, $b'_{3/2}(\alpha_L) \to \infty$ and $b^2_{3/2}(\alpha_L) \to \infty$

NB



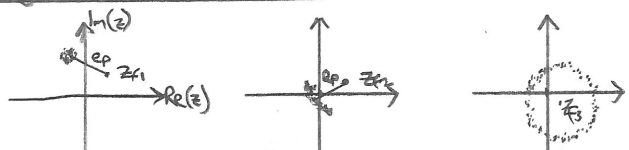
$\therefore$ $A_L \to \infty$ and $\sum_{j=1}^{N_{pl}} A_j e_{ji} \to A_L e_{Li}$

Also, $A \to \infty$ and $g_i - A \to -A = -B_L$

$$\therefore\ z_f \to \sum_{i=1}^{N_{pl}} \left(\frac{A_L}{-B_L}\right) e_{Li}\, e^{i(g_i t + \beta_i)} = \left(\frac{A_L}{-B_L}\right) z_L = \underbrace{\frac{b^2_{3/2}(\alpha \to 1)}{b'_{3/2}(\alpha \to 1)}}_{\text{see MD99 eqs 7.87,88 for proof that this} \to 1} z_L = z_L \qquad \boxed{7.17}$$

$\therefore$ forced elements are planet's orbit

Hirayama asteroid families



i. Asteroid w/ e_p breaks into fragments

ii. z_f moves and fragments precess at slightly different rates

iii. Fragments end up distributed around circle of radius e_p centred on forced elements

Expansions for high e, I :

Take additional terms in disturbing function, or use hierarchy if $a_2 \gg a_1$ and expand in α using Legendre...

- Quadrupole expansion is to $O(\alpha^2)$
  eg. Kozai (1962) for $m_0 \gg m_2 \gg m_1$ and $e_2 = 0$
- Octupole expansion is to $O(\alpha^3)$
  eg. Lee & Peale (2003) for coplanar system, Ford et al. (2000)
- Higher orders, eg. Migaszewski & Goździewski (2008)

Secular terms are found by double averaging $\langle\langle R \rangle\rangle = \left(\frac{1}{2\pi}\right)^2 \iint R\, dM\, dM'$

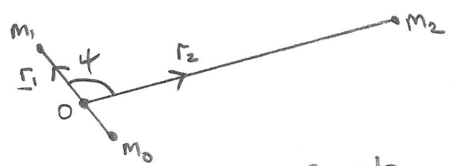
Secular perturbations with...

The disturbing function is additive, so can take equ^s of motion $\boxed{7.9}$ or $\boxed{7.12}$ and add effects of other perturbations.

$\boxed{eg}$ • Murray & Dermott showed the effect of resonant forces between two of the planets
• EX2010 Q3 considered effect of oblateness.

Kozai Mechanism (see ch.9 of Valtonen & Karttunen)

For hierarchical systems, Jacobi coordinates are used to remove high order variation in orbital elements of m_2 due to motion of m_1 around m_0 (EX2010Q1)



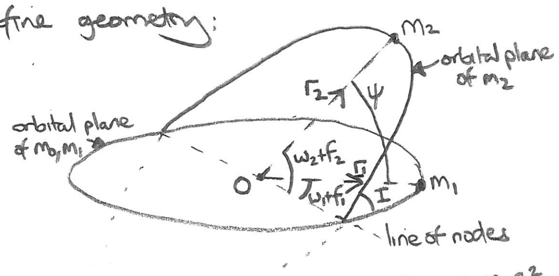
$\underline{r}_1$ = pos^n of m_1 w.r.t. m_0

$\underline{r}_2$ = pos^n of m_2 w.r.t. c.o.m. of m_0 and m_1

To second order in expansion of r_1/r_2, the disturbing f^n is (ie quadrupole):

$$R = \frac{G m_0 m_1 m_2}{2(m_0+m_1)}\left(\frac{1}{r_2}\right)\left(\frac{r_1}{r_2}\right)^2(3\cos^2\psi - 1) \qquad \boxed{7.18}$$

Define geometry:



$$\therefore \cos\psi = \cos(\omega_2+f_2)\cos(\omega_1+f_1) + \sin(\omega_1+f_1)\sin(\omega_2+f_2)\cos I \qquad \boxed{7.19}$$

Double averaging: $$\langle\langle R\rangle\rangle = \left[\frac{G m_0 m_1 m_2 a_1^2}{8(m_0+m_1)a_2^3(1-e_2^2)^{3/2}}\right]\left[2+3e_1^2 - 3\sin^2 I\,(5e_1^2\sin^2\omega_1 + 1 - e_1^2)\right] \qquad \boxed{7.20}$$

Use Lagrange's planetary equations:

Outer orbit m_2: $\dot{a}_2 = \dot{e}_2 = \dot{I}_2 = 0$ and only orientation of orbit vary (but not shape)

Inner orbit m_1: $\dot{a}_1 = 0$ (by def^n of sec. pert^s)

$$\left.\begin{aligned}
\dot{e}_1 &= \tfrac{15}{8} e_1\sqrt{1-e_1^2}\,\sin 2\omega_1 \sin^2 I\\
\dot{\omega}_1 &= \tfrac{3}{4}(1-e_1^2)^{-1/2}\left[2(1-e_1^2) + 5\sin^2\omega_1\,(e_1^2 - \sin^2 I)\right]\\
\dot{I} &= -\tfrac{15}{8} e_1^2 (1-e_1^2)^{-1/2}\sin 2\omega_1 \sin I\cos I\\
\dot{\Omega}_1 &= -\tfrac{1}{4}\cos I\,(1-e_1^2)^{-1/2}\left[3+12e_1^2 - 15e_1^2\cos^2\omega_1\right]
\end{aligned}\right\} \quad \boxed{7.21}$$

where unit of time is $G m_0 m_2\left[(m_0+m_1)n_1 a_2^3(1-e_2^2)^{3/2}\right]^{-1}$

This means that $\sqrt{1-e_1^2}\,\cos I = \text{const}$ $\boxed{7.22}$

Can be solved, but for qualitative sol^n, consider evol^n when $e_1 \ll 1$ (ie, ignore terms $O(e_1^2)$)

$\therefore \dot{I} \simeq 0$

$\therefore \dot{\omega}_1 \simeq \frac{3}{4}(2 - 5\sin^2 I\sin^2\omega_1)$

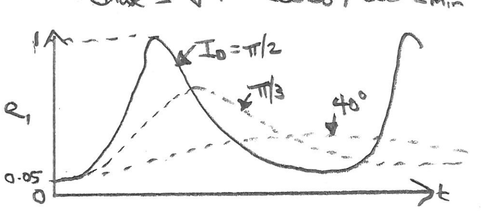
This can be solved leading to qualitatively different behaviour depending on I wrt $I_{crit} = 39.2^\circ$ $\boxed{7.23}$

- For $\sin^2 I < 0.4$, $\dot{\omega}_1 > 0$ always, get sinusoidal oscillation in e_1
- For $\sin^2 I > 0.4$, as $t\to\infty$, $\sin\omega_1 \to \sqrt{0.4}/\sin I$

$\therefore \dot{e}_1 \propto e_1$ s.t. $e_1 = e_{10}\, e^{2.38\sqrt{\sin^2 I - 0.4}\, t}$

So e_1 grows rapidly, but by $\boxed{7.22}$ I must decrease and there is a maximum eccentricity

$e_{max} = \sqrt{1 - \cos^2 I_0/\cos^2 I_{min}}$ where $\sin^2 I_{min} = 0.4$ $\boxed{7.24}$



Kozai cycle where eccentricity (and inclination) oscillate with a period

$$P_{Kozai} \simeq \left(\frac{a_2}{a_1}\right)^3\left(\frac{m_c}{m_2}\right)t_{per} \qquad \boxed{7.25}$$

$\boxed{eg}$ If $I > 39.2^\circ$, an outer binary would subject an inner planetary system to large increases in eccentricity, with correspondingly large changes in I, on the timescale given by $\boxed{7.25}$

Although formally it does this regardless of m_2, the timescale is long for small m_2, and secular precession can be much smaller than that due to gravity of other planets etc, preventing this behaviour.

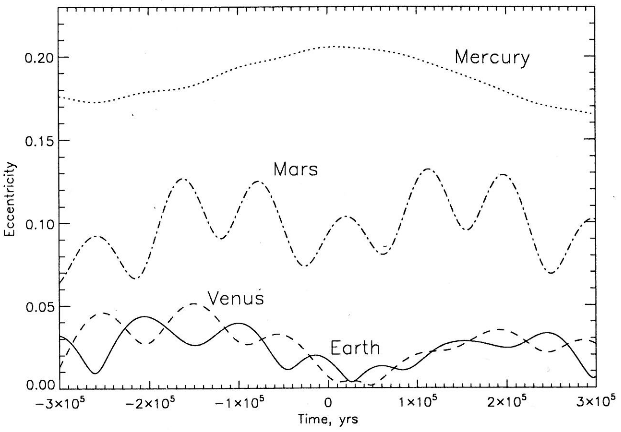
Another application is to circumplanetary orbits: the secular pert^s of Sun's gravity cause Kozai cycles in irregular satellites with $I = 39^\circ - 141^\circ$

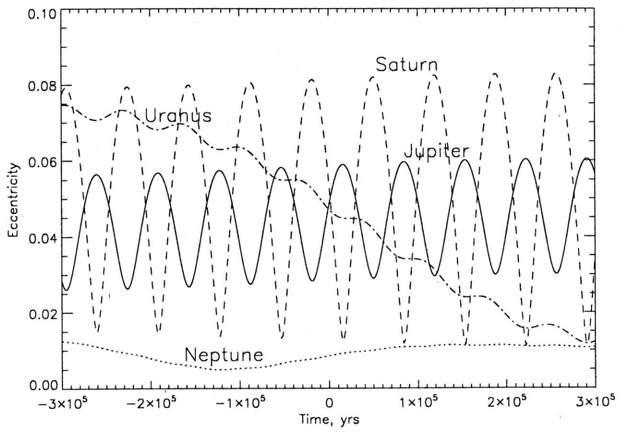
<u>Secular Perturbations</u>

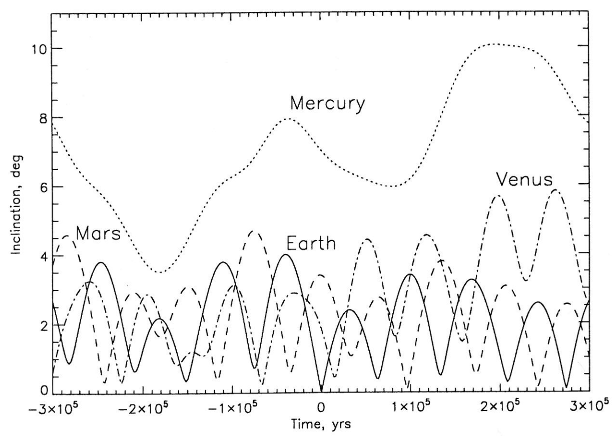
Secular Evolution of Solar System Planet Orbits

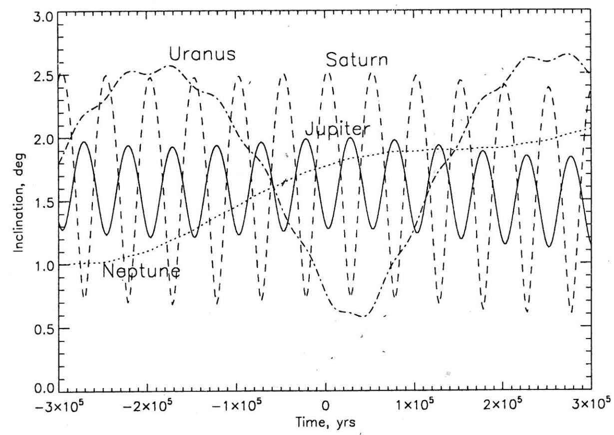
Inner Solar System

Outer Solar System









Secular Perturbations

Secular Evolution of Test Particles in the Solar System