Scanned 01/02/2017

Solutions POA Sheet 2

i) $$\frac{J_E}{J_{EM}} \sim \frac{\frac{2}{5} \overset{81}{M_E}\, r_E^2\, \Omega_E}{M_m\, r_{EM}^2\, \underset{28}{\Omega_{EM}}}$$

$$r_{EM} = \left(\frac{G M_{earth} P_m^2}{4\pi^2}\right)^{1/3} \sim 4 \times 10^8 \text{ m}.$$

$$\frac{J_E}{J_{EM}} \sim \frac{2}{5}\, 81 \times \left(\frac{6\times 10^6}{4 \times 10^8}\right)^2 \times 28 \sim 0.2$$

$$\frac{J_m}{J_{EM}} = \frac{2}{5}\left(\frac{r_m}{r_{EM}}\right)^2$$ (because $\Omega_m = \Omega_{EM}$ – synchronised)

Use $\frac{r_m}{r_{EM}} = \frac{R_\odot}{1\,\text{AU}}$ same angular diameter sun and moon

$= 10^{-5}$ negligible.

When synchronised $\frac{\Omega_E}{\Omega_{EM}} = 1 \Rightarrow \frac{J_E}{J_{EM}} \ll$ value now

So negligible.

Ang. mom. conservation $\underbrace{J_{EM}\left(1 + \frac{J_E}{J_{EM}}\right)}_{\text{now}} = J_{EM}\big|_{end}$.

$J_{EM}\big|_{end} = 1.2\, J_{EM}\big|_{now} \Rightarrow a_{end} = 1.4\, a_{now}$.

$P_{end} = P_{EM} \times (1.4)^{3/2}$

$$\frac{E_{spin}}{|E_{EM}|} = \frac{\overset{\swarrow\ \text{spin}}{E_E}}{E_{EM\,\&\,K.E.}} = \frac{J_E}{J_{EM}}\,\frac{\Omega_E}{\Omega_M}$$

$$= 28 \times 0.2 \sim 5$$

most of energy currently in Earth's spin

$$\left[\ \frac{E_m}{|E_{em}|} = \frac{J_m}{J_{EM}} = 10^{-5}\ \right]$$

raising moon from its present to final orbit requires an energy

$$= -\frac{G M_E M_{moon}}{2}\left[\frac{1}{a_{final}} - \frac{1}{a_{now}}\right]$$

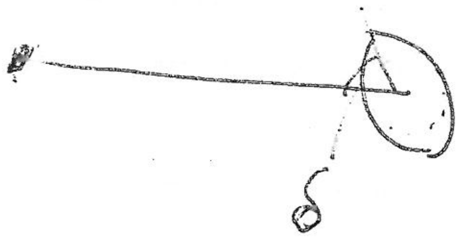
$$= \text{K.e now}\left[1 - \frac{a_{now}}{a_{final}}\right]$$

$$= |E_{EM}| \times \frac{0.\cancel{7}\ 4}{1.\cancel{7}\ 4}$$

cf. energy in Earth spin now $= 5\,E_{em}$

$\Rightarrow$ most of energy is dissipated.

2)



Tidal bulge has mass $\sim \left(\frac{h}{R_E}\right) M_E$

$$h = \frac{G M_m R_E^4}{R_{EM}^3 \, G M_E}$$

$$M_b = M_{\cancel{E}m} \left(\frac{R_E}{R_{EM}}\right)^3$$

$$\text{Torque} = \frac{G M_m M_b R_E^2 \sin\delta}{R_{EM}^3}$$

$$= \frac{G M_m^2 R_E^5 \sin\delta}{R_{EM}^6}$$

$$\tau_E = \frac{M_E R_E^2 \Omega \, R_{EM}^6}{G M_m^2 R_E^5 \sin\delta} \propto \frac{M_E}{M_m^2 R_E^3}$$

$$\tau_m \propto \frac{M_m}{M_E^2 R_m^3}$$

$\propto M_m/\rho_m$

$$\Rightarrow \frac{\tau_E}{\tau_m} = \frac{M_E^3 R_m^3}{M_m^3 R_E^3} \approx \frac{M_E^2}{M_m^2} \gg 1$$

$\propto \frac{M}{\rho}$

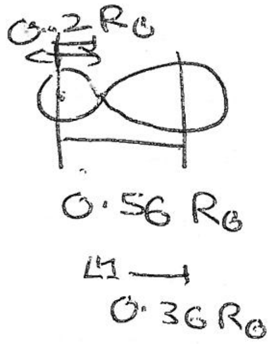
Hence why moon synchronised & earth not.

3) Secondary fills Roche lobe $\Rightarrow \dfrac{M_2}{R_2^3} = \dfrac{(3\times)\,M_{ns}}{a^3}$

$$\Rightarrow a = 3^{1/3} \times \left(\frac{1.4}{0.2}\right)^{1/3} R_2$$

$$= 2.8\, R_2 = 2.8 \times 0.2 \times 7\times 10^8\,\mathrm{m}$$

$$= 4 \times 10^8\,\mathrm{m}.$$



$$(0.36\, R_\odot)^2\, \Omega_b = (G M_{ns} R_{circ})^{1/2}$$

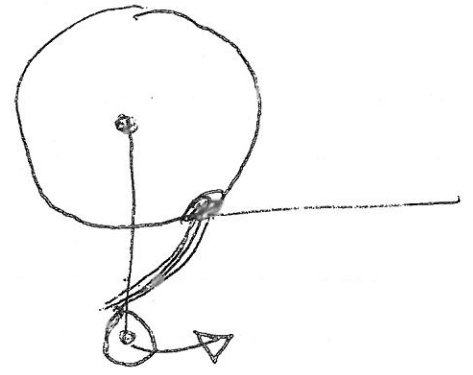
$$R_{circ} = \frac{(0.36\, R_\odot)^4 (M_{ns}+M_2)}{(0.56\, R_\odot)^3\, M_{ns}}$$

$$= \left(\frac{9}{14}\right)^3 \times 0.36 \times \frac{1.6}{1.4}\, R_\odot$$

$$\sim 0.12\, R_\odot$$

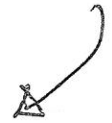
$$\sim 7 \times 10^7\,\mathrm{m}.$$

Spreads due to viscous transport of ang. momentum down Ω gradient in disc
Disc outer radius set by tidal removal of ang. mom (transfer to secondary)



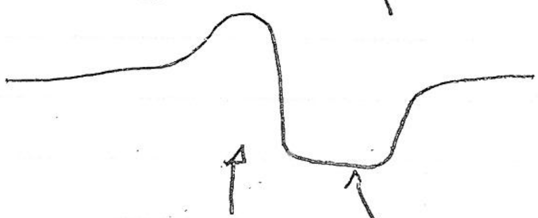
hot spot downstream of L1 point due to ang mom conservation

Observer

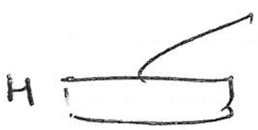


in corotating frame

sees hotspot at max just before secondary eclipses disc



Luminosity of hot spot $= GM_{ns}\dot{M}\left[\frac{1}{0.3R_\odot}-\frac{1}{0.36R_\odot}\right]$



$\frac{10}{180}\times\pi\times R_{disc}$,

$$= \frac{GM_{ns}\dot{M}}{R_\odot}\times\frac{0.2}{0.36}$$

$$\text{Area} = \frac{10}{180}\times\pi\times 0.3R_\odot\times (H/R)R_\odot$$

$$\sim 0.05\cdot(H/R)\,R_\odot^2$$

$$0.05\left(\frac{H}{R}\right)^2 R_\odot^2\,\sigma T^4 = \frac{GM_{ns}\dot{M}\times 0.5}{R_\odot}$$

$M_\odot y^{-1}$

$$T^4 = \frac{10\,GM_{ns}\dot{M}\left(\frac{R}{H}\right)^2}{R_\odot^3\,\sigma} = \frac{10^{-9}\;3\times10^{30}\times2\times10^{20}\,M_\odot s^{-1}}{5\times10^{7}\;3\times(7\times10^{8})^{3}\times10^{-4}}$$

10^{16} g s^{-1}

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4. To be a binary primary, star needs to be in a cluster where all N−1 members are less massive than it.

Prob. star mass M is a binary primary = fraction of randomly assembled clusters of size N−1 with all members less massive than M

$$= \left[\int_0^M f(m)\,dm\right]^{N-1}$$

//

To be a binary primary with companion in the mass range m to m+dm, it's necessary that it's in a cluster in which next most massive member is in that mass range. The prob. of this is equal to the prob. of a cluster of N−1 stars having most massive member in that mass range which implies ~~a) that all stars in that cluster have mass < M~~ and a) that one star is in range m → m+dm. ~~Prob of a)~~ and b) that all the other stars are less massive than this.

Prob of a) = $(N-1)\, f(m)\, dm$
(as dm→0).

because there are (N−1) drawings.

Prob of b) $= \left[\int_0^m f(m')\,dm'\right]^{N-2}$

Hence answer //

Significant feature of F_M is that it is independent of M both wrt functional form and normalisation.

So eg fraction of 10 $M_\odot$ stars with companion 0.3-0.4 $M_\odot$

= fraction of 1 $M_\odot$ " " " "

So one could test the hypothesis without knowing f(m) if one had a survey of companions to 10 $M_\odot$ and 1 $M_\odot$ stars that was complete for companions ~~of~~ in a given mass range.

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5.

$1'' \ @ \ 1\,pc = 1\,AU$

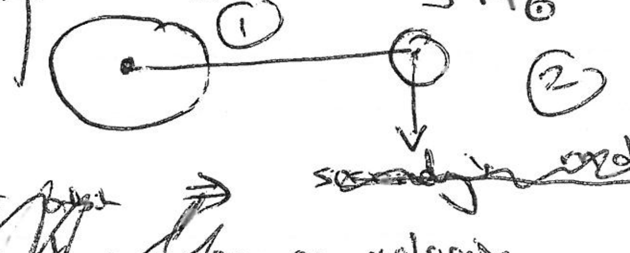
$20\,mas \ @ \ 500\,pc = 10\,AU$

$= 1.5 \times 10^{13}\,m$.

$$speed = \frac{1.5 \times 10^{12}}{3 \times 10^{7}} = 5 \times 10^{4}\,m/s$$

$= 50\,km/s$.

lower limit — component of motion along line of sight measured with Doppler shift.



~~secondary in orbital~~

~~GM ... $\sqrt{2}$ ... escape velocity.~~

relative velocity of ② and ①

$$= \sqrt{\frac{G \cdot 25 M_\odot}{a}}$$

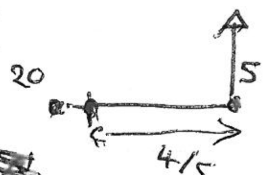
After explosion this is still the relative velocity of ② and the remnant of ①.

k.e of ② in frame co-moving with ①

$$= \frac{1}{2} \times 5 M_\odot \times \frac{G\, 25 M_\odot}{a}$$

binding energy $= \dfrac{G \times 1.4 \times 5\, M_\odot^2}{a}$

k.e ≫ binding energy so after explosion can neglect ①'s gravitational influence on ②, which escapes at speed $= V_2 = \left(\frac{4}{5}\right)\sqrt{\frac{G\,25 M_\odot}{a}}$



$= 50 \text{ km s}^{-1}$

$$\left.\begin{aligned} \Rightarrow 4\sqrt{\frac{GM_\odot}{a}} &= 50 \text{ km/s} \\ \sqrt{\frac{GM_\odot}{1\,\text{AU}}} &= 30 \text{ km/s} \end{aligned}\right\} \Rightarrow a = \left(\frac{30}{12.5}\right)^2 \text{AU}$$

$a \approx 6$ AU

Neutron star travels at original speed of ①

$= \frac{1}{4}$ of ② $\cancel{\frac{1}{5}\sqrt{\frac{G\,25 M_\odot}{a}}} = 12.5$ km/s.

So expect to see it diametrically opposite at separation 0.25 arc minutes

[Can discuss SN induced kicks on neutron stars]

~~XX~~ **6.** Tidal energy dissipation

$$= \left(\frac{h}{R_*}\right) M_* g h$$

$$h = \frac{G M_{ns} R_*^4}{a_{peri}^3 G M_*} \quad \text{(from lecture)}.$$

$$\Rightarrow \frac{h}{R_*} = \frac{M_{ns}}{M_*}\left(\frac{R_*}{a_{peri}}\right)^3.$$

$$= \left(\frac{M_{ns}}{M_*}\right)^2 \left(\frac{R_*}{R_{aperi}}\right)^6 R_* \frac{G M_*^2}{R_*^2}$$

$$= \left(\frac{R_*}{R_{\frac{1}{2}}}\right)^6 \frac{G M_*^2}{R_*}$$

Tidal capture: equate $\frac{1}{2} M_* V_*^2$.

$$\Rightarrow \frac{G M_*}{R_*} \left(\frac{M_{ns}}{m_*}\right)^2 \left(\frac{R_*}{R_{peri}}\right)^6 = V_*^2$$

$$\Rightarrow R_{peri} = R_* \left(\frac{G M_*}{R_* V_*^2}\right)^{1/6} \left(\frac{M_{ns}}{m_*}\right)^{1/3}$$

grav. focused $\Rightarrow b \sim \sqrt{\frac{2 G M_{ns} R_{peri}}{V^2}}$

Numbers $R_{peri} = \text{few } R_*$
$\sim 10^7$ m

$V \sim 10$ km/s.

$$b \sim \sqrt{10^{12}\ 10^9} \sim 3 \times 10^{10} \text{ m}$$

rate of capture per Myr $= \pi n b^2 v$ ← in appropriate units

$$= 6 \times n_{pc^{-3}} \; b^3_{pc} \; v_{\text{in pc per Myr}}$$

$$= 6\times10^{4} \times 10^{-12} \times 10 \quad (\text{equiv } 10\ \text{km/s})$$

$$= 10^{-6}\ \text{Myr}^{-1}!$$

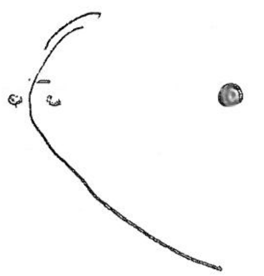
In 10^{10} years expect 10^{-2} captures per neutron star.

[How many neutron stars are expected in a globular cluster? For standard IMF, ~ 100s of stars formed per supernova – may be indicative of number of stars per neutron star produced – depends on detailed understanding of mapping progenitor masses → remnant outcome which is lacking partly because of role of mass loss pre-supernova. On this assumption ~ 1% of all stars in globular cluster are neutron stars so for $M_{gc} \sim 10^6 M_\odot \Rightarrow 10^4$ neutron stars. Would imply 100 capture events....]

XX 7. Tidal disruption of binary s-t

$$a_{bin} = \left(\frac{M_{bin}}{M_{bh}}\right)^{1/3} r_{min}.$$

$\Rightarrow r_{min} \sim 100\, a_{bin} \sim 100$ AU $\sim 10^{11}$ m.



After disruption, star ① travels at $V_{peri} + \Delta V$

star ② at $V_{peri} - \Delta V$

$$V_{\substack{esc \\ bin}} \sim \sqrt{\frac{2GM}{a_{bin}}} \sim 60 \text{ km/s}.$$

$$V_{peri} = \sqrt{\frac{2GM_h}{r_{peri}}} \sim \sqrt{\frac{2\times10^6}{100}} \cdot 30 \text{ km s}^{-1}.$$

$$\sim \cancel{5000} \; 5000 \text{ km s}^{-1}!$$

$$\text{Energy of 2 per unit mass} = -\frac{GM_h}{r_{peri}} + \frac{1}{2}\left(V_{peri} - V_{\substack{esc \\ bin}}\right)^2$$

$$\simeq -V_{peri}\, V_{\substack{esc \\ bin}}$$ [becomes parabolic originally]

so bound.

Not asked but could estimate its period :

$$\frac{GM_h}{2a_{bound}} = V_{peri}\, V_{\substack{esc \\ bin}}.$$

$$= 3\times10^5 \text{ km}^2 \text{s}^{-2}.$$

$$a_{\substack{bound \\ AU}} \sim \frac{10^6}{2\times10^{\cancel{}}\, 300} \sim \underline{10^3 \text{ AU}}$$

$$P_{bound} = 1 \text{ yr} \left(\frac{10^9}{10^6}\right)^{1/2} = 30 \text{ years}.$$

Eccentricity: $a(1-e) = r_{peri}$.

a: 1000 AU r_{peri}: 100 AU

$e \sim 0.9$]

For star ① $\underset{\text{per unit mass}}{\text{Energy}} = + V_{peri} V_{esc\,bin}$.

$$= \frac{1}{2} V_\infty^2$$

$$\Rightarrow V_\infty = \sqrt{2 \times 60 \times 5000} \text{ km s}^{-1}$$

$$V_\infty \sim 800 \text{ km s}^{-1} \gg V_{esc\,galaxy}$$

so not retained.

$$\frac{\Delta \nu}{\nu} = \frac{v}{c} \sim \frac{8 \times 10^2}{3 \times 10^5} \sim 3 \times 10^{-3}$$

shift $\sim 3 \times 10^{-3} \times 5000 \sim 15$ Å

$$a_{peri} = \left(\frac{M_h}{M_{bin}}\right)^{1/3} \text{ A.U. for a 1AU binary.}$$

$$a_{schw} \sim \frac{2GM_h}{c^2} \sim \frac{10^{-10} \times 2 \times 10^{30}}{10^{17}} \left(\frac{M_h}{M_\odot}\right)$$

$$\sim 2000\text{m} \left(\frac{M_h}{M_{bin}}\right)$$

$$\sim 10^{-8} \left(\frac{M_h}{M_{bin}}\right) \text{ AU}$$

$$\text{equate } \left(\frac{M_h}{M_{bin}}\right)^{1/3} \sim 10^{-8} \left(\frac{M_h}{M_{bin}}\right) \Rightarrow \frac{M_h}{M_{bin}} \sim 10^{12}$$

Because this binary is much more weakly bound than a star, any plausible mass bh will tidally disrupt such a binary outside its event horizon

POA Ex. 2

8. ang. mom conservation

$$\sin\theta \, v r = \sqrt{2GMr_t}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{2GMr_t}}{v r}\right).$$

To deviate a star by angle θ need to acquire mean square velocity $\perp r$ orbit of $v^2 \sin^2\theta$. Takes N encounters at impact parameter b where

$$N \Delta v^2 = v^2 \sin^2\theta$$

and $\Delta v(b) = \dfrac{Gm}{bv}$

$$\Rightarrow N = \frac{v^2 \sin^2\theta \; b^2 v^2}{G^2 m^2}$$

rate of encounters at impact parameter b

$$= n\pi b^2 v$$

so time for N encounters $= \dfrac{v^4 \cancel{b^2} \sin^2\theta}{G^2 m^2 n\pi \cancel{b^2} v}$

time (θ) $= \dfrac{v^3 \sin^2\theta}{G^2 m^2 n\pi}$.

r_{crit} s.t.

$$\frac{r_{crit}}{v} = \frac{v^3 \sin^2\theta}{G^2 m^2 n \pi} = \frac{v^3}{G^2 m^2 n \pi} \cdot \frac{2 G M r_t}{r_{crit}^2 v^2}$$

$$\Rightarrow r_{crit} = \left(\frac{2 G M r_t v^2}{G m^2 n \pi} \right)^{1/3}$$

(dimension check: $\left(\frac{M L L^2 T^{-2}}{M T^{-2}} \right)^{1/3}$ ✓).

- $r \ll r_{crit}$ swallow stars within loss cone but then take many crossing times to re-fill loss cone.

- $r \gg r_{crit}$: loss cone is full but star can diffuse out of loss cone before being swallowed.

- $r \sim r_{crit}$. Sweet spot where loss cone refills on crossing time so rate of feeding (= rate disrupting) = no. stars within loss cone at r_{crit} ÷ t_{cross} (r_{crit}).

Numbers ~~$r_t \left(\frac{M}{m_*}\right)^{1/3}$~~

~~$r_{crit} = 2 \times 10^6 \times 2 \times 10^{30} \times 6 \times 10^{10}$~~

$r_t \simeq \left(\frac{M}{m_*} \right)^{1/3} r_*$
$\sim 6 \times 10^{10}$ m

P.T.O. ~~$v = \sqrt{\frac{GM(r)}{r}}$~~

$$G n m r_{crit}^3 = \frac{2}{\pi}\,\frac{M}{m} \times 6\times 10^{10}\,\underbrace{\frac{G n m}{r_{crit}}}_{v^2}$$

$$\Rightarrow r_{crit} = 10^6 \times 6 \times 10^{10}$$

$$\sim 2\,pc.$$

[Note have calculated v by noting $n \propto r^{-2} \Rightarrow$ cluster with v independent of n so can calculate it at r_{crit} when enclosed mass is $\sim G n m r_{crit}^3$]

$$\sin\theta \Big|_{r_{crit}} = \sqrt{\frac{2 G M r_t\, r_{crit}}{G(\text{mass within } r_{crit})}} \div r_{crit}$$

$$= \sqrt{\frac{10^6}{\cancel{2\times}\,10^8}} \times \left(\frac{6\times 10^{10}}{6\times 10^{16}}\right)^{1/2}$$

$$\sim 10^{-4}$$

$$\frac{r_{crit}}{V} = \frac{\left(6\times10^{16}\right)^{3/2}}{\sqrt{G\times(\text{mass within } r_{crit})}}$$

$$= \frac{\left(6\times10^{16}\right)^{3/2}}{\left[6.7\times10^{-8}\times2\times10^{38}\right]^{1/2}}$$

$$= \left[\frac{6^3\times10^{48}}{10^{31}}\right]^{1/2} = 3\times10^{9}\ \text{s}$$

$\sim$ 100 years.

number of stars in loss cone at r_{crit}

$$\simeq \left(10^{-4}\right)^2 \times \text{number of stars @ } r_{crit}$$

Θ^2

$$\simeq \left(10^{-4}\right)^2 \times 10^{8} \sim 1.$$

so rate is 1 per 100 years.

to see 1 in 10 years need to monitor > 10 systems.