UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 7 June, 2021 12:00 pm to 3:00 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Starting from the equation of relative motion, show that the relative velocity v between two bodies of masses M_1 and M_2 at separation r is given by

$$0.5v^2 - \mu/r = C,$$

where $\mu = G(M_1 + M_2)$ and C is a constant.

(ii) Given that $r = \pm a(1 - e^2)/(1 + e\cos f)$ for bound orbits (positive sign) and hyperbolic orbits (negative sign), where $a > 0$ is the semimajor axis, e is the eccentricity and f the true anomaly of the orbit, show that $C = \mp 0.5\mu/a$.

(iii) A planet of mass M_p is on a circular orbit a distance a_p from a star of mass $M_\star \gg M_\mathrm{p}$. A comet of mass $M_\mathrm{c} \ll M_\mathrm{p}$ is on an orbit about the star that is coplanar with that of the planet, and that has semimajor axis a_c and eccentricity e_c. The comet has a close encounter with the planet. Show that the relative velocity of the encounter is

$$\Delta v_\mathrm{pc} = v_\mathrm{p}\left[3 - (a_\mathrm{p}/a_\mathrm{c}) - 2\sqrt{(a_\mathrm{c}/a_\mathrm{p})(1 - e_\mathrm{c}^2)}\right]^{1/2},$$

where v_p is the orbital velocity of the planet.

(iv) During the encounter the gravity of the star can be ignored and the comet's trajectory relative to the planet is hyperbolic with impact parameter b. A moon of mass $M_\mathrm{m} \ll M_\mathrm{p}$ is on a circular orbit a distance a_m from the planet. Its orbit is prograde in the same plane as the planet's orbit. The comet has a close encounter with the moon. Determine the radial and tangential components of the comet's velocity relative to the planet and hence show that its velocity relative to the moon is

$$\Delta v_\mathrm{mc} = \sqrt{3v_\mathrm{m}^2 - 2v_\mathrm{m}\Delta v_\mathrm{pc}(b/a_\mathrm{m}) + \Delta v_\mathrm{pc}^2},$$

where v_m is the orbital velocity of the moon.

(v) Determine the largest impact parameter for which the comet could hit the moon.

(vi) What is the smallest impact parameter for the comet to avoid hitting the planet if its radius is R_p?

(vii) How would the above calculations be affected if the comet's circumstellar orbit had been inclined to that of the planet by an inclination I_c?

2

(i) Consider a test particle in the vicinity of a binary comprised of bodies of mass M_1 and M_2. The binary follows a circular orbit about its centre of mass O. Units are chosen such that both the separation and mean motion of the binary are unity. The location of the particle is given by (x, y, z) in the rotating frame $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ that is centred on O with $\hat{\boldsymbol{x}}$ pointing towards M_2 and $\hat{\boldsymbol{z}}$ parallel with the binary angular momentum vector. Derive expressions for r_1 and r_2, the distance of the particle from M_1 and M_2, respectively, in terms of x, y, z and $\mu_i = GM_i$, where G is the gravitational constant.

(ii) In addition to the gravity of the two bodies, the test particle is subjected to acceleration due to radiation pressure from the mass M_1, the magnitude of which is a constant β_1 times the acceleration towards that body due to gravity. Derive the particle's equation of motion and show that this can be written in the form

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \partial U/\partial x, \\ \ddot{y} + 2\dot{x} &= \partial U/\partial y, \\ \ddot{z} &= \partial U/\partial z, \end{aligned}$$

where $U = \frac{1}{2}(x^2 + y^2) + \mu_1(1 - \beta_1)/r_1 + \mu_2/r_2$.

(iii) Derive a constant of motion for the particle's motion.

(iv) Show that the equilibrium points of the particle's motion satisfy

$$\begin{aligned} Az &= 0, \\ (1 - A)y &= 0, \\ (1 - A)x &= \mu_1\mu_2[(1 - \beta_1)r_1^{-3} - r_2^{-3}], \end{aligned}$$

where an expression for A should be determined.

(v) Use the result from (iv) to sketch how the locations of the triangular Lagrange equilibrium points (i.e., L_4 and L_5) change as the value of β_1 is increased from 0 to 1.

(vi) Consider the situation where the test particle is also subject to radiation pressure from M_2, which is characterised by a constant β_2. Describe how this affects the location of the triangular equilibrium points and the constraints on β_1 and β_2 for such points to exist.

(vii) What other physical process is likely to be relevant to the above calculations?

3

(i) Sketch the ratio β of the radiation pressure force to stellar gravity acting on dust grains near a main sequence star of mass $M_\star \approx 1M_\odot$ as a function of their diameter D. Explain the physical origin and location of any changes in slope on the figure.

(ii) A comet of mass $m \ll M_\star$ is orbiting the star with semimajor axis a and eccentricity e. A dust grain with a radiation pressure coefficient β is released from the comet with zero relative velocity at a distance r from the star. Show that the grain's new orbit has a semimajor axis and eccentricity of

$$\begin{aligned} a_\mathrm{d} &= a(1-\beta)/(1-2\beta a/r), \\ e_\mathrm{d} &= \left[(e-\beta)^2 + 2\beta a(1-e^2)/r\right]^{1/2}/(1-\beta). \end{aligned}$$

You may use without proof the standard two-body results that $0.5v^2 - \mu/r = -0.5\mu/a$ and $h = \sqrt{\mu a(1-e^2)}$.

(iii) The comet is on a near parabolic orbit with $e = 1-\delta$, where $\delta \ll 1$. Show that, to lowest order in δ, the distance within which dust grains are placed on unbound orbits is $\sim Q\beta/(1-\beta)$, where Q is the comet's separation from the star at apocentre.

(iv) Show that dust released at pericentre is unbound if $\beta > \delta/2$.

(v) A synchrone is a line connecting particles released from the comet at the same time in the past. Consider the trajectories of particles released at pericentre with $\beta = 1$, as well as those with slightly larger and smaller values of β. Explaining your reasoning, sketch the synchrone of particles with a wide range of β released at pericentre when the comet has reached a true anomaly $f = \pi/2$.

(vi) A syndyne is a line connecting particles with the same dynamics (i.e., the same β) that were released at different times. Explaining your reasoning, sketch the syndynes when the comet is at $f = \pi/2$ for particles with $\beta = 1$, and for those with slightly smaller and larger β, extending these back to the particles released when the comet was at $f = -\pi/2$,

(vii) Comment on the implications for the shape of cometary dust tails and their orientation as a function of particle size.

4

(i) Derive the equation of motion for a body of mass M_1 moving in the gravitational potential of a star of mass $M_\star$ and another body of mass M_2 in the form

$$\ddot{\boldsymbol{r}}_1 = \nabla(\mathcal{U}_1 + \mathcal{R}_1),$$

where $\boldsymbol{r}_i$ is the vector from $M_\star$ to M_i, $\mathcal{U}_1$ is the two-body potential of the masses $M_\star$ and M_1, and $\mathcal{R}_1$ is the disturbing function arising from the presence of M_2, an equation for which should be given along with a meaning for its different components.

(ii) Describe the form of the disturbing function when expanded in terms of the orbital elements of the two-body motion of M_i about $M_\star$, which can be given using the standard notation a_i, e_i, I_i, ϖ_i, Ω_i, λ_i.

(iii) Identify three classes for the terms in the disturbing function based on the timescales on which they vary. Provide a physical explanation for the origin of the perturbations associated with the different classes and the situations in which each might be relevant.

(iv) The two bodies are in an inclination resonance in which the resonant angle $\phi = (p+q)\lambda_2 - p\lambda_1 - q\Omega_2$ is librating. Give a constraint on q, and by considering the location of the two bodies relative to each other and the longitude of ascending node, describe the geometrical interpretation of ϕ/p and ϕ/q. You may assume that $r_2 > r_1$, where $r_i = |\boldsymbol{r}_i|$.

(v) Consider the situation that $M_2 \ll M_1$ and the inner body's orbital plane remains at $I_1 \approx 0$. Sketch the view along the ascending node Ω_2 towards the star, as well as the face-on view of the system, for the situation that conjunction between the two bodies occurs just after passing through the ascending node, and describe how the perturbing forces due to the encounter affect the orbit of M_2.

(vi) Use the above results to determine the value about which ϕ will oscillate for $q = 2$, explaining your reasoning.

(vii) Sketch the orbit of M_2 as viewed in a frame that is rotating with M_1 and with the line of sight parallel to the vector connecting M_1 and $M_\star$.

END OF PAPER