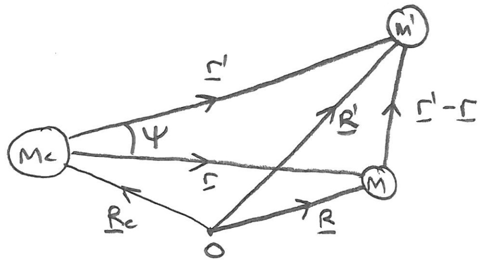
6 Disturbing Function



Equations of motion

$$m_c\ddot{\underline{R}}_c = Gm_cm\underline{r}/r^3 + Gm_cm'\underline{r}'/r'^3$$

$$m\ddot{\underline{R}} = -Gm_cm\underline{r}/r^3 + Gmm'(\underline{r}'-\underline{r})/|\underline{r}'-\underline{r}|^3$$

$$m'\ddot{\underline{R}}' = -Gm_cm'\underline{r}'/r'^3 + Gmm'(\underline{r}'-\underline{r})/|\underline{r}'-\underline{r}|^3$$

$\therefore$ eqn of rel. motion: $\ddot{\underline{r}} = \ddot{\underline{R}} - \ddot{\underline{R}}_c = -G(m_c+m)\underline{r}/r^3 + Gm'\left[\frac{\underline{r}'-\underline{r}}{|\underline{r}'-\underline{r}|^3} - \frac{\underline{r}'}{r'^3}\right]$ $\boxed{6.1}$

- This can be written $\ddot{\underline{r}} = \nabla(U+R)$
  where $U = -G(m_c+m)/r$ = 2 body potential
  $R = \underbrace{\mu'/|\underline{r}'-\underline{r}|}_{\text{direct terms}} - \underbrace{\mu'\underline{r}.\underline{r}'/r'^3}_{\text{indirect terms (from choice of origin)}}$ = Disturbing function $\boxed{6.2}$
  $\mu' = Gm'$

- Proof: $\nabla|\underline{r}'-\underline{r}|^{-1} = [\partial/\partial x, \partial/\partial y, \partial/\partial z]^T\,[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{-1/2}$
  $= -\frac{1}{2}|\underline{r}'-\underline{r}|^{-3}\,[-2(x'-x), -2(y'-y), -2(z'-z)]^T$
  $= (\underline{r}'-\underline{r})/|\underline{r}'-\underline{r}|^3$
  $\nabla\,\underline{r}.\underline{r}'/r'^3 = [\partial/\partial x, \partial/\partial y, \partial/\partial z]^T\,[(xx'+yy'+zz')/r'^3]$
  $= \underline{r}'/r'^3$

- By convention $r' > r$, and get R' by swapping dashed and undashed:
  $\ddot{\underline{r}}' = \nabla(U'+R')$, $U' = -G(m_c+m')/r'$, $R' = \mu/|\underline{r}'-\underline{r}| - \mu\underline{r}.\underline{r}'/r^3$ $\boxed{6.3}$

- U, 2 body potential, gives an orbit w $n^2a^3 = G(m_c+m)$ etc.
  R, the disturbing function, says how that orbit is perturbed by m'
  $\therefore$ we can define motion in terms of osculating elements = the two body orbit if disturbing fn removed

- Note 1: can add A/r to disturbing fn (and subtract from 2 body potential) to get eg.
  $U_{new} = -G(m_c+m+m')/r$
  $R_{new} = \mu'/|\underline{r}'-\underline{r}| - \mu'\underline{r}.\underline{r}'/r'^3 + \mu'/r$
  which simply changes the reference orbit to $n^2a^3 = G(m_c+m+m')$

- Note 2: this eqn is also valid for non-point-mass gravitational perts, as long as they can be expressed as a potential, such as perts due to planet's oblateness $\boxed{\text{EG 3.5}}$

- Remove vector notation using angle ψ

Cos rule: $|\underline{r}'-\underline{r}|^2 = [r^2 + r'^2 - 2rr'\cos\psi]$

$$\therefore \frac{1}{|\underline{r}'-\underline{r}|} = \frac{1}{r'}\left[1 + \left(\frac{r}{r'}\right)^2 - 2\left(\frac{r}{r'}\right)\cos\psi\right]^{-1/2} \quad \boxed{6.4}$$

$$\underline{r}.\underline{r}' = rr'\cos\psi \quad \boxed{6.5}$$

No exact solution, so expand taking advantage of hierarchy of system studied

Expansion for $r/r' \ll 1$

Binomial expansion of $\boxed{6.4}$

$$\frac{1}{|\underline{r}'-\underline{r}|} = \frac{1}{r'}\left[1 - \tfrac{1}{2}\left[-2\left(\tfrac{r}{r'}\right)\cos\psi + \left(\tfrac{r}{r'}\right)^2\right] + \tfrac{3}{8}\left[-2\left(\tfrac{r}{r'}\right)\cos\psi + \left(\tfrac{r}{r'}\right)^2\right]^2 + \ldots\right]$$

This can be written more succinctly using Legendre polynomials P_l

$$= \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) \qquad \boxed{6.6}$$

where $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$, $P_3(x)=\frac{1}{2}(5x^3-3x)$, . .

So $\boxed{6.2, 6.5} \Rightarrow R = \frac{\mu'}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) - \frac{\mu'}{r'}\left(\frac{r}{r'}\right)\cos\psi$

$$= \frac{\mu'}{r'}\sum_{l=2}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) \qquad \boxed{6.7}$$

where $l=0$ term was dropped as it is independent of r (and it's ∇R that matters)

Typically we want this in terms of orbital elements, for which it is useful to express in terms of r/a and r'/a' that are both $1+O(e,e')$

$$\therefore\ R = \left(\frac{\mu'}{a'}\right)\sum_{l=2}^{\infty}\left(\frac{r}{a}\right)^l\left(\frac{a'}{r'}\right)^{l+1}\alpha^l P_l(\cos\psi) \qquad \boxed{6.8}$$

where $\alpha = a/a'$

Similar expression for R', except can't just swap dashed/undashed as want to expand in r/r', not r'/r

$\boxed{6.6}$ still holds, so from $\boxed{6.3}$

$$R' = \frac{\mu}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) - \frac{\mu}{r}\left(\frac{r'}{r}\right)\cos\psi$$

$$= \frac{\mu}{r'}\sum_{l=2}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\psi) + \mu\cos\psi\left[r/r'^2 - r'/r^2\right] \qquad \boxed{6.9}$$

To get disturbing function, need to know r/a and $\cos\psi$ in terms of orbital elements.

Expansion of $\cos\psi$

$\boxed{6.5} \rightarrow \cos\psi = \underline{r}\cdot\underline{r}'/rr' = (xx' + yy' + zz')/rr'$

But $\boxed{1.12, 1.13, 1.14} \Rightarrow$
$$\begin{pmatrix} x/r \\ y/r \\ z/r \end{pmatrix} = \begin{pmatrix} \cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos I \\ \sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos I \\ \sin(\omega+f)\sin I \end{pmatrix}$$
and similarly for x'/r' etc.

$$\therefore \cos\psi = [C\Omega C\omega{+}f - S\Omega S\omega{+}f\,CI][C\Omega' C\omega'{+}f' - S\Omega' S\omega'{+}f'\,CI'] + [S\Omega C\omega{+}f + C\Omega S\omega{+}f\,CI][S\Omega' C\omega'{+}f' + C\Omega' S\omega'{+}f'\,CI']$$
$$+ [S\omega{+}f\,SI][S\omega'{+}f'\,SI']$$

To expand to lowest orders in I and I', note that for coplanar case $\psi = \theta' - \theta$ where $\theta = \varpi + f$ and $\theta' = \varpi' + f'$

$\therefore$ Let $\cos\psi = \cos(\theta' - \theta) + \Psi \qquad \boxed{6.10}$

$$\therefore\ \Psi = \cos\psi - \cos\psi(I = I' = 0)$$

This removes terms indep of (I, I'), retains unaffected those dep on $(\sin I, \sin I')$, turns $\cos I \rightarrow \cos I - 1$

$$\therefore\ \Psi = SI\,SI'\,S\omega{+}f\,S\omega'{+}f' + (CI-1)S\omega{+}f\,C\omega'{+}f'\,S\Omega'{-}\Omega + (CI'-1)C\omega{+}f\,S\omega'{+}f'\,S\Omega{-}\Omega'$$
$$+ (CI\,CI'-1)S\omega{+}f\,S\omega'{+}f'\,C\Omega{-}\Omega' \qquad \boxed{6.11}$$

where we needed trig ids: $S\Omega' C\Omega - S\Omega C\Omega' = S\Omega'{-}\Omega$, $S\Omega S\Omega' + C\Omega C\Omega' = C\Omega{-}\Omega'$

Letting $s = \sin I/2$, $s' = \sin I'/2$ gives some more trig' ids:

$$CI - 1 = -2s^2, \quad SI = 2s(1-s^2)^{1/2}, \quad CI\,CI' - 1 = -2s^2 - 2s'^2 + 4s^2 s'^2$$

$$\therefore\ \Psi = O(s^2, s'^2, ss')$$

i.e. inclinations come into the disturbing function at second order terms.

NOTE: $\cos\psi$ also has a dependence on eccentricity if written in terms of M not f...

Expansion of r/a (EG13)

From $\boxed{1.22}$, $E - M = e\sin E$ is an odd periodic fn and so can be expanded as a Fourier sine series

$$= \sum_{s=1}^{\infty} b_s(e) \sin sM \qquad \boxed{6.12}$$

Can find $b_s(e)$ by multiplying by $\sin sM$ then integrating $\int_0^{\pi} dM$ as only non-zero term is that in $\sin^2 sM$

$$\therefore b_s(e) = \frac{2}{\pi}\int_0^{\pi} e\sin E \sin sM \, dM$$

Integrate by parts to get:

$$= \frac{2}{\pi}\left[\left[e\sin E\left(-\tfrac{1}{s}\cos sM\right)\right]_0^{\pi} - \int_0^{\pi}\left(-\tfrac{1}{s}\cos sM\right)\frac{d\, e\sin E}{dM}\, dM\right]$$

From geometry $E = 0, \pi$ when $M = 0, \pi$, so first term cancels

And $de\sin E/dM = dE/dM - 1$ (from $\boxed{1.22}$)

$$\therefore b_s(e) = \frac{2}{\pi s}\left[\int_0^{\pi}\cos sM\, dE - \cancel{\int_0^{\pi}\cos sM\, dM}^{\,0}\right]$$

$$= \frac{2}{s}\frac{1}{\pi}\int_0^{\pi}\cos(sE - se\sin E)\, dE \qquad \boxed{6.13}$$

$$= \frac{2}{s} J_s(se) \qquad \boxed{6.14}$$

where $J_s(x)$ is the Bessel function, which to $O(x^5)$ is:

$$J_1(x) = \tfrac{1}{2}x - \tfrac{1}{16}x^3, \quad J_2(x) = \tfrac{1}{8}x^2 - \tfrac{1}{96}x^4, \quad J_3(x) = \tfrac{1}{48}x^3, \quad J_4(x) = \tfrac{1}{384}x^4$$

$$\therefore E - M = \sum_{s=1}^{\infty}\left(\frac{2}{s}\right)\sin sM\, J_s(se)$$

$$= 2\sin M\left[\tfrac{1}{2}e - \tfrac{1}{16}e^3\right] + \sin 2M\left[\tfrac{1}{2}e^2 - \tfrac{1}{6}e^4\right] + \tfrac{2}{3}\sin 3M\left[\tfrac{9}{16}e^3\right] + \tfrac{1}{2}\sin 4M\left[\tfrac{2}{3}e^4\right] + \dots$$

$$= e\sin M + \tfrac{1}{2}e^2\sin 2M + e^3\left[-\tfrac{1}{8}\sin M + \tfrac{3}{8}\sin 3M\right] + O(e^4) \qquad \boxed{6.15}$$

From $\boxed{1.21}$ $\quad r/a = 1 - e\cos E = 1 - e\cos\left[M + e\sin M + \tfrac{1}{2}e^2\sin 2M + \dots\right]$

$$= 1 - e\cos M\cos\left[e\sin M + \tfrac{1}{2}e^2\sin 2M + \dots\right] + e\sin M\sin\left[e\sin M + \tfrac{1}{2}e^2\sin 2M + \dots\right]$$

$$= 1 - e\cos M + e^2\tfrac{1}{2}(1 - \cos 2M) + e^3\tfrac{3}{8}\left[\cos M - \cos 3M\right] + O(e^4) \qquad \boxed{6.16}$$

using $\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\sin A = A - A^3/3! + \dots$, $\cos A = 1 - A^2/2! + \dots$, $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$

Expansion of sin f, cos f

Differentiate $\boxed{1.22}$ $\rightarrow dM/dE = 1 - e\cos E$

So from $\boxed{1.21}$ $\rightarrow r = a\, dM/dE$

Differentiating $\boxed{1.22}$ also $\rightarrow dM/dt = n$

Rewrite $\boxed{1.4, 1.6}$ $\rightarrow r^2 df/dt = na^2\sqrt{1-e^2}$

$$\therefore \cancel{a^2}(dM/dE)^2(df/dt) = (dM/dt)\cancel{a^2}\sqrt{1-e^2}$$

$$\therefore \int_0^f df = \sqrt{1-e^2}\int_0^M (dE/dM)^2\, dM$$

From $\boxed{6.15}$ $\rightarrow dE/dM = 1 + e\cos M + e^2\cos 2M + \dots$

$$\therefore f = \left(1 - \tfrac{1}{2}e^2 + \dots\right)\int_0^M 1 + 2e\cos M + e^2\left[2\cos 2M + \overbrace{\cos^2 M}^{\frac{1}{2}+\frac{1}{2}\cos 2M}\right] + \dots\, dM$$

$$= \left(1 - \tfrac{1}{2}e^2 + \dots\right)\left[M + 2e\sin M + e^2\left[\tfrac{5}{4}\sin 2M + \tfrac{1}{2}M\right] + \dots\right]$$

$$\therefore f - M = 2e\sin M + \tfrac{5}{4}e^2\sin 2M + O(e^3) \qquad \boxed{6.17}$$

= eqn of centre

Using the same trig ids and expansions used to get $\boxed{6.16}$, find:

$$\left.\begin{aligned} \sin f &= \sin M + e\sin 2M + e^2\left[\tfrac{9}{8}\sin 3M - \tfrac{7}{8}\sin M\right] + O(e^3) \\ \cos f &= \cos M + e(\cos 2M - 1) + \tfrac{9}{8}e^2\left[\cos 3M - \cos M\right] + O(e^3) \end{aligned}\right\} \boxed{6.18}$$

Sundman Criterion Can now expand R to arbitrary order in e, e', s, s', but note that series only converges for $e < 0.6627434$

Expansion for arbitrary r/r' EG34

Instead of expanding $|\underline{r}'-\underline{r}|^{-1}$ in terms of (r/r'), rewrite $\boxed{6.4}$ + $\boxed{6.10}$ as:

$$\frac{1}{|\underline{r}'-\underline{r}|} = \left[r^2 + r'^2 - 2rr'[\cos(\theta'-\theta) + \Psi]\right]^{-1/2}$$

$$= \frac{1}{\Delta_0}\left[1 - 2rr'\Psi/\Delta_0^2\right]^{-1/2}$$

where $\Delta_0 = [r^2 + r'^2 - 2rr'\cos(\theta'-\theta)]^{1/2}$

Binomial expansion in $\Psi = O(s^2, s'^2, ss', s^2e, s^2e', \ldots)$ (see $\boxed{6.11, 6.18}$)

$$\therefore \frac{1}{|\underline{r}'-\underline{r}|} = \frac{1}{\Delta_0}\left[1 + \left(\frac{rr'}{\Delta_0^2}\right)\Psi + \frac{3}{2}\left(\frac{rr'}{\Delta_0^2}\right)^2\Psi^2 + O(\Psi^3)\right] \qquad \boxed{6.19}$$

To get $\Delta_0^{-(1+2i)}$, do Taylor expansion about $r=a$, $r'=a'$:

$$\Delta_0^{-(1+2i)} = \Delta_0^{-(1+2i)}\Big|_{\substack{r'=a'\\ r=a}} + (r-a)\,\partial\Delta_0^{-(1+2i)}/\partial r\Big|_{\substack{r'=a'\\ r=a}} + (r'-a')\,\partial\Delta_0^{-(1+2i)}/\partial r'\Big|_{\substack{r'=a'\\ r=a}} + \ldots$$

Let $\rho_0 = \Delta_0\big|_{\substack{r'=a'\\ r=a}} = [a^2 + a'^2 - 2aa'\cos(\theta'-\theta)]^{1/2}$

$$\therefore \rho_0^{-(1+2i)} = a'^{-(1+2i)}\left[1 - 2\alpha\cos(\theta'-\theta) + \alpha^2\right]^{-(i+1/2)} \qquad \boxed{6.20}$$

where $\alpha = a/a'$

Also let $r/a = 1+\epsilon$ and $r'/a' = 1+\epsilon'$, where $\epsilon = O(e)$, $\epsilon' = O(e')$ and is given by $\boxed{6.16}$

$$\therefore \Delta_0^{-(1+2i)} = \rho_0^{-(1+2i)} + a\epsilon\,\partial\rho_0^{-(1+2i)}/\partial a + a'\epsilon'\,\partial\rho_0^{-(1+2i)}/\partial a' + \ldots \qquad \boxed{6.21}$$

From here we could do binomial expansion of $\rho_0^{-(1+2i)}$ in α...
Instead, note that $\rho_0^{-(1+2i)}$ is an even periodic fn and so can be expressed as a Fourier cosine series (see eg $\boxed{6.12}$)

$$\therefore \rho_0^{-(1+2i)} = a'^{-(1+2i)}\left[\tfrac{1}{2}b^0_{i+1/2}(\alpha) + \textstyle\sum_{j=1}^{\infty} b^j_{i+1/2}(\alpha)\cos j(\theta'-\theta)\right] \qquad \boxed{6.22}$$

We can find the $b^j_{i+1/2}(\alpha)$ by multiplying by $\cos j(\theta'-\theta)$ then integrating $\int_0^{2\pi} d(\theta'-\theta)$, as only the $\cos^2 j(\theta'-\theta)$ terms $\neq 0$

$$\therefore b^j_s(\alpha) = \frac{1}{\pi}\int_0^{2\pi}\cos j\psi\left[1 - 2\alpha\cos\psi + \alpha^2\right]^{-s}d\psi \qquad \boxed{6.23}$$

This is a Laplace coefficient, and means that we can express R in terms of a function of α, rather than a series expansion.
Laplace coefficients can be expressed as a series expansion:

$$b^j_s(\alpha) = 2\,\frac{s(s+1)\ldots(s+j-1)}{1.2.3.\ldots j}\,\alpha^j\left[1 + \frac{s(s+j)}{1(1+j)}\alpha^2 + O(\alpha^4)\right] \qquad \boxed{6.24}$$

which is convergent for all α, but can also be calculated numerically (eg. Murray & Dermott 1999, Q6.2)
Laplace coeffs are well studied, and it is also easy to determine $Db^j_s(\alpha) \equiv \partial b^j_s(\alpha)/\partial\alpha$ and $D^2 b^j_s(\alpha)$ etc

As $b^{-j}_s(\alpha) = b^j_s(\alpha)$ $\qquad \boxed{6.25}$

$$\rho_0^{-(1+2i)} = \tfrac{1}{2}a'^{-(1+2i)}\textstyle\sum_{j=-\infty}^{\infty} b^j_{i+1/2}(\alpha)\cos j(\theta'-\theta)$$

$$\partial\rho_0^{-(1+2i)}/\partial a = \tfrac{1}{2}a'^{-(2+2i)}\textstyle\sum_{j=-\infty}^{\infty} Db^j_{i+1/2}(\alpha)\cos j(\theta'-\theta)$$

$$\partial\rho_0^{-(1+2i)}/\partial a' = -\tfrac{1}{2}a'^{-(2+2i)}\textstyle\sum_{j=-\infty}^{\infty}\left[\alpha Db^j_{i+1/2}(\alpha) + (1+2i)b^j_{i+1/2}(\alpha)\right]\cos j(\theta'-\theta)$$

which can be substituted into $\boxed{6.21}$ and $\boxed{6.19}$, and ultimately $\boxed{6.2}$ to give R as an infinite sum, that is also a series expansion in inclinations and eccentricities, in terms of Laplace coeffs.

General expansion

Using Laplace coeffs, can expand R as infinite series in e, e', s, s' that can be written

$$R = \mu' \sum S \cos\psi$$

where $S = f(\alpha)\, a'^{-1} e'^{|j_3|} e^{|j_4|} s'^{|j_5|} s^{|j_6|}$ = strength of term (NB $\alpha = a/a'$)

$\psi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$ = argument

$\sum_{i=1}^{6} j_i = 0$ is d'Alembert relation

$|j_1 + j_2|$ = order of term

$\boxed{6.26}$

A similar expression holds for R'. [NOTE that $e<0.66$, $r<r'$ for convergence]

The strength of the various terms is given in Appendix B of Murray & Dermott 1999, for a disturbing function $\boxed{6.2, 6.3}$ rewritten as:

$$R = \frac{\mu'}{a'}\left[R_D + \alpha R_E\right] \qquad \boxed{6.27}$$

$$R' = \frac{\mu}{a}\left[\alpha R_D + \frac{1}{\alpha} R_I\right]$$

where $R_D = a'/|\underline{r}' - \underline{r}|$ = direct term

$R_E = -(r/a)(a'/r')^2 \cos\psi$
$R_I = -(r'/a')(a/r)^2 \cos\psi$
} indirect terms for E = External perturber, I = Internal perturber

Lagrange's planetary eqns

R can be used to determine evolution of osculating elements defined by 2 body potential U

Define $\lambda = \underbrace{nt}_{\text{unperturbed = linear increase with time}} + \epsilon$ $\boxed{6.28}$

where ϵ = mean longitude at epoch

Lagrange's planetary eqns are:

$$\frac{da}{dt} = \frac{2}{na}\partial R/\partial\epsilon$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\left[1 - \sqrt{1-e^2}\right]\partial R/\partial\epsilon - \frac{\sqrt{1-e^2}}{na^2 e}\partial R/\partial\varpi$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\partial R/\partial e + \frac{\tan I/2}{na^2\sqrt{1-e^2}}\partial R/\partial I$$

$$\frac{dI}{dt} = -\frac{\tan I/2}{na^2\sqrt{1-e^2}}\left[\partial R/\partial\epsilon + \partial R/\partial\varpi\right] - \frac{1}{na^2\sqrt{1-e^2}\sin I}\partial R/\partial\Omega \qquad \boxed{6.29}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\partial R/\partial I$$

$$\frac{d\epsilon}{dt} = -\frac{2}{na}\partial R/\partial a + \frac{\sqrt{1-e^2}\left(1-\sqrt{1-e^2}\right)}{na^2 e}\partial R/\partial e + \frac{\tan I/2}{na^2\sqrt{1-e^2}}\partial R/\partial I$$

These are exact assuming that orbital elements are slowly varying (though you often only take lowest orders) and all quantities can be "primed" to get evolⁿ of outer orbit

Problem 1: R is in terms of λ not ϵ

ok: $\underline{\partial R/\partial\epsilon = (\partial R/\partial\lambda)(\partial\lambda/\partial\epsilon) = \partial R/\partial\lambda}$ $\boxed{6.30}$

Problem 2: Let $d\epsilon/dt = -\frac{2}{na}\partial R/\partial a + X$

$$= -\frac{2}{na}(\partial R/\partial a)_{\text{explicit}} \underbrace{-\frac{2}{na}(\partial R/\partial\lambda)}_{\dot{a}}\underbrace{(\partial\lambda/\partial n)}_{t}(\partial n/\partial a) + X$$

$$= -\frac{2}{na}(\partial R/\partial a)_{\text{explicit}} - \dot{n}t + X$$

ie $d\epsilon/dt \to \infty$ as $t \to \infty$

$\therefore$ Define $\underline{\dot{\epsilon}^* = \dot{\epsilon} + \dot{n}t = -\frac{2}{na}(\partial R/\partial a)_{\text{explicit}} + X}$ $\boxed{6.31}$

Has to use ϵ^*: $\underline{\dot{\lambda}} = \dot{n}t + n + \dot{\epsilon} = \underline{n + \dot{\epsilon}^*}$ $\boxed{6.32}$

This means: $\lambda = \int n\,dt + \epsilon^*$

Defining $\rho = \int n\,dt \rightarrow \dot{\rho} = n$ and $\ddot{\rho} = \dot{n} = -3a^{-2}\partial R/\partial\lambda$

Classification of terms

But why is it good to have R as an infinite series? Because most terms are unimportant

	Unperturbed Problem	Perturbed Problem
λ, λ'	Increase linearly with time	Rapidly varying
$\varpi, \varpi', \Omega, \Omega', a, a', e, e', I, I'$	Fixed	Slowly varying

As terms involve $\psi = j_1\lambda' + j_2\lambda + j_3\varpi' + j_4\varpi + j_5\Omega' + j_6\Omega$, there are two types of terms: rapidly varying and slowly varying (which have a further subdivision).

$\boxed{\text{SHORT PERIOD}}$ Most terms are rapidly varying, but can invoke averaging principle to say that, as orbital elements are unchanged after one orbit, these average to zero.
Such terms become important during scattering, but this problem is treated differently

$\boxed{\text{SECULAR}}$ $j_1 = j_2 = 0$ terms are slowly varying and are important at all semimajor axes. $\boxed{6.33}$
These terms can alternatively be identified by double-averaging the disturbing function to identify terms independent of λ and λ' $\boxed{6.34}$
ie. $R_{sec} = \langle\langle R \rangle\rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} R \, dM \, dM'$

$\boxed{\text{RESONANT}}$ $\boxed{6.26}, \boxed{6.28} \Rightarrow \psi = (j_1 n' + j_2 n)t + f(\epsilon, \epsilon', \varpi', \varpi, \Omega', \Omega)$
$\therefore$ Terms with $j_1 n' + j_2 n \approx 0$ are also slowly varying $\boxed{6.35}$
As $n = \sqrt{\mu/a^3}$, these terms become important at semimajor axes close to
$$a/a' \sim (|j_2|/|j_1|)^{2/3} \quad \boxed{6.36}$$
(NB This is not a necessary or sufficient condition for objects to be "in" resonance, but does indicate where those terms may be important)
There are an infinite # of terms that satisfy condition $\boxed{6.36}$, but from $\boxed{6.26}$ we know that the strength decreases with the order of the resonance
ie., first order resonances (2:1, 3:2, 4:3, 5:4 etc) are strongest