EXERCISES: Set 1 of 4

Q1: On the basis of the ratio of the 'characteristic' distance between two of the objects below to their 'characteristic' size (radius), which are 'closer' together, planets in the solar system, stars in the Milky Way Galaxy, or galaxies in the local Universe?

Q2: Suppose the space density of stars within the Galaxy is $0.1\,\mathrm{pc}^{-3}$.

(i) How many stars would you see per square degree in a direction where the Galaxy extends to: (a) 100 pc, (b) 1000 pc and (c) 10 000 pc?

At a distance of 50 pc from the Sun, there is a cluster of 3000 stars, occupying (uniformly) a sphere of radius 1.25 pc.

(ii) If in that direction the Galaxy extends to 250 pc, how many field stars will occupy the same apparent area as the cluster? How many will be in front and how many behind?

About 2% of the stars, in both the field and the cluster, can be identified spectroscopically as being virtually identical to the Sun.

(iii) What is the apparent magnitude of the solar-type stars in the cluster? And what is the apparent magnitude of (a) the brightest and (b) the faintest, of the solar-type field stars projected on the cluster?

(iv) Sketch a histogram of the cumulative apparent magnitude distribution of the solar type stars, both in the cluster and the field, putting them in bins of width 0.5 mag. For the field component, show that the number of stars in successive bins increases by very nearly a factor of two per bin.

Q3:

(i) Explain what astronomers mean by the term *'proper motion'*.

(ii) Two stars are at distances of 10 pc and 1 kpc from the Sun, respectively. Which of the two stars would you expect to show the higher proper motion and why?

(iii) Two solar-type stars are both at a distance of 50 pc from the Sun. One star is a member of the halo population, while the other is a disk star. Which of the two stars would you expect to show the higher proper motion? Which other physical property would you expect to be different between the two stars?

(iv) A star at a distance of 10 pc is travelling at 5 km s^{-1} along a path perpendicular to our line of sight. What is its proper motion, in seconds of arc per century?

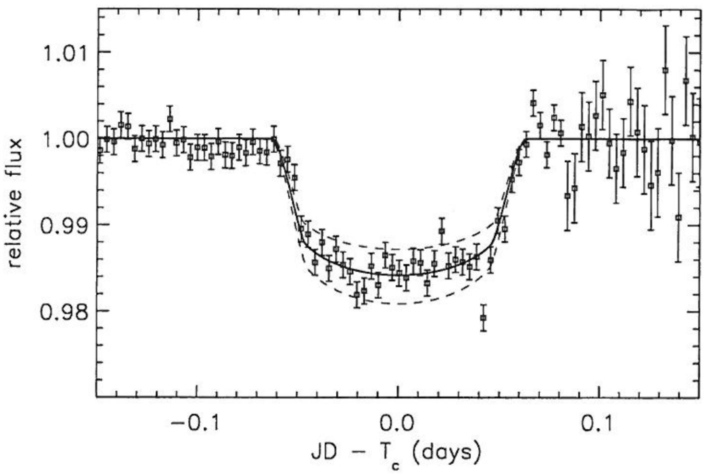
(v) Suppose the Galaxy is rotating rigidly with a period of 10^8 years. Find the proper motion of any star in the plane of the Galaxy, as measured relatively to a fixed background of extragalactic objects.

(vi) An astronomer measures the following positions for a bright star at yearly intervals:

Date	RA	Dec
1 Jan 2000	12h 30m 15s.1	$+51^\circ\,20'\,15''.0$
1 Jan 2001	12h 30m 16s.9	$+51^\circ\,20'\,32''.3$
1 Jan 2002	12h 30m 18s.0	$+51^\circ\,20'\,42''.5$
1 Jan 2003	12h 30m 18s.2	$+51^\circ\,20'\,44''.5$
1 Jan 2004	12h 30m 19s.0	$+51^\circ\,20'\,52''.1$
1 Jan 2005	12h 30m 20s.4	$+51^\circ\,21'\,05''.7$

From these data, derive the proper motion of the star. What can you conclude regarding the accuracy of the positional measurements?

Q4: A gas planet orbits a $1 M_{\odot}$ solar-type star and transits in front of the star once every 3.5 days. The light curve of the transit is shown below:



(i) From the depth of the eclipse, calculate the radius of the planet, assuming that it is completely dark.

(ii) What other information might be inferred from the shape of the light curve?

(iii) Suppose the brightness of the star is measured with a CCD (charge-coupled device) detector in which each photon generates one measurable electron (or 'count'). How many counts are needed to get the same accuracy as shown in the plot (i.e. errors of ~ 0.002 on the relative flux, assuming Poisson statistics)? How does that compare to the maximum counts in a CCD of $\sim 60\,000$ counts per pixel?

(iv) If the mass of the planet is $0.001 M_{\odot}$ (approximately one Jupiter mass), calculate the radial velocity amplitude of the star due to the orbiting planet.

(v) What wavelength shift would that give to an absorption line in the star's spectrum at 5000 Å? How does that compare to the typical resolution of 0.1 Å of the spectrographs normally used for these kinds of observations?

(vi) Another, similar, planet, which was in a wider orbit around the star, survives during the late stages of stellar evolution and ends up orbiting the stellar remnant—a white dwarf. With a (future) very sensitive instrument the orbit of the planet around the white dwarf can be followed. The planet's period is 244 yr, and the mass of the white dwarf is $0.6\, M_{\odot}$. What is the semi-major axis of the white dwarf–planet system?

(vii) As projected onto the sky, the orbit of the planet around the white dwarf appears to be a perfect circle of radius $1''$, but the white dwarf, instead of being in the centre of the circle, is 60% of the way to the edge. Show that the true orbit is an ellipse, with eccentricity $e = 3/5$, and calculate the size of the semi-minor axis and the distance to the system.

Q5a: Starting from the Planck function for blackbody radiation, derive:

(i) the Stefan-Boltzmann law

$$\int_0^\infty \pi B_\nu \mathrm{d}\nu = \sigma T^4$$

and

(ii) Wien's law

$$\lambda_{\mathrm{max}} T = 0.290\,\mathrm{cm\ K}$$

Q5b: A person is standing in a room at a temperature of 20 C. The average human body has a temperature of 36 C and a surface area of skin of $1.4\,\mathrm{m}^2$. Assuming that the average person absorbs and emits radiation according to the Planck formula for blackbody radiation:

(i) Calculate the energy per second radiated by an average person.

(ii) Determine the peak wavelength of the blackbody radiation emitted by the average person. In what region of the electromagnetic spectrum is this wavelength found?

(iii) Calculate the energy per second absorbed by the average person from the room.

(iv) Calculate the net energy difference between (i) and (iii), and compare its value with the power consumption of an everyday household item. What is the source of this energy?

Q6a: A star has an apparent magnitude $m_V = 2.5$. You measure its parallax to be $\theta = 0.002$ arcseconds.

(i) What is the star's absolute magnitude in V?

(ii) Given that a main-sequence A0 star has $M_V = +0.6$, what can you deduce about the nature of the star whose parallax you have measured?

Q6b: The star explodes as a supernova, increasing its luminosity by a factor of 50 000. What are the new values of apparent and absolute magnitude?

Q6c: The supernova remnant, in the shape of a ring, is expanding with a speed $v = 10,000\ \mathrm{km\ s^{-1}}$. After one day, would it be possible to resolve the ring with a ground-based telescope? And with the *Hubble Space Telescope*?

Q7:

(i) X-ray observations have shown that the outer atmosphere of the Sun (the corona) reaches a temperature of nearly 10^6 K. Why doesn't the Sun appear as a blackbody with $T_{\rm eff} \simeq 10^6$ K?

(ii) If the temperature gradient of a star's atmosphere were reversed, so that the temperature increased *outwards*, what type of spectral line would you expect to see in the star's spectrum at wavelengths where the opacity is greatest?

(iii) Consider a star surrounded by a large hollow spherical shell of hot gas. Under what circumstances would you see this shell as a ring around the star? If you observed the ring with a spectrograph, what type of spectrum would you see?

EXERCISES, Set 1: Solutions

Q1: On the basis of the ratio of the 'characteristic' distance between two of the objects below to their 'characteristic' size (radius), which are 'closer' together, planets in the solar system, stars in the Milky Way Galaxy, or galaxies in the local Universe?

A1: Taking the radius of Neptune ($\sim 25\,000$ km) as a typical planetary size, and the radius of the orbit of Jupiter (5 AU) as the typical distance between planets, the typical distance between planets in units of planetary radii is:

$$\left(\frac{d}{R}\right)_{\rm planet} = \frac{5 \times 1.5 \times 10^{11}\,{\rm m}}{25 \times 10^3 \times 10^3\,{\rm m}} = 0.3 \times 10^5 \tag{1.1}$$

Taking the Sun to be a typical star, and the distance to our nearest star, α Cen (1.3 pc), as the typical separation between stars, we have

$$\left(\frac{d}{R}\right)_{\rm star} = \frac{1.3 \times 3.1 \times 10^{16}\,{\rm m}}{7.0 \times 10^8\,{\rm m}} \simeq 4/7 \times 10^8 \tag{1.2}$$

The nearest galaxy in M31 (the Andromeda galaxy) at $d \simeq 780$ kpc (ignoring the Magellanic Clouds, which are at $d \simeq 50$ kpc but can be considered as companions to the Milky Way), and a typical galaxy like the Milky Way has a radius $R \simeq 15$ kpc. Thus:

$$\left(\frac{d}{R}\right)_{\rm galaxy} = \frac{780\,{\rm kpc}}{15\,{\rm kpc}} = 52 \tag{1.3}$$

Therefore, stars are really far apart, three orders of mgnitude more so than planets, and galaxies are quite close together. This suggests that stars basically never interact with each other (but we have not considered multiple stars here), whereas galaxies interact frequently. For comparison, the Bohr radius divided by the electron 'radius' is $\sim 20\,000$, similar to the typical planetary value.

Q2: Suppose the space density of stars within the Galaxy is $0.1\,\text{pc}^{-3}$.

Q2(i): How many stars would you see per square degree in a direction where the Galaxy extends to: (a) 100 pc, (b) 1000 pc and (c) 10 000 pc?

A2(i):

The solid angle in steradians (π^2 steradians $= 180^2$ sq. deg) subtended by area A at distance d is $\Omega = A/d^2$.
The total number of stars is $N = n \times V$, where n is the number density and V is the volume.
The *surface density* of stars is $\Sigma = N/\Omega$.

Thus, in a cone of height d and base πr^2, we have:

$$\Sigma = \frac{N}{\Omega} = \frac{nVd^2}{A} = \frac{n\left(\frac{1}{3}\pi r^2 d\right)d^2}{\pi r^2} = \frac{1}{3}nd^3 \text{ stars steradian}^{-1} \qquad (1.4)$$

and

$$\Sigma = \frac{1}{3}nd^3\,\frac{\pi^2}{180^2} \text{ stars sq. deg}^{-1}$$

Therefore:

$$\Sigma = \frac{1}{3}0.1\,\text{pc}^{-3}\,100^3\,\text{pc}^3\,\frac{\pi^2}{180^2} \simeq 10\,\text{stars sq. deg}^{-1} \qquad \text{(a)}$$

$$\Sigma = \frac{1}{3}0.1\,\text{pc}^{-3}\,1000^3\,\text{pc}^3\,\frac{\pi^2}{180^2} \simeq 10^4\,\text{stars sq. deg}^{-1} \qquad \text{(b)}$$

$$\Sigma = \frac{1}{3}0.1\,\text{pc}^{-3}\,10\,000^3\,\text{pc}^3\,\frac{\pi^2}{180^2} \simeq 10^7\,\text{stars sq. deg}^{-1} \qquad \text{(c)}$$

Q2(ii): At a distance of 50 pc from the Sun, there is a cluster of 3000 stars, occupying (uniformly) a sphere of radius 1.25 pc.

If in that direction the Galaxy extends to 250 pc, how many field stars will occupy the same apparent area as the cluster? How many will be in front and how many behind?

A2(ii): From A2(i), we have:

$$\Sigma = \frac{1}{3}0.1\,\text{pc}^{-3}\,250^3\,\text{pc}^3\,\frac{\pi^2}{180^2} \simeq 160\,\text{stars sq. deg}^{-1} \qquad (1.5)$$

Projected area of the cluster is:

$$\Omega = \frac{A}{d_{\rm cl}^2} = \frac{\pi r^2}{d_{\rm cl}^2} \cdot \frac{180^2}{\pi^2} = \frac{1.25^2\,{\rm pc}^2}{50^2\,{\rm pc}^2} \cdot \frac{180^2}{\pi} = 6.45\,{\rm sq.\,deg.}, \qquad (1.6)$$

and therefore, the number of field stars seen projected against the cluster is:

$$N = \Sigma \cdot \Omega = 159\,{\rm stars\;sq.\,deg.}^{-1} \cdot 6.45\,{\rm sq.\,deg.} = 1025\,{\rm stars}\,.$$

Denoting N_1 and N_2 as the number of stars respectively in front and behind the cluster, we have:

$$\frac{N_1}{N_2} = \left(\frac{d_1}{d_2}\right)^3 = \left(\frac{50}{250}\right)^3 = \left(\frac{1}{5}\right)^3 = \frac{1}{125}, \qquad (1.7)$$

and since $N = N_1 + N_2 \simeq N_2 = 1025$, we have:

$$N_1 = \frac{1025}{125} \simeq 8, \quad N_2 \simeq 1025 - 8 = 1017$$

Q2(iii): About 2% of the stars, in both the field and the cluster, can be identified spectroscopically as being virtually identical to the Sun.

What is the apparent magnitude of the solar-type stars in the cluster? And what is the apparent magnitude of (a) the brightest and (b) the faintest, of the solar-type field stars projected on the cluster?

A2(iii): The Sun has absolute magnitude $M_{\rm V} = 4.8$.
The distance modulus has been defined as: $m - M = 5 \log d/10$, where d is in pc.
Thus, at $d = 50\,{\rm pc}$, $m - M = 5 \log 5 = 3.5$ and $m_{\rm V} = 4.8 + 3.5 = 8.3$.

The faintest star is likely to be at $d = 250\,{\rm pc}$. Therefore, its distance modulus will be $m - M = 5 \log 25 = 7.0$ and $m_{\rm V} = 4.8 + 7.0 = 11.8$

To deduce the apparent magnitude of the closest solar-type star, we have to work out how far we need to look, on average, until we find a solar-type star in the solid angle subtended by the cluster.

We know that $n = 0.1\,{\rm pc}^{-3}$, and that 2% of the stars are of solar type. Thus, $n_\odot = 0.01 \times 0.02 = 0.002\,{\rm pc}^{-3}$.

We also know that the number of stars projected onto an area of sky is:

$$N = \sigma\Omega = \frac{1}{3} n d^3 \frac{\pi^2}{180^2} \Omega$$

and we know that $\Omega = 6.45\,\mathrm{sq.\,deg}$. So, the problem reduces to working out the value of d at which $N = 1$:

$$d = \left(\frac{3 \cdot 180^2}{n\,\pi^2\,\Omega} \right)^{1/3} = \left(\frac{3 \cdot 180^2}{0.002\,\mathrm{pc}^{-3}\,\pi^2\,6.45\,\mathrm{sq.\,deg}} \right)^{1/3} = 91\,\mathrm{pc}\,. \qquad (1.8)$$

A solar-type star at a distance of 91 pc will therefore have a distance modulus $m - M = 5 \log 91/10 = 4.8$ and an apparent magnitude $m_V = 4.8 + 4.8 = 9.6$.

Q2(iv): Sketch a histogram of the cumulative apparent magnitude distribution of the solar type stars, both in the cluster and the field, putting them in bins of width 0.5 mag. For the field component, show that the number of stars in successive bins increases by very nearly a factor of two per bin.

A2(iv):

For the cluster, we have $m_V = 8.3$ with $0.02 \times 3000 = 60$ stars. For the field, we have from A2(iii), $9.6 \le m_V \le 11.8$. Thus, in order to build the histogram required, we need to work out distances and numbers of stars corresponding to $m_V = 9.5, 10, 10.5, 11, 11.5, 12$.

From the definition of the distance modulus, we have:

$$d = 10^{\left(\frac{m-M+5}{5}\right)} \qquad (1.9)$$

and for the number of stars:

$$N = \Sigma\Omega = \frac{1}{3} n \frac{\pi^2}{180^2} \Omega\, d^3 = \frac{1}{3} 0.002\,\mathrm{pc}^{-3} \frac{\pi^2}{180^2} 6.45\, d^3 = 1.3 \times 10^{-6} d^3\,. \qquad (1.10)$$

Using the above two equations, we have:

m_V	d (pc)	N
9.5	87	0.9
10.0	110	1. 7
10.5	138	3.4
11.0	174	7
11.5	219	14
12.0	275	27

As can be seen from the Table, N indeed increases by about a factor of 2 every 0.5 mag interval in m_V. That this is the case generally, it can be seen by considering the following:

$$m_1 = 5\log\left(\frac{d_1}{10}\right) + M$$
$$m_2 = 5\log\left(\frac{d_2}{10}\right) + M$$

Therefore:

$$\begin{aligned} m_1 - m_2 &= 5\log\left(\frac{d_1}{10}\right) - 5\log\left(\frac{d_2}{10}\right) \\ &= 5\log\left(\frac{d_1}{d_2}\right) \end{aligned}$$

Hence:

$$\frac{0.5}{5} = 0.1 = \log\left(\frac{d_1}{d_2}\right)$$

and

$$\frac{d_1}{d_2} = 1.26$$

and

$$\frac{N_1}{N_2} \simeq \left(\frac{d_1}{d_2}\right)^3 \simeq 2.0$$

Q3(i): Explain what astronomers mean by the term *'proper motion'*.

A3(i): The term *'proper motion'* reflects the change in the positions of some stars on the celestial sphere, relative to a fixed frame of reference based on the unchanging coordinates of very distant objects, usually quasars at cosmological distances.

Q3(ii): Two stars are at distances of 10 pc and 1 kpc from the Sun, respectively. Which of the two stars would you expect to show the higher proper motion and why?

A3(ii): Proper motion arises primarily (but not exclusively) because stars move in 3D relative to one another. Proper motion measures the projection on the plane of the sky of the 3D velocity between a star and the Sun as an angle per unit time (usually measured in arcsec yr^{-1}). From this it follows that the closer a star is to the Sun, the larger its proper motion (again in arcsec yr^{-1}) *for the same value of its projected 3D velocity.*

Q3(iii): Two solar-type stars are both at a distance of 50 pc from the Sun. One star is a member of the halo population, while the other is a disk star. Which of the two stars would you expect to show the higher proper motion? Which other physical property would you expect to be different between the two stars?

A3(iii): Stars in the Galactic halo have a higher velocity dispersion (they move with random motions relative to one another) than stars in the disk (with ordered rotation around the centre of the Galaxy). Thus, in general, we would expect a halo star to have a higher proper motion than a disk star, and indeed members of the Galactic halo population can be identified by their high proper motions (if close to the Sun).

We would also expect an average halo star to be older than an average disk star and to exhibit a lower proportion of elements heavier than He, that are synthesised through successive generations of stars.

Q3(iv): A star at a distance of 10 pc is travelling at 5 km s^{-1} along a path perpendicular to our line of sight. What is its proper motion, in seconds of arc per century?

A3(iv): We want to work out the proper motion:

$$\frac{\Delta\theta}{\Delta t} = \frac{v_t}{d}$$

where $\Delta\theta$ is the angle subtended by the movement $\Delta r = v_t \cdot \Delta t \perp$ to the line of sight of a star at distance d travelling with speed v_t. Thus,

$$\frac{\Delta\theta}{\Delta t} = \frac{5 \times 10^3 \text{ m s}^{-1}}{10 \times 3.1 \times 10^{16} \text{ m}} = 1.6 \times 10^{-14} \text{ rad s}^{-1}$$

$$1'' = \frac{\pi}{180 \times 60 \times 60} = \frac{\pi}{648000} \text{ radians}$$

$$1 \text{ year} = 3.16 \times 10^7 \text{ s}$$

Thus,

$$\mu \equiv \frac{\Delta\theta}{\Delta t} = \frac{648000}{\pi} \times 3.16 \times 10^7 \times 100 \times 1.6 \times 10^{-14} = 10.4 \text{ arcsec } (100 \text{ yr})^{-1}$$

Q3(v): Suppose the Galaxy is rotating rigidly with a period of 10^8 years. Find the proper motion of any star in the plane of the Galaxy, as measured relatively to a (supposedly) fixed background of extragalactic objects.

A3(v): The angular speed is:

$$\omega = \frac{2\pi}{P} = \frac{d\theta}{dt} \text{ rad yr}^{-1}$$

$$\omega = \frac{2\pi}{10^8} \cdot \frac{648000}{\pi} \cdot 100 = 1.3 \text{ arcsec } (100 \text{ yr})^{-1}$$

Q3(vi): An astronomer measures the following positions for a bright star at yearly intervals:

Date	RA	Dec
1 Jan 2000	12h 30m 15s.1	+51° 20′ 15″.0
1 Jan 2001	12h 30m 16s.9	+51° 20′ 32″.3
1 Jan 2002	12h 30m 18s.0	+51° 20′ 42″.5
1 Jan 2003	12h 30m 18s.2	+51° 20′ 44″.5
1 Jan 2004	12h 30m 19s.0	+51° 20′ 52″.1
1 Jan 2005	12h 30m 20s.4	+51° 21′ 05″.7

From these data, derive the proper motion of the star. What can you conclude regarding the accuracy of the positional measurements?

A3(vi): The point of this question is to: (a) familiarise the students with celestial coordinates, and (b) foster a critical attitude to data.

To deduce the proper motion from year to year, we need to calculate the angular distance between two successive positions, as given by the above values of Right Ascension, α, and Declination, δ. The angular distance Δ is obtained by summing in quadrature the differences in RA and Dec: $\Delta = \left(\Delta\alpha^2 + \Delta\delta^2\right)^{1/2}$. Potential pitfalls to avoid are forgetting that: (i) on the equator (i.e. when $\delta = 0$) 1 s of time (1 s in RA) is equivalent to 15 arcsec , and (ii) away from the equator, one has to multiply $\Delta\alpha$ by $\cos\delta$ in order to obtain the correct angular distance in RA.

With these points in mind, the angular distances between two successive measurements of RA and Dec are as follows:

Interval	Δ (arcsec)
2000-1	24.1
2001-2	14.5
2002-3	2.7
2003-4	10.7
2004-5	18.9

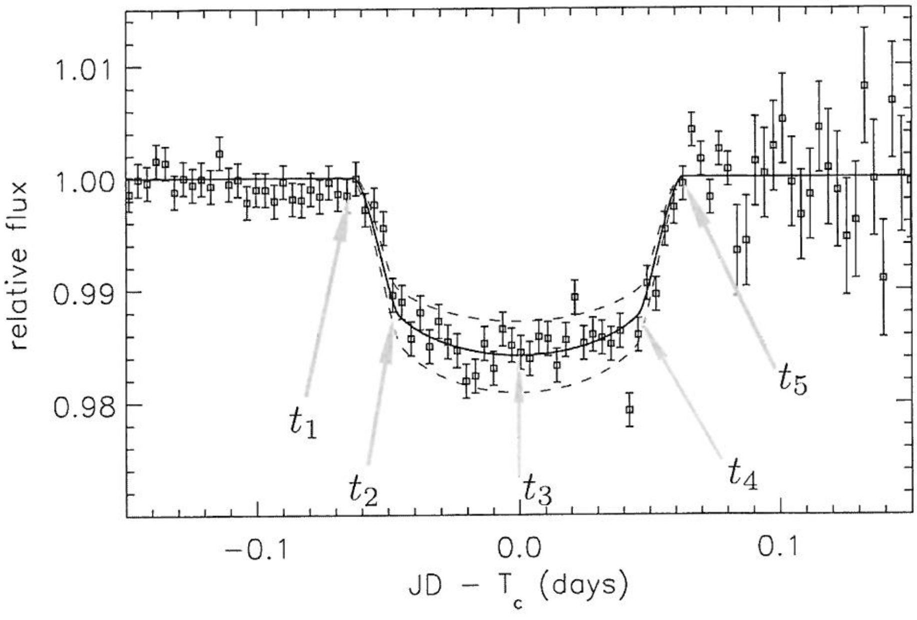
One could average these values to deduce a mean proper motion $\langle \mu \rangle = 14.2\,\mathrm{arcsec\ yr^{-1}}$.

However, more importantly, the student should realise that there are large differences between the values of Δ measured from year to year. These are unlikely to be real: stars do not accelerate or decelerate from year to year! Proper motion is smooth (at least on these timescales), and the large scatter between the above values must therefore arise from large errors in the measurements, of order $\sim 10\,\mathrm{arcsec}$.

Furthermore, in the lecture notes the students were told that the largest value of proper motion measured for any star is $\mu = 10.4\,\mathrm{arcsec\ yr^{-1}}$ (Barnard's star; see Figure 2.2 of lecture notes). This is one more reason for the above measurements to be suspect.

Final point: if the accuracy of the positional measurements is $\pm \sim 10\,\mathrm{arcsec}$, the values in the table compiled by the hypothetical astronomer are quoted to far too many (in)significant figures. Thus, the first entry of declination should really be quoted as $+51^\circ\,20'.2$ and so on. This sort of 'sloppiness' is often encountered in the astronomical literature.

Q4(i): A gas planet orbits a $1M_{\odot}$ solar-type star and transits in front of the star once every 3.5 days. The light curve of the transit is shown below:



From the depth of the eclipse, calculate the radius of the planet, assuming that it is completely dark.

A4(i): When the planet is in front of the star, the stellar light is dimmed by 0.015 (relative flux = 0.985 of uneclipsed star). Since $R \propto \sqrt{A}$:

$$\frac{R_p}{R_{\odot}} = \sqrt{0.015} = 0.12$$

Q4(ii): What other information might be inferred from the shape of the light curve?

A4(ii):

(a) From the time difference $\Delta t_{1,2} = t_2 - t_1$ and R_p it is possible to deduce the relative velocity v of the star and planet.

(b) Knowing v and $R_{\odot}$ we can deduce the inclination of the orbital plane to the line of sight, by comparing $v \cdot \Delta t_{1,4}$ with $R_{\odot}$.

(c) From v and the period P we can deduce the average distance of the planet from the star

(d) We can measure limb darkening of the star by comparing the relative fluxes at t_2 and t_3.

Q4(iii): How many counts are needed to get the same accuracy as shown in the plot (i.e. errors of ~ 0.002 on the relative flux, assuming Poisson statistics)? How does that compare to the maximum counts in a CCD of $\sim 60\,000$ counts per pixel?

A4(iii): Photons obey Bose-Einstein statistics, whereby the 1σ uncertainty is given by $\sqrt{N}/N$, where N is the number of photons recorded. Thus, to obtain a 1σ uncertainty $\sigma(N)/N = 0.002$, we require $(0.002)^{-2} =$ 250 000 counts.

Since we are told that each CCD pixel saturates at $\sim 60\,000$ counts, we need to spread the star's light on at least 5 pixels on the detector in order to achieve the statistical accuracy of the photometric data shown in the plot.

Q4(iv): If the mass of the planet is $0.001 M_\odot$ (approximately one Jupiter mass), calculate the radial velocity amplitude of the star due to the orbiting planet.

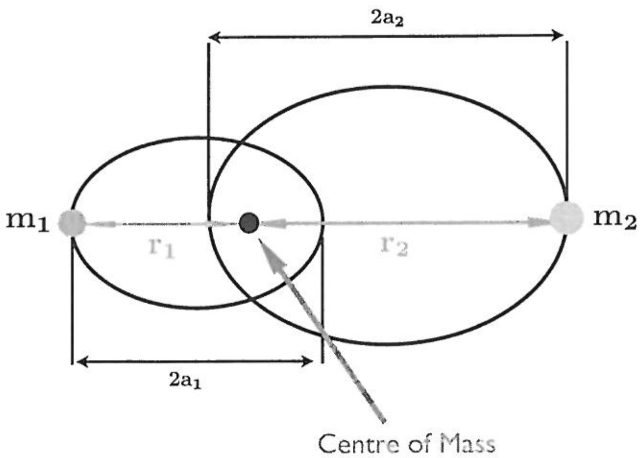
A4(iv): The star and the planet orbit about their common centre of mass:

with $M_\odot r_1 = M_p r_2$, and therefore:

$$r_1 = r_2 \frac{M_p}{M_\odot} = 0.001 r_2 \tag{1.11}$$

From Kepler's third law, we have:

$$P^2 = \frac{4\pi^2}{G(M_\odot + M_p)} (r_1 + r_2)^3 = \frac{4\pi^2}{G} \frac{(1.001 r_2)^3}{(1.001 M_\odot)} \tag{1.12}$$



With $P = 3.5$ days, we have:

$$r_2 \simeq \left(\frac{GP^2 M_\odot}{4\pi^2}\right)^{1/3} = 6.7 \times 10^9\,\mathrm{m} \tag{1.13}$$

and

$$r_1 = 0.001 r_2 = 6.7 \times 10^6\,\mathrm{m} \tag{1.14}$$

The speed of the solar-type star relative to the common centre of mass is therefore:

$$v_1 = \frac{2\pi r_1}{P} = \frac{2\pi \cdot 6.7 \times 10^6\,\mathrm{m}}{3.5 \times 24 \times 3600\,\mathrm{s}} = 139\,\mathrm{m\ s}^{-1} \tag{1.15}$$

Q4(v): What wavelength shift would that give to an absorption line in the star's spectrum at 5000 Å? How does that compare to the typical resolution of 0.1 Å of the spectrographs normally used for these kinds of observations?

A4(v): From the Doppler formula, we have $v/c = \Delta\lambda/\lambda$. Therefore,

$$\Delta\lambda = 5000 \cdot \frac{139}{3 \times 10^8} = 0.0023\,\text{Å} \tag{1.16}$$

which is only $0.0023/0.1 \lesssim 1/40$ of the typical spectral resolution. Hence, means must be devised to keep the spectrograph very stable and measure its wavelength scale with great precision.

Q4(vi): Another, similar, planet, which was in a wider orbit around the star, survives during the late stages of stellar evolution and ends up orbiting the stellar remnant—a white dwarf. With a (future) very sensitive instrument the orbit of the planet around the white dwarf can be followed. The planet's period is 244 yr, and the mass of the white dwarf is $0.6\,M_\odot$. What is the semi-major axis of the white dwarf–planet system?

A4(vi): We have:

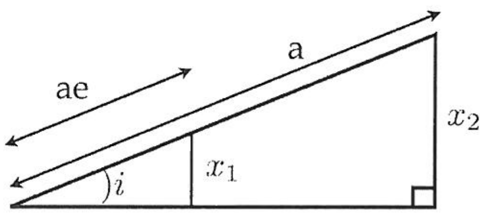
$$a^3 = MP^2 \Rightarrow a = \left(MP^2\right)^{1/3} \tag{1.17}$$

Scaling to the solar system, where the Earth's orbit has a semi-major axis of $a = 1\,\mathrm{AU}$ and period $P = 1\,\mathrm{yr}$ aroud a star of mass $M = 1M_\odot$, we have:

$$a = \left[(0.6)\,(244)^2\right]^{1/3} = 33\,\mathrm{AU} \tag{1.18}$$

Q4(vii): As projected onto the sky, the orbit of the planet around the white dwarf appears to be a perfect circle of radius $1''$, but the white dwarf, instead of being in the centre of the circle, is 60% of the way to the edge. Show that the true orbit is an ellipse, with eccentricity $e = 3/5$, and calculate the size of the semi-minor axis and the distance to the system.

A4(vii): Imagine that we are viewing the ellipse edge on, along the semi-major axis:



Then,

$$\sin i = \frac{x_1}{ae} = \frac{x_2}{a} \tag{1.19}$$

and hence:

$$\frac{x_1}{x_2} = \frac{ae}{a} = e \tag{1.20}$$

and for $x_1/x_2 = 0.6$, $e = 3/5$.

The semi-major axis is:

$$b = a\sqrt{1-e^2} = 33\,\mathrm{AU}\sqrt{1-(3/5)^2} = 26.4\,\mathrm{AU} \tag{1.21}$$

The distance is just:

$$d = \frac{b}{\tan\theta} = \frac{26.4\,\mathrm{AU}}{\tan\left(1''\,\frac{1^\circ}{3600''}\right)} = 5.45\times10^6\,\mathrm{AU} \tag{1.22}$$

Q5a: Starting from the Planck function for blackbody radiation, derive:

(i) the Stefan-Boltzmann law

$$\int_0^\infty \pi B_\nu \mathrm{d}\nu = \sigma T^4$$

and

(ii) Wien's law: $\lambda_{\max} T = 0.290\,\mathrm{cm\ K}$.

A5a(i): The Planck function gives the dependence of blackbody radiation on temperature and frequency; it may be transformed by introducing a new variable $x = h\nu/kT$, so that $\nu = x \cdot kT/h$ and $\mathrm{d}\nu = (kT/h)\,\mathrm{d}x$:

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp[h\nu/kT]-1} = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^3 \frac{x^3}{e^x-1}. \tag{1.23}$$

Integrating over all frequencies:

$$F = \int_0^\infty \pi B_\nu \,\mathrm{d}\nu$$

$$= \int_0^\infty \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp[h\nu/kT]-1}\,\mathrm{d}\nu \tag{1.24}$$

$$= \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1}\,\mathrm{d}x\,.$$

From numerical integration tables (or by numerical calculation), the value of the above integral is found to be:

$$\int_0^\infty \frac{x^3}{e^x-1}\,\mathrm{d}x = \frac{\pi^4}{15} \tag{1.25}$$

giving us the required result:

$$F = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1}\,\mathrm{d}x = \frac{2\pi k^4 T^4}{c^2 h^3}\frac{\pi^4}{15} = \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 = \sigma T^4 \tag{1.26}$$

where all the physical constants have been grouped together under the Stefan-Boltzmann constant:

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} \simeq 5.67 \times 10^{-5}\,\text{erg cm}^{-2}\ \text{K}^{-4}\ \text{s}^{-1}$$

A5a(ii): Wien's law gives the wavelength at which the Planck function:

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\exp[\frac{hc}{\lambda kT}] - 1} \tag{1.27}$$

is at its maximum as a function of temperature. By introducing a new variable: $x = hc/\lambda kT$ so that $\lambda = (hc/kT) \cdot (1/x)$, we can re-write B_λ as:

$$B_\lambda = \frac{2k^5T^5}{h^4c^3}\frac{x^5}{e^x - 1}\,. \tag{1.28}$$

To find the wavelength λ at which B_λ as a maximum, we differentiate eq. 1.28 and equate it to zero:

$$\frac{\mathrm{d}B_\lambda}{\mathrm{d}\lambda} = \frac{\mathrm{d}B_\lambda}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\lambda} = -\frac{2k^5T^5}{h^4c^3}\left(\frac{x^4\,[5\,(e^x-1) - xe^x]}{(e^x-1)^2}\right)\frac{hc}{\lambda^2kT} = 0 \tag{1.29}$$

A solution exists at $x = 0$ ($\lambda = \infty$), but that is simply due to the fact that the slope of the Planck function at long wavelengths is nil. However, this solution is not of interest since it represents a minimum. To find the solution of interest, the equation that has to be solved is

$$5\,(e^x - 1) - xe^x = 0 \tag{1.30}$$

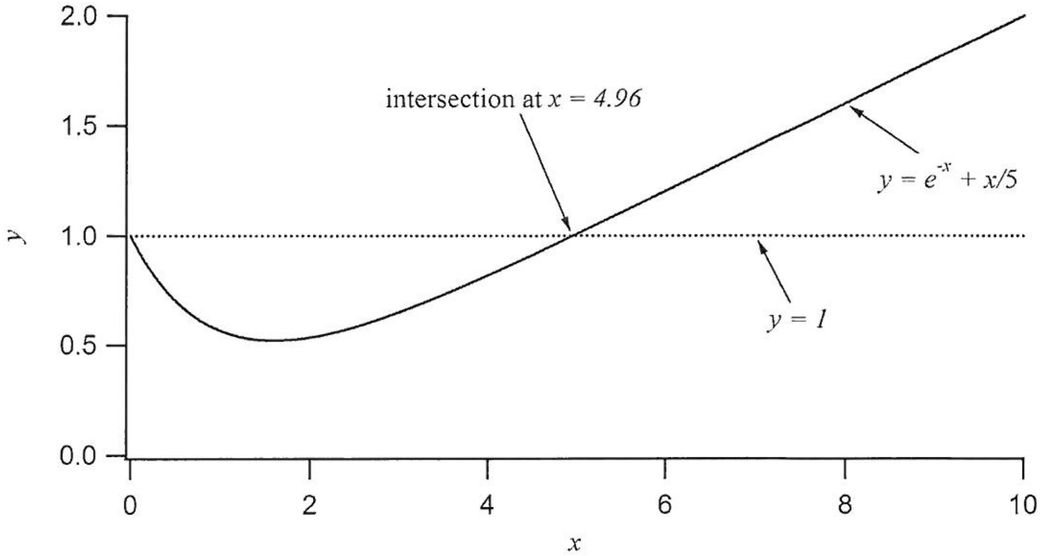
which can be re-arranged as:

$$e^{-x} + \frac{x}{5} = 1\,. \tag{1.31}$$

The equation may be solved numerically (see Figure below for a graphical solution) to give $x_{\text{max}} = 4.96$.

Thus:

$$\begin{aligned}\lambda_{\text{max}} &= \frac{hc}{x_{\text{max}}kT} \\ &= \frac{6.626 \times 10^{-27}\,\text{erg s} \times 2.9979 \times 10^{10}\,\text{cm s}^{-1}}{4.96 \times 1.3807 \times 10^{-16}\text{erg K}^{-1}T} \\ &= \frac{0.290\,\text{cm K}}{T}\,.\end{aligned} \tag{1.32}$$



Q5b: A person is standing in a room at a temperature of 20 C. The average human body has a temperature of 36 C and a surface area of skin of 1.4 m^2. Assuming that the average person absorbs and emits radiation according to the Planck formula for blackbody radiation:

(i) Calculate the energy per second radiated by an average person.

A5b(i): Using

$$L_{\text{person}} = 4\pi R^2 \sigma T^4 \tag{1.33}$$

with $4\pi R^2 = 1.4\,\text{m}^2$, $\sigma = 5.67 \times 10^{-8}\,\text{W m}^{-2}\,\text{K}^{-4}$, and $T = 36 + 273 = 309\,\text{K}$, we have:

$$L_{\text{person}} = 1.4 \times 5.67 \times 10^{-8} \times 309^4 = 724\,\text{W} \tag{1.34}$$

(ii) Determine the peak wavelength of the blackbody radiation emitted by the average person. In what region of the electromagnetic spectrum is this wavelength found?

A5b(ii): Using Wien's displacement law, $\lambda_{\rm max}T = 0.290\,{\rm cm\ K}$, we have:

$\lambda_{\rm max} = 0.290/309 = 9.4\,\mu{\rm m}$ which is in the mid-infrared range.

(iii) Calculate the energy per second absorbed by the average person from the room.

A5b(iii): Repeating the calculation at (i), we find:

$$L_{\rm room} = 1.4 \times 5.67 \times 10^{-8} \times 293^4 = 585\,{\rm W} \qquad (1.35)$$

(iv) Calculate the net energy difference between (i) and (iii), and compare its value with the power consumption of an everyday household item. What is the source of this energy?

A5b(iv): Comparing the answers at (i) and (iii), we see that in this situation the average person loses energy at a rate of 139 W — similar to a bright light-bulb! This power is provided by the metabolism of the human body.

Q6a: A star has an apparent magnitude $m_V = 2.5$. You measure its parallax to be $\theta = 0.002$ arcseconds.

(i) What is the star's absolute magnitude in V?

(ii) Given that a main-sequence A0 star has $M_V = +0.6$, what can you deduce about the nature of the star whose parallax you have measured?

A6a:

(i) Since for $\theta = 1$ arcsec $d = 1$ pc, $d = 500$ pc for $\theta = 0.002$ arcsec. The distance modulus is defined as:

$$m_V - M_V = 5 \log d - 5 = 8.5\,.$$

Hence, the star's absolute magnitude is $M_V = 2.5 - 8.5 = -6.0$

(ii) Thus, this star is 6.6 magnitudes (a factor of 437) more luminous than a main-sequence A0 star. The only stars with such high luminosities are supergiants.

Q6b: The star explodes as a supernova, increasing its luminosity by a factor of 50 000. What are the new values of apparent and absolute magnitude?

A6b: Recalling that fluxes are magnitudes are related by the expression:

$$\frac{F_2}{F_1} = 10^{0.4 \times (m_1 - m_2)}$$

it can easily be seen than an increase in luminosity by a factor of 50 000 corresponds to a *decrease* of both the apparent and absolute magnitudes by 11.75 magnitudes. Hence, $M_V = -17.75$ and $m_V = -9.25$.

Q6c: The supernova remnant, in the shape of a ring, is expanding with a speed $v = 10,000$ km s^{-1}. After one day, would it be possible to resolve the ring with a ground-based telescope? And with the *Hubble Space Telescope*?

A6c: If the ring-like structure has an expansion velocity of 10 000 km s^{-1}, then the ring diameter after one day is

$$D_{SN} = 2 \times 10\,000 \times 1000 \times 60 \times 60 \times 24 = 1.73 \times 10^{12}\,\text{m}$$

At a distance of $500 \times 3.1 \times 10^{16}$ m, the SN remnant diameter will subtend an angle:

$$\begin{aligned}\theta &= \frac{1.73 \times 10^{12}}{500 \times 3.1 \times 10^{16}} \\ &= 1.1 \times 10^{-7}\,\text{radians} \\ &= 1.1 \times 10^{-7} \times \frac{180}{\pi} \times 60 \times 60 = 0.023\,\text{arcsec}\end{aligned}$$

With the typical atmospheric seeing of FWHM = 1 arcsec, the ring will *not* be resolved by ground-based telescopes, and will appear as a point source.

Although there is no atmosphere above the *Hubble Space Telescope*, its optics produce images with a resolution of FWHM = 0.05 arcsec at best. Thus, we'll have to wait for a few days before the *HST* will be able to resolve the supernova remnant.

Q7(i): X-ray observations have shown that the outer atmosphere of the Sun (the corona) reaches a temperature of nearly 10^6 K. Why doesn't the Sun appear as a blackbody with $T_{\rm eff} \simeq 10^6$ K?

A7(i): When looking into a star's atmosphere, we always look to an optical depth $\tau_\lambda \simeq 2/3$. The fact that the Sun does not appear as a blackbody with $T_{\rm eff} \simeq 10^6$ K tells us that the corona is optically thin ($\tau \ll 1$).

Q7(ii): If the temperature gradient of a star's atmosphere were reversed, so that the temperature increased *outwards*, what type of spectral line would you expect to see in the star's spectrum at wavelengths where the opacity is greatest?

A7(ii): The fact that we look down to $\tau_\lambda \simeq 2/3$ implies that we see down to a depth:

$$\int_0^s \kappa_\lambda \rho \, \mathrm{d}s \simeq 2/3$$

Thus, at wavelengths where κ_λ is greatest, s is smallest. If the temperature of the star's atmosphere increases outwards, then a smaller value of s corresponds to looking at gas at a higher temperature. At wavelengths where the opacity is highest, one would therefore see *emission* lines.

Q7(iii): Consider a star surrounded by a large hollow spherical shell of hot gas. Under what circumstances would you see this shell as a ring around the star? If you observed the ring with a spectrograph, what type of spectrum would you see?

A7(iii): A large spherical shell would look like a ring if we could see straight through the middle of the shell. Thus, the shell must be *optically thin*. Near the edge of the shell, where the line of sight passes through more gas, the shell appears brighter. Hot gas would produce an *emission line* spectrum.

EXERCISES: Set 2 of 4

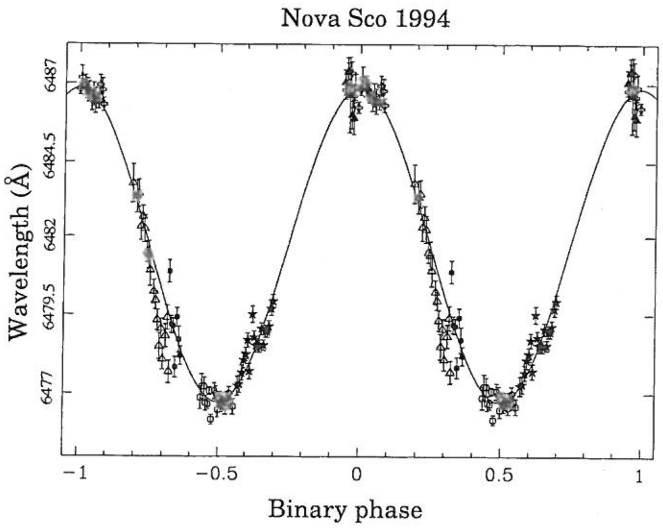
Q1: In 1994 a new bright star was found, Nova Sco 1994. It turned out that it was a sudden brightening of an X-ray binary in which an F star is orbiting a compact, unseen, object. The mass function of the unseen companion is defined as

$$f(M_1, M2) = \frac{(M_1 \sin i)^3}{(M_1 + M_2)^2},$$

and can be estimated from the radial velocity amplitude of the F star v_2, the binary period P, and the inclination of the plane of the orbit to the line of sight, $\sin i$. The mass function has the units of mass, and is the minimum mass of the companion should the star for which we have orbital information be a test particle (or effectively massless). When additional information is available about the mass of the star with the orbital information, more accurate estimates of the companion mass can be obtained.

(i) Derive the formula giving the mass function of the unseen companion in terms of v_2, P and $\sin i$, assuming that the orbit is circular.

(ii) The Figure below shows the observed wavelength of the Fe II λ6485.10 line as function of the orbital period.



Justify the assumption that the orbit is circular using the data shown in the Figure, and deduce the value of v_2. Why is the mean wavelength of the Fe line not equal to its laboratory wavelength?

(iii) When the system returns to quiescence, the light of the binary system is dominated by the F-type star. In such close binaries, the stars become deformed by the gravitational pull of the companion. This leads to their projected surface on the sky—and therefore the total brightness—varying with the orbital period. From such variations, the orbital inclination can be deduced. In the case shown here, $i \simeq 70°$ and $P = 2.62\,\mathrm{days}$. From these parameters, determine the value of the mass function.

(iv) Knowing that the mass of an F-type star is $M_2 \simeq 2.4 M_\odot$, deduce M_1. It is generally thought that in X-ray binaries where the mass of the unseen companion is $M_1 \gtrsim 3M_\odot$, the compact object is likely to be a black hole. Is the unseen companion in Nova Sco 1994 a black hole?

Q2: The gas density within a star decreases from the centre to the surface as a function of radial distance r according to

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R}\right)^2\right]$$

where ρ_c is the core density and R is the star's radius.

(i) Find $m(r)$.

(ii) Derive the relation between the total mass of the star M and R.

(iii) What is the average density of the star in units of the core density ρ_c?

(iv) The gravitational potential energy of a star of mass M and radius R is given by:

$$U_g = -\alpha \frac{GM^2}{R}$$

where α is a constant of order unity determined by the distribution of matter within the star.

Find the value of α for the density profile given above.

Q3: The Lorentz profile:

$$\phi_\nu = \frac{\frac{\Gamma}{4\pi^2}}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

describes the '*natural*' broadening of an absorption line due to the inherent widths of the energy levels between which the atomic transition takes place following the absorption of a photon of rest frequency ν_0. In the above expression $\Gamma = 1/\Delta t$ is the radiative damping constant, inversely proportional to the mean lifetime Δt of the excited energy level to which absorption takes place.

(i) Show the the full width at half maximum of a Lorentzian line profile is $\Delta\nu_{FWHM} = \Gamma/2\pi$.

(ii) At what interval in units of $\Gamma/4\pi$ (in other words, in units of the half-width at half maximum) from the rest frequency does the Lorentz profile have a value of 1% of its central intensity?

(iii) Thermal broadening is described by a Gaussian distribution. For a Gaussian distribution with the same FWHM as the Lorentz profile, what is the probability of absorption at the same $\Delta\nu$ from the line centre as in part (ii)?

Q4: The temperature dependence of energy generation by the triple-alpha process is:

$$\mathcal{E}_{3\alpha} = k_0 T_8^{-3} e^{-(44/T_8)}$$

where T_8 is the temperature in units of 10^8 K and k_0 is a constant.

(i) By considering the energy generation near $T_8 = 1$ to scale as $\mathcal{E}(T) = k_1 T_8^\gamma$, show that: $\mathcal{E}_{3\alpha} \approx k_2 T_8^{41}$, where k_1 and k_2 are constants.

(ii) Calculate the change in energy output resulting from a 10% change in temperature.

Q5: An eclipsing-binary system has a parallax of 0.1 arcsec (with negligible error) and consists of two solar-type stars with a semi-major axis of $500R_\odot$. The period is known very accurately.

(i) What is the angular size of each of the stars and of the semi-major axis? If you can measure angles on the sky with a 1σ rms accuracy of 0.01 arcsec, what is the percentage accuracy of the measurement of the semi-major axis and of the radius of each star ?

(ii) Assume that the stars emit as blackbodies:

$$F_\nu(T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1},$$

where ν is the frequency is Hz, with an effective temperature $T_{\rm eff} \simeq 5800\,{\rm K}$. If you measure the flux ratio between $\log\nu = 14.0$ and 15.0 with an accuracy of 10%, with what precision can you determine the value of $T_{\rm eff}$?

(iii) If we now include an error in the measurement of the parallax of $\sigma_\Pi = 0.01$ arcsec, what is the percentage accuracy in the mass of the system?

Q6: A star is composed of H (mass fraction $X = 0.7$), He (mass fraction $Y = 0.3$) and negligible amounts of heavier elements.

(i) Calculate the mean molecular weight immediately above and below the radius in the star where hydrogen becomes ionized. Assuming the transition between ionized and neutral hydrogen takes place over a very small radial distance, such that the pressure and temperature can be considered constant across the zone, what would this imply about the dynamical stability of the zone?

(ii) Assuming that the pressure P has contributions βP from gas pressure and $(1-\beta)P$ from radiation pressure, where $0 \le \beta \le 1$, show that:

$$\beta^4 \left(\frac{P^3}{\rho^4}\right) = \left(\frac{\mathcal{R}}{\mu}\right)^4 \frac{3}{a}(1-\beta)$$

(iii) A polytrope of index n has central pressure $P_c = W_n GM^2/R^4$ and central density $\rho_c = X_n M/(\frac{4}{3}\pi R^3)$, where W_n and X_n are dimensionless constants that depend only on n. Write down the equation for β_c, the value of β at the centre of the polytrope of index n, and show that β_c depends only on M and n.

(iv) The Sun may be approximated by a polytrope of index $n = 3.25$, for which $W_n = 20.4$ and $X_n = 88.1$. With $\mu = 0.59$, evaluate the constant A in the equation $\beta_c^4 = A(1-\beta)$ deduced from part (iii). What can you conclude about the importance of radiation pressure at the centre of the Sun?

Q7(i): Two early-type stars in the same cluster start their lives on the H-burning main sequence with the same mass: $M_0(\mathrm{A}) = M_0(\mathrm{B}) = 5\,M_\odot$. Star A is single. Star B is a member of a binary system and throughout its life on the main sequence loses mass to a compact companion at an average rate $\dot{M} = 1 \times 10^{-8} M_\odot\ \mathrm{yr}^{-1}$. Which of the two stars do you think will leave the main sequence first and why?

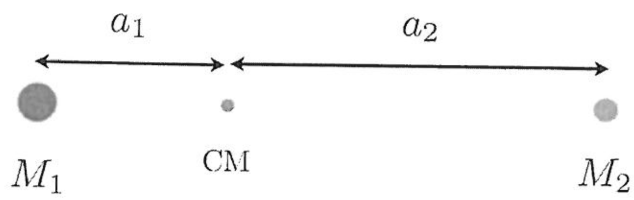
Q7(ii): An astronomer armed with a photometer and two broad-band filters, V and B, measures the following magnitudes for two stars in the constellation of Pegasus: $m_V(\alpha\,\mathrm{Peg}) = 2.45$, $m_B(\alpha\,\mathrm{Peg}) = 2.45$; and $m_V(\beta\,\mathrm{Peg}) = 2.40$, $m_B(\beta\,\mathrm{Peg}) = 4.04$. On the basis of this information alone, which of the two stars would you consider more likely to be the closer one to the Sun? What other information would you require to definitely establish which is closer?

Q8: Using the tabulation of solar photospheric abundances by Asplund et al. 2009 (ARAA, 47, 481) given at the end of Lecture 6, calculate the mass fractions of H, He, C, N, O, and Ne in the Sun.

EXERCISES, Set 2: Solutions

Q1(i): Derive the formula giving the mass function of the unseen companion in terms of v_2, P and $\sin i$, assuming that the orbit is circular.

A1(i): Referring the the sketch below:



we have, from Kepler's third law:

$$P^2 = \frac{4\pi^2}{G\,(M_1 + M_2)}\,(a_1 + a_2)^3 \tag{2.1}$$

with

$$M_1 a_1 = M_2 a_2$$

If the orbits are circular:

$$v_1 = \frac{2\pi}{P} a_1, \qquad v_2 = \frac{2\pi}{P} a_2 \tag{2.2}$$

and

$$\frac{v_1}{v_2} = \frac{a_1}{a_2} = \frac{M_2}{M_1}, \tag{2.3}$$

so that:

$$v_1 = v_2 \frac{M_2}{M_1}, \qquad a_1 = a_2 \frac{M_2}{M_1}. \tag{2.4}$$

Substituting into (2.1):

$$\begin{aligned}
P^2 &= \frac{4\pi^2}{G\,(M_1 + M_2)} \left(a_2 \frac{M_2}{M_1} + a_2 \right)^3 \\
&= \frac{4\pi^2 a_2^3}{G\,(M_1 + M_2)} \left(1 + \frac{M_2}{M_1} \right)^3 \\
&= \frac{4\pi^2 a_2^3}{G\,(M_1 + M_2)\,M_1^3} (M_1 + M_2)^3 \\
&= \frac{4\pi^2 a_2^3}{G M_1^3} (M_1 + M_2)^2 \,.
\end{aligned} \tag{2.5}$$

Thus:

$$\frac{GM_1^3}{(M_1+M_2)^2} = \frac{8\pi^3 a_2^3}{P^3}\frac{P}{2\pi} = v_2^3\frac{P}{2\pi} \tag{2.6}$$

Recognising that we do not observe v_2 directly, because of the unknown inclination of the orbital plane to our line of sight, we have $v_{\rm obs} = v_2 \sin i$, leading to the usual formula for the reduced mass:

$$\boxed{f(M_1, M2) = \frac{M_1^3 \sin^3 i}{(M_1+M_2)^2} = \frac{P v_{2,\rm obs}^3}{2\pi G}} \tag{2.7}$$

which groups the observables on the right-hand side of the equation.

Q1(ii): Justify the assumption that the orbit is circular using the data shown in the Figure, and deduce the value of v_2. Why is the mean wavelength of the Fe line not equal to its laboratory wavelength?

A1(ii): The orbit is close to circular because the curve describing the wavelength shift of the spectral line is symmetric.

The wavelength of the line varies from 6477.0 Å to 6487.0 Å, that is, it shifts by ±5.0 Å from a mean wavelength $\langle\lambda\rangle = 6482.0$ Å. The value of v_2 follows directly from the Doppler formula:

$$\frac{v}{c} = \frac{\Delta\lambda}{\lambda} \Rightarrow v_2 = \frac{5}{6482.0} 3\times10^8\,\mathrm{m\ s^{-1}} = 231\,\mathrm{km\ s^{-1}} \tag{2.8}$$

By the same reasoning, the fact that $\langle\lambda\rangle - \lambda_{\rm lab} = -3.1$ Å implies that the Nova Sco 1994 binary system is approaching us with a velocity $v_{\rm sys} = (-3.1/6485.1)\times c = 143\,\mathrm{km\ s^{-1}}$.

Q1(iii): In the case shown here, $i \simeq 70°$ and $P = 2.62$ days. From these parameters, determine the value of the mass function.

A1(iii): Plugging the appropriate values into eq. 2.7, we have:

$$\begin{aligned}
\frac{M_1^3}{(M_1+M_2)^2} &= \frac{P v_{2,\mathrm{obs}}^3}{2\pi G \sin^3 i} \\
&= \frac{(2.62 \times 24 \times 3600\,\mathrm{s})}{2\pi \times 6.67 \times 10^{-11}\,\mathrm{m^3\ s^{-2}\ kg^{-1}}} \frac{\left(2.31 \times 10^5\,\mathrm{m\ s^{-1}}\right)^3}{\sin^3 70^\circ} \qquad (2.9) \\
&= \frac{8.0 \times 10^{30}\,\mathrm{kg}}{2 \times 10^{30}\,\mathrm{kg}\ M_\odot^{-1}} \\
&= 4.0 M_\odot
\end{aligned}$$

Q1(iv): Knowing that the mass of an F-type star is $M_2 \simeq 2.4 M_\odot$, deduce M_1.

A1(iv): If $f(M_1, M_2) = 4.0M_\odot$ and $M_2 = 2.4M_\odot$, we can solve numerically for $M_1 = 7.1M_\odot$, making the unseen companion of Nova Sco 1994 one of the best candidates for a stellar mass black hole.

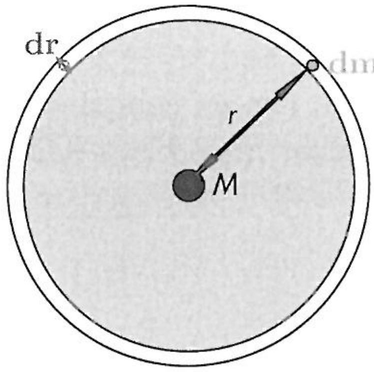
Q2(i): The gas density within a star decreases from the centre to the surface as a function of radial distance r according to

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R}\right)^2\right]$$

where ρ_c is the core density and R is the star's radius.

(i) Find $m(r)$.

A2(i): Consider a spherical shell of radius r and thickness dr, density ρ and mass dm:



The mass within the spherical shell is:

$$dm = 4\pi r^2 \rho \, dr \tag{2.10}$$

and the mass within a sphere of radius r is obtained by integrating eq. 2.10:

$$\begin{aligned} m(r) &= 4\pi \int_0^r r^2 \rho \, \mathrm{d}r \\ &= 4\pi \rho_c \int_0^r r^2 \left[1 - \left(\frac{r}{R}\right)^2\right] \mathrm{d}r \\ &= 4\pi \, \rho_c \left[\int_0^r r^2 \mathrm{d}r - \frac{1}{R^2} \int_0^r r^4 \mathrm{d}r\right] \\ &= 4\pi \, \rho_c \left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right] \end{aligned} \tag{2.11}$$

Q2(ii): Derive the relation between the total mass of the star M and R.

A2(ii): To obtain the total mass of the star, we set $r = R$:

$$M(r = R) = 4\pi\rho_c \left[\frac{R^3}{3} - \frac{R^3}{5}\right] \tag{2.12}$$

or

$$\boxed{M(R) = \frac{8\pi\rho_c R^3}{15}} \tag{2.13}$$

Q2(iii): What is the average density of the star in units of the core density ρ_c?

A2(iii): The average density is obtained by dividing the mass just found by the volume of the spherical star:

$$\langle\rho\rangle = \frac{8\pi\rho_c R^3}{15} \cdot \frac{3}{4\pi R^3} = \frac{2\rho_c}{5}\,. \tag{2.14}$$

Q2(iv): The gravitational potential energy of a star of mass M and radius R is given by:

$$U_g = -\alpha\frac{GM^2}{R}$$

where α is a constant of order unity determined by the distribution of matter within the star.

Find the value of α for the density profile given above.

A2(iv): The gravitational potential energy of the star is given by:

$$U_g = -\int_0^M \frac{Gm}{r} dm \tag{2.15}$$

With $m(r)$ given by eq. 2.11, and $dm = \rho(r)\, 4\pi r^2\, dr$ we have:

$$
\begin{aligned}
U_g &= -4\pi G \rho_c^2 \int_0^R \left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right) \left[1 - \left(\frac{r}{R}\right)^2\right] 4\pi r \,\mathrm{d}r \\
&= -16\pi^2 \rho_c^2 G \int_0^R \frac{r^4}{3} - \frac{r^6}{5R^2} - \frac{r^6}{3R^2} + \frac{r^8}{5R^4} \,\mathrm{d}r \\
&= -16\pi^2 \rho_c^2 G \left[\frac{R^5}{15} - \frac{R^5}{35} - \frac{R^5}{21} + \frac{R^5}{45}\right] \\
&= -16\pi^2 \rho_c^2 G \,\frac{4R^5}{315}
\end{aligned}
\tag{2.16}
$$

Recalling the expression for the total mass of a star with the given density profile (eq. 2.13), we have:

$$
M(R)^2 = \frac{64\pi^2 \rho_c^2 R^6}{225} \tag{2.17}
$$

and therefore

$$
\begin{aligned}
U_g &= -\frac{GM^2}{R} \cdot \frac{225}{315} \\
&= -\frac{5}{7} \frac{GM^2}{R}
\end{aligned}
\tag{2.18}
$$

or $\alpha = 0.71$.

Q3(i): Show the the full width at half maximum of a Lorentzian line profile is $\Delta\nu_{FWHM} = \Gamma/2\pi$.

A3(i): The maximum value of the Lorentz profile is achieved at $\nu = \nu_0$, where:

$$\phi(\nu = \nu_0) = \frac{4}{\Gamma}. \tag{2.19}$$

The half maximum is reached at some frequency such that:

$$\phi(\nu_{hm}) = \frac{1}{2}\frac{4}{\Gamma} = \frac{2}{\Gamma} = \frac{\frac{\Gamma}{4\pi^2}}{(\nu_{hm} - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} \tag{2.20}$$

After simplifying this equation, we have:

$$(\nu_{hm} - \nu_0)^2 = \frac{\Gamma^2}{8\pi^2} - \left(\frac{\Gamma}{4\pi}\right)^2 = \left(\frac{\Gamma}{4\pi}\right)^2 \tag{2.21}$$

or

$$\nu_{hm} - \nu_0 = \frac{\Gamma}{4\pi} \tag{2.22}$$

which is the half-width at half maximum. The FWHM is just twice this value:

$$\Delta\nu_{FWHM} = \frac{\Gamma}{2\pi}. \tag{2.23}$$

Note that in terms of the angular frequency $\omega \equiv 2\pi\nu$, $\Delta\nu_{FWHM} = \Gamma$.

Q3(ii): At what interval in units of $\Gamma/4\pi$ (in other words, in units of the half-width at half maximum) from the rest frequency does the Lorentz profile have a value of 1% of its central intensity?

A3(ii): Since at peak a Lorentz profile has a value $\phi(\nu = \nu_0) = 4/\Gamma$ [part (i)], we want to know the frequency at which:

$$\phi(\nu) = \frac{0.04}{\Gamma} \tag{2.24}$$

We thus need to solve the equation:

$$\frac{\frac{\Gamma}{4\pi^2}}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} = \frac{0.04}{\Gamma} \tag{2.25}$$

which may be simplified to:

$$\Delta\nu^2 \equiv (\nu - \nu_0)^2 = \left(\frac{1}{0.01} - 1\right)\left(\frac{\Gamma}{4\pi}\right)^2 = 99\left(\frac{\Gamma}{4\pi}\right)^2 \tag{2.26}$$

Therefore, in units of $\Gamma/4\pi$, $\Delta\nu = \sqrt{99} \simeq 10$. Thus, a Lorentz profile falls to 1% of its peak value at 10 times the half-width at half maximum from its central frequency.

Q3(iii): Thermal broadening is described by a Gaussian distribution. For a Gaussian distribution with the same FWHM as the Lorentz profile, what is the probability of absorption at the same $\Delta\nu$ from the line centre as in part (ii)?

A3(iii): A Gaussian distribution function is defined as:

$$\phi(\nu) = \exp\left[-\frac{(\nu - \nu_0)^2}{2\sigma^2}\right] \tag{2.27}$$

where σ is the one-dimensional projection of the velocity dispersion of the absorbing atoms along the line of sight.

At the line centre, $\nu = \nu_0$, $\phi(\nu) = 1$. At half maximum, $\phi = 0.5$ and thus:

$$0.5 = \exp\left[-\frac{(\nu_{hm} - \nu_0)^2}{2\sigma^2}\right] \tag{2.28}$$

and

$$-\ln 0.5 = \frac{(\nu_{hm} - \nu_0)^2}{2\sigma^2} \tag{2.29}$$

$$\Delta\nu \equiv \nu_{hm} - \nu_0 = \sqrt{2(-\ln 0.5)}\,\sigma = 1.18\,\sigma \tag{2.30}$$

When $\Delta\nu = 10 \times (\nu_{hm} - \nu_0) = 11.8\sigma$, we have:

$$\phi(\nu) = \exp\left[-\frac{(11.8\sigma)^2}{2\sigma^2}\right] = \exp\left[-\frac{11.8^2}{2}\right] = \exp\left[-\frac{139.2}{2}\right] = 5.8 \times 10^{-31} \tag{2.31}$$

Conclusion: away from the line centre, a Gaussian distribution falls off much more rapidly than the inverse-square behaviour of the Lorentz function. Thus, a frequencies well away from the resonant frequency of an atomic transition, absorption takes place via natural broadening, rather than Doppler broadening.

Q4(i): By considering the energy generation near $T_8 = 1$ to scale as $\mathcal{E}(T) = k_1 T_8^{\gamma}$, show that: $\mathcal{E}_{3\alpha} \approx k_2 T_8^{41}$, where k_1 and k_2 are constants.

A4(i): Differentiating the energy generation equation:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}_{3\alpha}}{\mathrm{d}T_8} &= -3k_0 T_8^{-4} e^{-44/T_8} + 44 k_0 T_8^{-5} e^{-44/T_8} \\ &= \left(\frac{44}{T_8} - 3\right) \frac{\mathcal{E}_{3\alpha}}{T_8}\end{aligned} \tag{2.32}$$

Near $T_8 = 1$,

$$\frac{\mathrm{d}\mathcal{E}_{3\alpha}}{\mathrm{d}T_8} = 41 \frac{\mathcal{E}_{3\alpha}}{T_8} \tag{2.33}$$

$$\frac{\mathrm{d}\mathcal{E}_{3\alpha}}{\mathcal{E}_{3\alpha}} = 41 \frac{\mathrm{d}T_8}{T_8} \tag{2.34}$$

Integrating:

$$\ln \mathcal{E}_{3\alpha} = 41 \ln T_8 + C \tag{2.35}$$

or

$$\mathcal{E}_{3\alpha} = k_2 T_8^{41} \tag{2.36}$$

Q4(ii): Calculate the change in energy output resulting from a 10% change in temperature.

A4(ii): We have:

$$\frac{\mathcal{E}'_{3\alpha}}{\mathcal{E}_{3\alpha}} = \left(\frac{T'_8}{T_8}\right)^{41} = 1.1^{41} = 50! \tag{2.37}$$

Clearly, the energy generation rate depends very sensitively on the temperature.

Q5: An eclipsing-binary system has a parallax of 0.1 arcsec (with negligible error) and consists of two solar-type stars with a semi-major axis of $500R_\odot$. The period is known very accurately.

Q5(i): What is the angular size of each of the stars and of the semi-major axis? If you can measure angles on the sky with a 1σ rms accuracy of 0.01 arcsec, what is the percentage accuracy of the measurement of the semi-major axis and of the radius of each star ?

A5(i): A parallax of 0.1 arcsec implies that the stars are at a distance of $d = 10$ pc$= 3.1 \times 10^{17}$ m. Thus we have:

$$\theta_{2R} = \frac{2R_\odot}{d} = \frac{2 \times 7.0 \times 10^8}{3.1 \times 10^{17}} = 4.5 \times 10^{-9} \times \frac{180}{\pi} \times 3600 = 9.3 \times 10^{-4} \,\text{arcsec} \tag{2.38}$$

for the stellar radii, and

$$\theta_a = 250 \times \theta_{2R} = 250 \times 9.3 \times 10^{-4} = 0.23 \,\text{arcsec} \tag{2.39}$$

for the semi-major axis.

The percentage accuracy is simply:

$$\frac{\sigma_{\theta_{2R}}}{2R} = \frac{0.01}{9.3 \times 10^{-4}} \times 100 = 1075\% \tag{2.40}$$

and

$$\frac{\sigma_{\theta_a}}{a} = \frac{0.01}{0.23} \times 100 = 4.3\% \tag{2.41}$$

Q5(ii): Assume that the stars emit as blackbodies:

$$F_\nu(T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1}, \tag{2.42}$$

where ν is the frequency is Hz, with an effective temperature $T_{\text{eff}} \simeq 5800$ K. If you measure the flux ratio between $\log \nu = 14.0$ and 15.0 with an accuracy of 10%, with what precision can you determine the value of T_{eff}?

A5(ii): We have:

$$x \equiv \frac{F_\nu(10^{14}\,\mathrm{Hz})}{F_\nu(10^{15}\,\mathrm{Hz})} = \frac{2h\nu_{14}^3/c^2\; e^{h\nu_{15}/kT}-1}{2h\nu_{15}^3/c^2\; e^{h\nu_{14}/kT}-1} = \left(\frac{\nu_{14}}{\nu_{15}}\right)^3 \frac{e^{h\nu_{15}/kT}-1}{e^{h\nu_{14}/kT}-1} \tag{2.43}$$

To find the uncertainty in T from x, we need dx/dT:

$$\frac{dx}{dT} = -x\left(\frac{h\nu_{15}}{kT^2} - \frac{h\nu_{14}}{kT^2}\frac{e^{h\nu_{14}/kT}}{e^{h\nu_{14}/kT}-1}\right) \approx -x\,\frac{h\nu_{15}}{kT^2} \tag{2.44}$$

since $\nu_{14} = 0.1\nu_{15}$. Therefore, since $\sigma_x = \frac{dx}{dt}\sigma_T$, we have:

$$\frac{\sigma_T}{T} \approx \frac{kT}{h\nu_{15}}\frac{\sigma_x}{x} \approx \frac{1.4\times10^{-23}\,\mathrm{JK^{-1}}\cdot 5800\mathrm{K}}{6.6\times10^{-34}\,\mathrm{J\,s}\cdot 10^{15}\,\mathrm{s^{-1}}}\times 10\% \approx \frac{1.4\cdot 5.8\times10^{-20}}{6.6\times10^{-19}}\times10\% \approx 1.2\times10^{-1}\times10\% \approx 1.2\% \tag{2.45}$$

Q5(iii): If we now include an error in the measurement of the parallax of $\sigma_\Pi = 0.01\,$arcsec, what is the accuracy with which we can determine the mass of the system?

A5(iii): From Kepler's third law, we have:

$$GM\left(\frac{P}{2\pi}\right)^2 = a^3 \tag{2.46}$$

where P is the period and $M = m_1 + m_2$ (eq. 4.3 of lecture notes). Therefore,

$$M = \frac{4\pi^2}{GP^2}\,a^3 \tag{2.47}$$

$$\frac{dM}{da} = \frac{12\pi^2}{GP^2}a^2 = 3\frac{M}{a} \tag{2.48}$$

$$\frac{\sigma_M}{M} = 3\,\frac{\sigma_a}{a}\,. \tag{2.49}$$

In part (i) of this question, we estimated $\sigma_a/a = 4.3\%$ ignoring the uncertainty in the measurement of the parallax. If we include the latter, the two sources of error should be combined in quadrature:

$$\begin{aligned} \frac{\sigma_a}{a} &= \sqrt{\left(\frac{\sigma_{\theta_a}}{a}\right)^2 + \left(\frac{\sigma_\Pi}{\Pi}\right)^2} \\ &= \sqrt{\left(\frac{0.01}{0.23}\right)^2 + \left(\frac{0.01}{0.1}\right)^2} \\ &= 11\% \end{aligned} \tag{2.50}$$

And therefore,

$$\frac{\sigma_M}{M} = 33\%. \tag{2.51}$$

Q6(i): Calculate the mean molecular weight immediately above and below the radius in the star where hydrogen becomes ionized. Assuming the transition between ionized and neutral hydrogen takes place over a very small radial distance, such that the pressure and temperature can be considered constant across the zone, what would this imply about the dynamical stability of the zone?

A6(i): The mean molecular weight μ_n of a fully neutral gas is given by:

$$\frac{1}{\mu_n} = \sum_j \frac{1}{A_j} F_j \tag{2.52}$$

where A is the atomic mass number of element with mass fraction F. Thus, for $X = 0.7$. $Y = 0.3$, $Z \simeq 0$:

$$\frac{1}{\mu_n} = \frac{X}{1} + \frac{Y}{4} + \left\langle \frac{1}{A} \right\rangle Z = 0.70 + \frac{0.3}{4} + \sim 0; \qquad \mu_n \simeq 1.29 \tag{2.53}$$

On the other hand, the mean molecular weight $\mu_{\rm ion}$ of a fully ionised gas is given by:

$$\frac{1}{\mu_{\rm ion}} = \sum_j \frac{1 + Z_j}{A_j} F_j \tag{2.54}$$

where Z_j is the atomic number of element j, i.e. the number of electrons liberated in complete ionisation of the atom. Thus:

$$\frac{1}{\mu_{\rm ion}} = 2X + \frac{3}{4} Y; \qquad \mu_{\rm ion} \simeq 0.62 \tag{2.55}$$

To answer the second part of this question, let us first consider where in a star like the Sun such a transition might take place. Qualitatively, we expect the interior of the Sun to be fully ionised (and incidentally for a fully ionised gas of solar composition $\mu_{\rm ion} \simeq 0.59$ taking into account a mass fraction of elements heavier than He of $Z = 0.12$). On the other hand, at the surface the gas is at least partly neutral (we see absorption lines of neutral H and He). Thus, in our hypothetical star, such a sudden transition zone would occur somewhere near the surface.

For an ideal gas, the pressure is given by:

$$P = \frac{1}{\mu m_{\rm H}} \rho k T \,. \tag{2.56}$$

Therefore, if the transition is really so sudden that P and T remain constant, as the gas goes from fully ionised to fully neutral its density must increase by a factor $1.29/0.62 \simeq 2$. Thus, we would have a layer of denser, neutral, gas overlaying a less dense layer of ionised gas: a highly unstable situation!

Q6(ii): Assuming that the pressure P has contributions βP from gas pressure and $(1-\beta)P$ from radiation pressure, where $0 \leq \beta \leq 1$, show that:

$$\beta^4 \left(\frac{P^3}{\rho^4}\right) = \left(\frac{\mathcal{R}}{\mu}\right)^4 \frac{3}{a}(1-\beta) \tag{2.57}$$

A6(ii): We have,

$$P = \frac{P_{\rm rad}}{1-\beta} = \frac{P_{\rm gas}}{\beta}, \tag{2.58}$$

and hence:

$$P = \frac{\frac{1}{3}aT^4}{1-\beta} = \frac{\mathcal{R}\rho T}{\mu\beta} \tag{2.59}$$

where $\mathcal{R} = k/m_{\rm H}$. Writing the density in terms of the temperature from (2.59) leads to:

$$\rho = \frac{\frac{1}{3}aT^3\mu\beta}{(1-\beta)\mathcal{R}}. \tag{2.60}$$

Substituting back:

$$P = \frac{1}{3}a\left(\frac{3(1-\beta)\mathcal{R}}{\mu\beta a}\right)^{4/3} \frac{\rho^{4/3}}{1-\beta} \tag{2.61}$$

$$P^3 = (1-\beta)\frac{3}{a}\left(\frac{\mathcal{R}}{\mu\beta}\right)^4 \rho^4 \tag{2.62}$$

which can be rearranged in the form of (2.57).

Q6(iii): A polytrope of index n has central pressure $P_c = W_n GM^2/R^4$ and central density $\rho_c = X_n M/(\frac{4}{3}\pi R^3)$, where W_n and X_n are dimensionless constants that depend only on n. Write down the equation for β_c, the value of β at the centre of the polytrope of index n, and show that β_c depends only on M and n.

A6(iii): Substituting the expressions given for P_c and ρ_c into (2.57), we have:

$$\beta_c^4 \left(\frac{W_n G M^2}{R^4}\right)^3 \left(\frac{\frac{4}{3}\pi R^3}{X_n M}\right)^4 = \left(\frac{\mathcal{R}}{\mu}\right)^4 \frac{3}{a}(1-\beta_c) \tag{2.63}$$

Simplifying:

$$\beta_c^4 \left(W_n G\right)^3 M^2 \left(\frac{4\pi}{3X_n}\right)^4 = \left(\frac{\mathcal{R}}{\mu}\right)^4 \frac{3}{a}(1-\beta_c) \tag{2.64}$$

Hence:

$$\beta_c^4 = \left(\frac{3\mathcal{R}X_n}{4\pi\mu}\right)^4 \frac{3(1-\beta_c)}{aM^2(W_n G)^3} \tag{2.65}$$

From which it can be seen that indeed β_c depends only on M and n, since all the other quantities in the above equation are constants.

Q6(iv): The Sun may be approximated by a polytrope of index $n = 3.25$, for which $W_n = 20.4$ and $X_n = 88.1$. With $\mu = 0.59$, evaluate the constant A in the equation $\beta_c^4 = A(1-\beta)$ deduced from part (iii). What can you conclude about the importance of radiation pressure at the centre of the Sun?

A6(iv): Writing eq. 2.65 as $\beta_c^4 = A(1-\beta)$, we have:

$$A = \left(\frac{3\mathcal{R}X_n}{4\pi\mu}\right)^4 \frac{3}{aM^2(W_n G)^3} \tag{2.66}$$

With $X_n = 88.1$, $\mathcal{R} = 8314\,\mathrm{J\,K^{-1}\,mol^{-1}}$, $\mu = 0.59$, $a = 7.6\times10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$, $M_\odot = 2.0\times10^{30}\,\mathrm{kg}$, $W_n = 20.4$, and $G = 6.7\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, $A = 2988$ (dimensionless).

The equation:

$$x^4 = 2988(1-x) \tag{2.67}$$

clearly has a solution very close to 1. The exact (positive) solution is $x \equiv \beta_c = 0.999666$.

We conclude that radiation pressure is unimportant in the core of the Sun, where gas pressure dominates because of the very high values of temperature and density (recall that $P_{\rm gas} \propto \rho T$).

Q7(i): Two early-type stars in the same cluster start their lives on the H-burning main sequence with the same mass: $M_0(\mathrm{A}) = M_0(\mathrm{B}) = 5\,M_\odot$. Star A is single. Star B is a member of a binary system and throughout its life on the main sequence loses mass to a compact companion at an average rate $\dot{M} = 1 \times 10^{-8} M_\odot\ \mathrm{yr}^{-1}$. Which of the two stars do you think will leave the main sequence first and why?

A7(i): Referring to Figure 4.10 of the lecture notes, one can see that the main sequence lifetime of a $5\,M_\odot$ star is $\tau \simeq 1.25 \times 10^8$ years. Thus, we expect this to be the lifetime of star A. On the other hand, star B will have lost some of its mass during this interval of time. By the time star A moves off the main sequence, the mass of star B will have been reduced to $\sim (5 - 1.25) = 3.75\,M_\odot$. Given the steep (inverse) dependence of stellar lifetimes on mass, $\tau \propto M^{-2.47}$, star B will have consumed its core hydrogen at a slower rate. Consequently, star B will leave the main sequence *after* star A.

Q7(ii): An astronomer armed with a photometer and two broad-band filters, V and B, measures the following magnitudes for two stars in the constellation of Pegasus: $m_V(\alpha\,\mathrm{Peg}) = 2.45$, $m_B(\alpha\,\mathrm{Peg}) = 2.45$; and $m_V(\beta\,\mathrm{Peg}) = 2.40$, $m_B(\beta\,\mathrm{Peg}) = 4.04$. On the basis of this information alone, which of the two stars would you consider more likely to be the closer one to the Sun? What other information would you require to definitely establish which is closer?

A7(ii): The two stars have essentially the same magnitude in the V band, but have very different $(B - V)$ colours: $(B - V) = 0.00$ for $\alpha\,$Peg and $(B-V) = +1.64$ for $\beta\,$Peg. This question tests the students' understanding of the introductory material given in the first few lectures on several levels:

(1) Hopefully, everyone will realise that $\beta\,$Peg is *fainter* than $\alpha\,$Peg in the B-band (i.e. that the magnitude scale runs backwards).

(2) The difference in $(B-V)$ colour implies that $\alpha\,$Peg is *bluer* than $\beta\,$Peg.

(3) The blue colour implies that $\alpha\,$Peg has a higher effective temperature than $\beta\,$Peg.

(4) Since $L = 4\pi R^2 \sigma T_{\rm eff}^4$, the bluer colour would suggest that α Peg is intrinsically more luminous than β Peg, *if both stars are on the main sequence.*

(5) On the other hand, stars of the same $T_{\rm eff}$ can differ by 10 magnitudes in their absolute luminosity L, if their radii differ by two orders of magnitude. Thus, with the information provided, we do not know if the bluer star, α Peg, is a white dwarf (i.e. blue and intrinsically very faint). Conversely, the redder star, β Peg, may be an intrinsically luminous red giant or supergiant.

(6) Reasoning that most stars are on the main sequence, and that other loci of the H-R diagram are less densely populated, we would conclude that β Peg is more likely to be intrinsically fainter because of its red colour. In terms of their absolute magnitudes, $M(\beta\,\mathrm{Peg}) > M(\alpha\,\mathrm{Peg})$, *at all wavelengths.*

Turning to the distance moduli, we have:

$$[m_V - M_V]_{\beta\,\mathrm{Peg}} < [m_V - M_V]_{\alpha\,\mathrm{Peg}}$$

since $m_V(\beta\,\mathrm{Peg}) \simeq m_V(\alpha\,\mathrm{Peg})$. In other words, if both stars are on the main sequence, β Peg is closer to the Sun than α Peg.

(7) In order to establish if the two stars are on the main sequence, photometry is usually insufficient. We need to obtain *spectra* of the two stars, from which we could measure the *widths* of their absorption lines. Line widths reflect the pressure of the stellar atmosphere, and can differentiate between the extended atmospheres of giants and supergiants (narrow lines) at one extreme, and the very high pressures in the dense atmospheres of white dwarfs (broad lines) at the other.

(8) All of the above reasoning has ignored the complicating effects of interstellar dust, which have only been alluded to in the course (dust is discussed in more detail in the *"The Physics of Astrophysics"* course). Dust can make stars appear both redder and fainter than they intrinsically are.

Q8: Using the tabulation of solar photospheric abundances by Asplund et al. 2009 (ARAA, 47, 481) given at the end of Lecture 6, calculate the mass fractions of H, He, C, N, O, and Ne in the Sun.

A8: This is a simple question, just aimed at familiarising the students with the different ways in which solar abundances are quoted in the astronomical literature.

(i) The Asplund et al. 2009 compilation gives abundances in both the photosphere (derived from spectral modelling) and in carbonaceous chondrites (derived by chemical analysis in the laboratory). It is useful to discuss the reasons for the two sets of measurements, and how they compare (and why). The students have been told to use the photospheric scale.

(ii) The values tabulated by Asplund et al. are $\log(X/H) + 12$, where X/H is the *number* of atoms of element X per H atom. Here we are asked to calculate the *mass* fractions. For example, the entry in the Table for carbon is 8.43. This corresponds to $C/H = 2.7 \times 10^{-4}$. The mass fraction will be this value $\times 12$, since a carbon atom has $12\times$ the mass of the hydrogen atom (approximately).

(iii) Thus, consider a mass element Δm containing 10 000 H atoms and let the mass unit be the mass of the hydrogen atom. Then, we have:

$$\begin{aligned}\Delta m &= 10000 \times 1 + 851 \times 4 + 2.7 \times 12 + 0.7 \times 14 + 4.9 \times 16 + 0.85 \times 20 \\ &\simeq 13\,541\end{aligned} \tag{2.68}$$

neglecting elements heavier than Ne.

Now, by definition:

$$\mathrm{X} = \frac{10\,000 \times 1}{\Delta m} = 0.738$$
$$\mathrm{Y} = \frac{851 \times 4}{\Delta m} = 0.251$$
$$\mathrm{Z_C} = \frac{2.7 \times 12}{\Delta m} = 0.0024$$
$$\mathrm{Z_N} = \frac{0.7 \times 14}{\Delta m} = 0.00072$$
$$\mathrm{Z_O} = \frac{4.9 \times 16}{\Delta m} = 0.0058$$
$$\mathrm{Z_{Ne}} = \frac{0.85 \times 20}{\Delta m} = 0.0013$$

Conclusion: the total mass in elements heavier than He is very small, only of the order of $\sim 1.2\%$.

EXERCISES: Set 3 of 4

Q1:

(i) Show that the equation of hydrostatic equilibrium may be written as:

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln\tau} = \frac{g\tau}{\kappa P} = \frac{\mathrm{d}\log P}{\mathrm{d}\log\tau}$$

where τ is the optical depth, κ is the opacity and g is the gravitational acceleration.

(ii) We know from geological and fossil records that the Sun's luminosity has remained constant over the last billion years. From this statement, deduce the accuracy of the approximation of hydrostatic equilibrium as applied to the Sun. (At the surface of the Sun, the gravitational acceleration is $g = 2.5 \times 10^2$ m s^{-2}).

Q2: In a certain temperature regime, the p-p chain can be approximated by the following four reactions:

$$\begin{aligned}
{}^1\mathrm{H} + {}^1\mathrm{H} &\rightarrow {}^2\mathrm{H} + \beta^+ + \nu, \\
{}^1\mathrm{H} + {}^2\mathrm{H} &\rightarrow {}^3\mathrm{He} + \gamma, \\
{}^3\mathrm{He} + {}^4\mathrm{He} &\rightarrow {}^7\mathrm{Li}, \\
{}^7\mathrm{Li} + {}^1\mathrm{H} &\rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}\ .
\end{aligned}$$

(i) Write down the rate equations for the abundances N_1, N_2, N_3, N_4, and N_7 of the five nuclear species involved in terms of the rates r_{11}, r_{12}, r_{34}, and r_{17} of the four reactions.

(ii) If ^{2}H, ^{3}He and ^{7}Li are all in transient equilibrium, show that the equations can be simplified to

$$\dot{N}_1 = -4 r_{11} N_1^2\,, \qquad \dot{N}_4 = r_{11} N_1^2\,,$$

and find equations for the equilibrium abundances of the other species.

(iii) At lower temperatures, the last two reactions in the chain are replaced by the single reaction:

$$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + 2\ {}^1\text{H} \quad .$$

Making equivalent assumptions about transient equilibrium, show that the corresponding simplified rate equations are:

$$\dot{N}_1 = -2r_{11}N_1^2\,, \qquad \dot{N}_4 = \frac{1}{2}r_{11}N_1^2\,.$$

(iv) What is the underlying reason why these differ from the previous equations by a factor of 2?

Q3: The gravitational binding energy of a star of mass M and radius R is given by:

$$E = -\frac{\alpha G M^2}{R}$$

where α is a constant. Such a star contracts homologously at constant effective temperature, radiating at a rate $L(t)$.

(i) Derive expressions for $L(t)$ and $R(t)$ for a star of fixed mass which at time $t = 0$ had $L = L_0$ and $R = R_0$.

(ii) Show that at late times L and R display power law dependences on time.

(iii) Where would such a star be found in the Hertzsprung-Russell diagram?

Q4: Consider a convective star.

(i) Give an approximate derivation for the boundary condition at the photosphere in the form:

$$\kappa P \sim \frac{GM}{R^2}$$

In fully convective low-mass main-sequence stars, the equation of state is that of an ideal gas and the opacity is of the form $\kappa = \kappa_0 \rho T^8$. Show that the adiabatic constant $K \equiv PT^{-5/2}$ is such that $K \propto M^{1/2}R^{-1}T_{\text{eff}}^{-6}$. The energy generation rate is of the form $\mathcal{E} = \mathcal{E}_0 \rho T^6$.

(ii) Show that for a group of such stars of the same chemical composition the members satisfy the following relations:

$$R \propto M^{11/17}; \qquad L \propto M^{37/17}$$

(iii) Sketch the locus of such stars on the H-R diagram, marking the locus of the main sequence and the position of the Sun.

Q5: The H II region around an O star has radius $R = 5\,\mathrm{pc}$, temperature $T = 10\,000\,\mathrm{K}$, and density $n = 100\,\mathrm{cm}^{-3}$.

(i) Estimate whether such an interstellar cloud is stable against gravitational collapse.

(ii) If such an H II region were visible at redshift $z = 3$ and you recorded its spectrum with a ground-based telescope, which spectral feature would you expect to be strongest? Give reasons for your answer.

Q6:

(i) Find the temperature at which the number density of hydrogen atoms in the ground state is equal to that of atoms in the second excited state ($n = 3$).

(ii) Calculate the electron density, n_e, of a pure hydrogen gas at $T = 14\,000\,\mathrm{K}$ where 70% of the atoms are ionised. You may assume that the partition function is $Z(T) = 2$ for neutral hydrogen at this temperature.

Q7: Estimate the time the Sun will spend on the horizontal branch, if 15% of its mass is converted from $^4\mathrm{He}$ to $^{12}\mathrm{C}$ via the triple-alpha reaction.

EXERCISES, Set 3: Solutions

Q1(i): Show that the equation of hydrostatic equilibrium may be written as:

$$\frac{d\ln P}{d\ln\tau} = \frac{g\tau}{\kappa P} = \frac{d\log P}{d\log\tau} \tag{3.1}$$

where τ is the optical depth, κ is the opacity and g is the gravitational acceleration.

A1(i): The equation of hydrostatic equilibrium is:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{3.2}$$

Since $Gm/r^2 \equiv g$ the equation can also be written as:

$$\frac{dP}{dr} = -\rho g \tag{3.3}$$

In the lectures, we have defined the optical depth τ (at a given wavelength or frequency) as:

$$d\tau = -\kappa\,\rho\,dr\,. \tag{3.4}$$

Substituting into (3.3)

$$\frac{dP}{d\tau} = \frac{g}{\kappa}\,. \tag{3.5}$$

Since $d\ln P = \frac{dP}{P}$ and $d\ln\tau = \frac{d\tau}{\tau}$, we have:

$$\frac{d\ln P}{d\ln\tau} = \frac{\tau}{P}\frac{dP}{d\tau} = \frac{g\tau}{\kappa P} \tag{3.6}$$

And, since $\log x = \frac{\ln x}{\ln(10)}$,

$$\frac{d\log P}{d\log\tau} = \frac{\ln(10)}{\ln(10)}\frac{d\ln P}{d\ln\tau} = \frac{d\ln P}{d\ln\tau} = \frac{g\tau}{\kappa P} \tag{3.7}$$

Q1(ii): We know from geological and fossil records that the Sun's luminosity has remained constant over the last billion years. From this statement, deduce the accuracy of the approximation of hydrostatic equilibrium as applied to the Sun. (At the surface of the Sun, the gravitational acceleration is $g = 2.5 \times 10^2\,\mathrm{m\ s^{-2}}$).

A1(ii): For a blackbody, $L = 4\pi\sigma R^2 T_{\mathrm{eff}}^4$. Since it is extremely unlikely that R and T_{eff} would have changed in such a way that the product $R^2 \cdot T_{\mathrm{eff}}^4$ has remained constant, the fact that $\mathrm{d}L/\mathrm{d}t = 0$ (over the last 10^9 years) implies that $\mathrm{d}R/\mathrm{d}t = 0$ (as well as $\mathrm{d}T_{\mathrm{eff}}/\mathrm{d}t = 0$).

In hydrostatic equilibrium,

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} + g\rho(r) = 0 \tag{3.8}$$

If hydrostatic equilibrium does not hold, then on the right-hand side of eq. 3.9 will have a non-zero term, corresponding to a residual acceleration a:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \rho(r)g = \rho(r)a\,. \tag{3.9}$$

Expressing a as a (small) fraction of g, we have:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \rho(r)g = \rho(r)\beta g\,. \tag{3.10}$$

If the Sun had been perturbed from hydrostatic equilibrium 10^9 years ago, a small mass element originally at rest at the surface would have moved a distance:

$$\delta R = \frac{1}{2}at^2 = \frac{1}{2}\beta g t^2\,. \tag{3.11}$$

The accuracy to which the approximation of hydrostatic equilibrium applies is equivalent to finding an upper limit to β:

$$\beta < 2\frac{\delta R}{g t^2}\,. \tag{3.12}$$

With $g = 250\,\mathrm{m\ s^{-2}}$ and $t = 3.156 \times 10^7\,\mathrm{yr}$, we have:

$$\beta < 2\delta R \frac{1}{250\times \sim 10 \times 10^{14}} \lesssim \frac{\delta R}{1.25 \times 10^{17}} \tag{3.13}$$

or

$$\frac{\delta R}{R_\odot} \gtrsim \frac{1.25 \times 10^{17}}{R_\odot}\beta \gtrsim \frac{1.25 \times 10^{17}}{7 \times 10^{8}}\beta \gtrsim 2 \times 10^{8}\beta \tag{3.14}$$

Since $\delta R/R_\odot$ is negligible, $\beta \ll 5 \times 10^9$ or, equivalently, the assumption of hydrostatic equilibrium must hold to an accuracy of better than 1/200 000 000.

Q2(i): In a certain temperature regime, the p-p chain can be approximated by the following four reactions:

$$
\begin{array}{ll}
{}^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{H} + \beta^+ + \nu & (1) \\
{}^1\mathrm{H} + {}^2\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma, & (2) \\
{}^3\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^7\mathrm{Li} & (3) \\
{}^7\mathrm{Li} + {}^1\mathrm{H} \rightarrow {}^4\mathrm{He} + {}^4\mathrm{He} & (4)
\end{array}
$$

Write down the rate equations for the abundances N_1, N_2, N_3, N_4, and N_7 of the five nuclear species involved in terms of the rates r_{11}, r_{12}, r_{34}, and r_{17} of the four reactions.

A2(i): The rates at which these nuclear reactions proceed is proportional to the product of the abundances of the two nuclei on the left-hand sides of the four equations. Thus:

$$\dot{N}_1 = -2N_1^2 r_{11} - N_1 N_2 r_{12} - N_1 N_7 r_{17} \tag{3.15}$$

(hydrogen is used up in all three reactions and is not produced in any of the four). Similarly:

$$\dot{N}_2 = N_1^2 r_{11} - N_1 N_2 r_{12} \tag{3.16}$$

$$\dot{N}_3 = N_1 N_2 r_{12} - N_3 N_4 r_{34} \tag{3.17}$$

$$\dot{N}_4 = -N_3 N_4 r_{34} + 2N_1 N_7 r_{17} \tag{3.18}$$

$$\dot{N}_7 = N_3 N_4 r_{34} - N_1 N_7 r_{17} \tag{3.19}$$

Q2(ii): If ^{2}H, ^{3}He and ^{7}Li are all in transient equilibrium, show that the equations can be simplified to:

$$\dot{N}_1 = -4r_{11}N_1^2\,, \qquad \dot{N}_4 = r_{11}N_1^2\,,$$

and find equations for the equilibrium abundances of the other species.

A2(ii): If D, ^{3}He, and ^{7}Li are in equilibrium, their concentrations do not change with time, and therefore $\dot{N}_2 = \dot{N}_3 = \dot{N}_7 = 0$. Then:

$$\begin{aligned} N_1^2 r_{11} &= N_1 N_2 r_{12} \\ N_1 N_2 r_{12} &= N_3 N_4 r_{34} \\ N_3 N_4 r_{34} &= N_1 N_7 r_{17} \end{aligned}$$

Substituting into eq. (3.15), we have:

$$\dot{N}_1 = -2N_1^2 r_{11} - N_1^2 r_{11} - N_1^2 r_{11} = -4N_1^2 r_{11}\,. \tag{3.20}$$

Similarly, substituting into eq. (3.18):

$$\dot{N}_4 = N_3 N_4 r_{34} = N_1 N_2 r_{12} = N_1^2 r_{11}\,. \tag{3.21}$$

The equilibrium abundances of H, ^{3}He and ^{7}Li can be written in terms of H and ^{4}He as:

$$N_2 = N_1 \frac{r_{11}}{r_{12}} \tag{3.22}$$

(from eq. 3.16 with $\dot{N}_2 = 0$);

$$N_3 = \frac{N_1^2}{N_4}\frac{r_{11}}{r_{34}}; \tag{3.23}$$

$$N_7 = N_1 \frac{r_{11}}{r_{17}}\,. \tag{3.24}$$

Q2(iii): At lower temperatures, the last two reactions in the chain are replaced by the single reaction:

$$^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2\,{}^1\mathrm{H} \quad .$$

Making equivalent assumptions about transient equilibrium, show that the corresponding simplified rate equations are:

$$\dot{N}_1 = -2r_{11}N_1^2\,, \qquad \dot{N}_4 = \frac{1}{2}r_{11}N_1^2\,.$$

A2(iii): The new rate equations are as follows:

$$\dot{N}_1 = -2N_1^2 r_{11} - N_1 N_2 r_{12} + 2N_3^2 r_{33} \tag{3.25}$$

$$\dot{N}_2 = N_1^2 r_{11} - N_1 N_2 r_{12} \tag{3.26}$$

$$\dot{N}_3 = N_1 N_2 r_{12} - 2N_3^2 r_{33} \tag{3.27}$$

$$\dot{N}_4 = N_3^2 r_{33} \tag{3.28}$$

With $\dot{N}_2 = \dot{N}_3 = 0$, we then have:

$$\begin{aligned}\dot{N}_1 &= -2N_1^2 r_{11} - N_1^2 r_{11} + 2N_3^2 r_{33} \\ &= -2N_1^2 r_{11} - N_1^2 r_{11} + N_1^2 r_{11} = -2N_1^2 r_{11}\end{aligned}$$

and

$$\dot{N}_4 = \frac{1}{2}\,N_1 N_2 r_{12} = \frac{1}{2}N_1^2 r_{11}\,.$$

Q2(iv): What is the underlying reason why these differ from the previous equations by a factor of 2?

A2(iv):
In case 1, the net result of the reactions is 2 ^{4}He nucleus.
In case 2, the net result is 1 ^{4}He and 2 H nuclei.

Thus, the rate of decrease of H is halved, due to the production of 2 ^{1}H nuclei, and the rate of increase of ^{4}He is halved due to the production of only one ^{4}He nucleus.

Q3(i): The gravitational binding energy of a star of mass M and radius R is given by:

$$E = -\frac{\alpha G M^2}{R}$$

where α is a constant. Such a star contracts homologously at constant effective temperature, radiating at a rate $L(t)$.

Derive expressions for $L(t)$ and $R(t)$ for a star of fixed mass which at time $t = 0$ had $L = L_0$ and $R = R_0$.

A3(i): As we discussed in the lectures, the virial theorem tells us that:

$$\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle \tag{3.29}$$

only half the change in gravitational potential energy is available to be radiated away as the protostar contracts; the remaining potential energy supplies the thermal energy that heats the gas. Thus:

$$L = \frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha\frac{GM^2}{R}\right) = \frac{1}{2}\frac{\alpha G M^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t}. \tag{3.30}$$

We also know that:

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4 \tag{3.31}$$

and therefore:

$$4\pi R^2 \sigma T_{\mathrm{eff}}^4 = \frac{1}{2}\frac{\alpha G M^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t}. \tag{3.32}$$

With T_{eff} and M both constant, we have:

$$\frac{8\pi\sigma T_{\mathrm{eff}}^4}{\alpha G M^2}\int_0^t \mathrm{d}t = \int_{R_0}^{R} \frac{1}{R^4}\,\mathrm{d}R. \tag{3.33}$$

Integrating:

$$\frac{8\pi\sigma T_{\mathrm{eff}}^4}{\alpha G M^2}\,t = -\frac{1}{3R^3} + \frac{1}{3R_0^3} \tag{3.34}$$

$$\frac{1}{R^3} = \frac{1}{R_0^3} - \frac{24\pi\sigma T_{\mathrm{eff}}^4 t}{\alpha G M^2} \tag{3.35}$$

or

$$\boxed{R = \left[\frac{1}{R_0^3} - \frac{24\pi\sigma T_{\mathrm{eff}}^4 t}{\alpha G M^2}\right]^{-1/3}} \tag{3.36}$$

We can now solve for the luminosity using (3.31):

$$L = 4\pi\sigma T_{\rm eff}^4 \left[\frac{1}{R_0^3} - \frac{24\pi\sigma T_{\rm eff}^4 t}{\alpha G M^2}\right]^{-2/3} \tag{3.37}$$

We also know that:

$$L_0 = 4\pi\sigma R_0^2 T_{\rm eff}^4 \tag{3.38}$$

so that

$$\frac{1}{R_0^2} = \frac{4\pi\sigma T_{\rm eff}^4}{L_0} \tag{3.39}$$

and

$$\frac{1}{R_0^3} = \left(\frac{4\pi\sigma T_{\rm eff}^4}{L_0}\right)^{3/2} \tag{3.40}$$

which takes us to our final answer:

$$\boxed{L = 4\pi\sigma T_{\rm eff}^4 \left[\left(\frac{L_0}{4\pi\sigma T_{\rm eff}^4}\right)^{-3/2} - \frac{24\pi\sigma T_{\rm eff}^4}{\alpha G M^2}\,t\right]^{-2/3}} \tag{3.41}$$

Q3(ii): Show that at late times L and R display power law dependences on time.

A3(ii): At late times, i.e. at large values of t, the second term in square brackets of eqs. 3.36 and 3.41 dominates over the first term, and therefore:

$$R = \left[-\frac{24\pi\sigma T_{\rm eff}^4}{\alpha G M^2}\,t\right]^{-1/3} \qquad \text{i.e. } R \propto t^{-1/3} \tag{3.42}$$

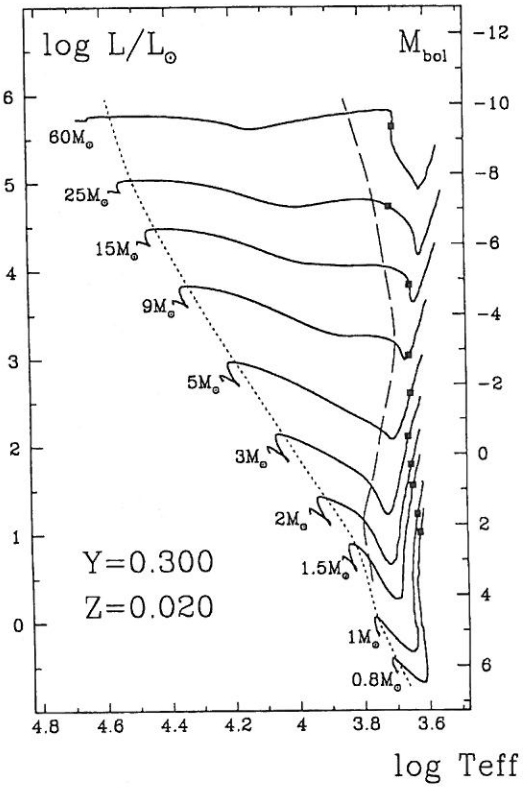
and

$$L \approx 4\pi\sigma T_{\rm eff}^4 \left[-\frac{24\pi\sigma T_{\rm eff}^4}{\alpha G M^2}\,t\right]^{-2/3} \qquad \text{i.e. } L \propto t^{-2/3} \tag{3.43}$$

Q3(iii): Where would such a star be found in the Hertzsprung-Russell diagram?

A3(iii): With L decreasing at constant $T_{\rm eff}$, the contracting star moves along a vertical track in the H-R diagram (see Figure below).

This is the Hayashi track followed by contracting pre-main sequence stars with a gravitational, rather than nuclear, internal energy source.



Q4(i): Give an approximate derivation for the boundary condition at the photosphere in the form:

$$\kappa P \sim \frac{GM}{R^2} \tag{3.44}$$

and then show that:

$$K \propto M^{1/2} R^{-1} T_{\rm eff}^{-6} \tag{3.45}$$

A4(i): We start from the equation of hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho}{r^2} \tag{3.46}$$

At the photosphere:

$$\frac{\mathrm{d}P}{\mathrm{d}r} \approx -\frac{GM\rho}{R^2} \tag{3.47}$$

where M and R are the stellar mass and radius respectively. Integrating from R to ∞:

$$P \approx \frac{GM}{R^2} \int_R^\infty \rho \, \mathrm{d}r \, . \tag{3.48}$$

We can define the photosphere of a star as the region extending down to optical depth $\tau = 1$. In lecture 5 (eq. 5.10) we also saw that the optical depth τ is related to the opacity κ via:

$$\tau = \int \kappa \rho \, \mathrm{d}r \tag{3.49}$$

Therefore:

$$\tau = \int_R^\infty \kappa \rho \, \mathrm{d}r = 1 \tag{3.50}$$

Taking the opacity κ to be a constant, we have:

$$\kappa \int_R^\infty \rho \, \mathrm{d}r = 1 \qquad \Longrightarrow \int_R^\infty \rho \, \mathrm{d}r = \frac{1}{\kappa} \tag{3.51}$$

Combining (3.51) and (3.48) gives the desired result:

$$\boxed{\kappa P \sim \frac{GM}{R^2}}$$

For the second part of this question, we are given the following:

$$P = \frac{R}{\mu m_{\rm H}} \rho T \tag{3.52}$$

$$\kappa = \kappa_0 \rho T^8 \tag{3.53}$$

$$K = P T^{-5/2} \tag{3.54}$$

Using the result of the first part of this question, we have:

$$\kappa_0 \, \rho \, T_{\rm eff}^8 \, P \approx \frac{GM}{R^2} \tag{3.55}$$

Expressing ρ in terms of P and T (eq. 3.52):

$$P^2 \, T_{\rm eff}^7 \propto \frac{M}{R^2} \qquad \Longrightarrow P \propto M^{1/2} R^{-1} \, T_{\rm eff}^{-7/2} \tag{3.56}$$

since all the other quantities have constant values. Substituting into (3.54), we arrive at the stated relationship:

$$\boxed{K \propto M^{1/2} R^{-1} T_{\rm eff}^{-6}}$$

which holds everywhere, since K is a constant.

Q4(ii): Show that for a group of such stars of the same chemical composition the members satisfy the following relations:

$$R \propto M^{11/17}; \qquad L \propto M^{37/17}$$

A4(ii): Since we are asked only to demonstrate the proportionality, we can use dimensional arguments.

From the equation of hydrostatic equilibrium, we have:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{GM}{r^2} \qquad \Longrightarrow \frac{P}{R} \propto \frac{\rho M}{R^2} \tag{3.57}$$

From the equation of mass continuity:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \qquad \Longrightarrow \frac{M}{R} \propto \rho \, R^2 \tag{3.58}$$

Hence,

$$\rho \propto \frac{M}{R^3} \text{ and } P \propto \frac{M^2}{R^4} \tag{3.59}$$

The ideal gas law:

$$P \propto \rho T \qquad T \propto \frac{P}{\rho} \tag{3.60}$$

We are also told that the energy generation rate:

$$\frac{\mathrm{d}L}{\mathrm{d}M} \equiv \mathcal{E} = \mathcal{E}_0 \rho T^6 \qquad \Longrightarrow \frac{L}{M} \propto \rho T^6 \propto \rho^{-5} P^6 \tag{3.61}$$

Therefore:

$$\frac{L}{M} \propto \left(\frac{R^3}{M}\right)^5 \left(\frac{M^2}{R^4}\right)^6 \tag{3.62}$$

giving:

$$L \propto R^{-9} M^8 \,. \tag{3.63}$$

We also know that:

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \tag{3.64}$$

Combining eqs. (3.54) and (3.45) from part (i) of this question, we have:

$$P T^{-5/2} \propto M^{1/2} R^{-1} T_{\text{eff}}^{-6} \tag{3.65}$$

and

$$P \left(\frac{P}{\rho}\right)^{-5/2} \propto M^{1/2} R^{-1} T_{\text{eff}}^{-6} \tag{3.66}$$

Using (3.59):

$$\frac{M^2}{R^4} \left(\frac{M^2}{R^4}\frac{R^3}{M}\right)^{-5/2} \propto M^{1/2} R^{-1} T_{\text{eff}}^{-6} \tag{3.67}$$

giving:

$$M^{-1/2} R^{-3/2} \propto M^{1/2} R^{-1} T_{\text{eff}}^{-6} \tag{3.68}$$

and therefore

$$T_{\text{eff}} \propto M^{1/6} R^{1/12} \,. \tag{3.69}$$

Substituting into (3.64):

$$L \propto R^2 M^{4/6} R^{4/12} \propto R^{7/3} M^{2/3} \,. \tag{3.70}$$

We now have two equations for L, eq. 3.70 and eq. 3.63. Equating them, leads to:

$$R^{-9}M^{8} \propto R^{7/3}M^{2/3} \Rightarrow R \propto M^{11/17} \tag{3.71}$$

Now plug this proportionality into (3.63) to obtain:

$$L \propto R^{-9}M^{8} \Rightarrow \propto \left(M^{11/17}\right)^{9} M^{8} \tag{3.72}$$

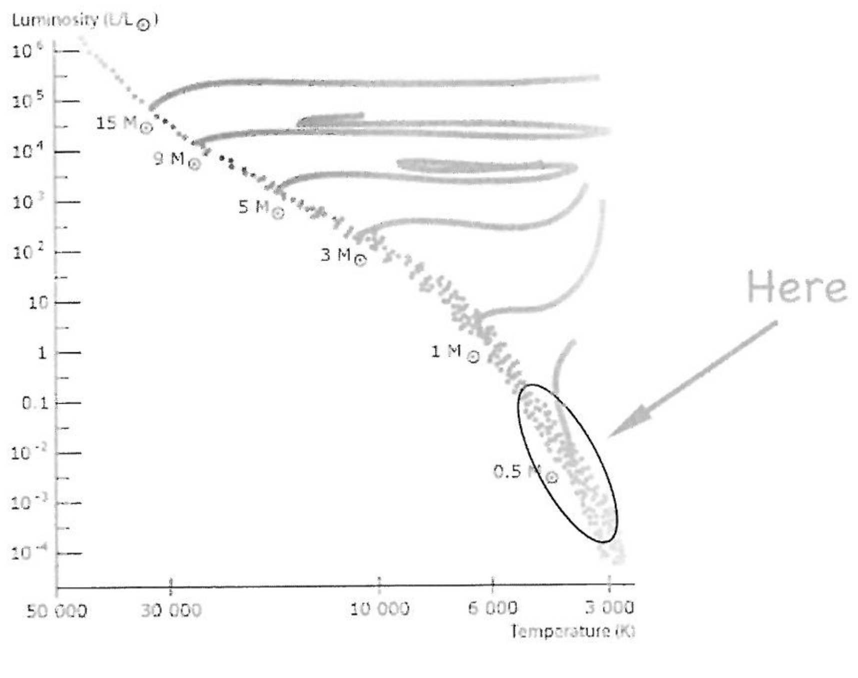
giving:

$$\boxed{L \propto M^{37/17}}$$

Q4(iii): Sketch the locus of such stars on the H-R diagram, marking the locus of the main sequence and the position of the Sun.

A4(iii):

Such stars are found on the main sequence of stars less massive than the Sun (which is only partly convective).



Q5(i): The H II region around an O star has radius $R = 5\,\mathrm{pc}$, temperature $T = 10\,000\,\mathrm{K}$, and density $n = 100\,\mathrm{cm}^{-3}$.

Estimate whether such an interstellar cloud is stable against gravitational collapse.

A5(i): The criterion for gravitational collapse is given by the Jeans mass:

$$M_\mathrm{J} = \left(\frac{5kT}{G\mu m_\mathrm{H}}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} , \qquad (3.73)$$

whereby if the mass of the H II region is greater than M_J the cloud will be unstable to gravitational collapse and vice versa.

An H II is defined as the volume around the star within which H is fully ionised. In general, He and other elements will be only partially ionised but, as a rough approximation, we can neglect all elements heavier than H. In this case, the mean molecular weight is $\mu = 1/2$.

The density of the cloud is:

$$\begin{aligned} \rho &= n_{\mathrm{H\,I}} m_\mathrm{H} + n_{\mathrm{H\,II}} m_\mathrm{p} + n_\mathrm{e} m_\mathrm{e} + \ldots \\ &\simeq n_{\mathrm{H\,II}} m_\mathrm{p} \\ &\simeq 100 \times 10^6 \cdot 1.7 \times 10^{-27} \simeq 1.7 \times 10^{-19}\,\mathrm{kg\ m^{-3}} \end{aligned}$$

Using S.I. units throughout, we have:

$$M_\mathrm{J} \simeq \left(\frac{5 \cdot 1.4 \times 10^{-23} \cdot 10^4}{6.7 \times 10^{-11} \cdot 1/2 \cdot 1.7 \times 10^{-27}}\right)^{3/2} \left(\frac{3}{4 \cdot 3.14 \cdot 1.7 \times 10^{-31}}\right)^{1/2} \qquad (3.74)$$

$$M_\mathrm{J} \simeq \left(\frac{1.4 \times 10^{-18}}{1.1 \times 10^{-37}}\right)^{3/2} \left(\frac{3}{2.1 \times 10^{-18}}\right)^{1/2} \qquad (3.75)$$

$$\begin{aligned} M_\mathrm{J} &\simeq \left(1.3 \times 10^{19}\right)^{3/2} \left(1.4 \times 10^{18}\right)^{1/2} \\ &\simeq \frac{5.5 \times 10^{37}\,\mathrm{kg}}{2 \times 10^{30}\,\mathrm{kg}\ M_\odot^{-1}} \simeq 2.8 \times 10^7\,M_\odot \end{aligned} \qquad (3.76)$$

The mass of the H II region, on the other hand, is:

$$\begin{aligned} M_{\rm HII} &= \frac{4\pi}{3} R^3 \rho \\ &\simeq \frac{4\pi}{3} \cdot \left(5 \times 3.1 \times 10^{16}\right)^3 \cdot 1.7 \times 10^{-19} \quad \mathrm{m^3\,kg\,m^{-3}} \\ &\simeq \frac{2.7 \times 10^{33}\,\mathrm{kg}}{2 \times 10^{30}\,\mathrm{kg}\ M_\odot^{-1}} \simeq 1.3 \times 10^3\, M_\odot \end{aligned} \tag{3.77}$$

Since $M_{\rm HII} \ll M_{\rm J}$, the cloud is stable against gravitational collapse.

Q5(ii): If such an H II region were visible at redshift $z = 3$ and you recorded its spectrum with a ground-based telescope, which spectral feature would you expect to be strongest? Give reasons for your answer.

A5(ii): With a ground-based telescope, we can expect to record the portion of the spectrum most easily transmitted by the Earth's atmosphere, typically from 3 500 Å to 9 000 Å.

At redshift $z = 3$, this wavelength interval corresponds to rest-frame wavelengths between 875 Å and 2 250 Å, in the far-ultraviolet.

The spectrum of an H II region is dominated by emission lines, produced as H^+ ions and free electrons recombine; the electron is generally captured in a high atomic energy level, and then cascades down to the ground state through intermediate levels. Each of these transitions will produce an emission line at the appropriate wavelength. Since most downward transitions end up with a transition from the $n = 2$ to the $n = 1$ level, the strongest emission line one would expect is the Lyman α line at a rest wavelength $\lambda_0 = 1215.67$ Å.

At redshift $z = 3$, this emission line would be recorded at an observed wavelength $\lambda_{\rm obs} = (1 + z)\lambda_0 = 4863$ Å.

Q6(i): Find the temperature at which the number density of hydrogen atoms in the ground state is equal to that of atoms in the second excited state ($n = 3$).

A6(i): The number of atoms in an atomic level n with energy E_n is given by the Boltzmann's equation:

$$N_n = A\, e^{-E_n/kT}\, g_n\,, \tag{3.78}$$

where k is Boltzmann's constant ($k = 8.62 \times 10^{-5}$ eV deg^{-1}), A is a constant of proportionality and g_n is the statistical weight of atomic level n denoting the number of particles which can be in atomic state n.

The relative populations of the two levels in question are therefore:

$$\frac{N_1}{N_3} = \frac{g_1}{g_3}\, e^{-(E_1-E_3)/kT} = 1\,. \tag{3.79}$$

The values of the atomic parameters required to solve this equation were not given explicitly in the lectures. It is left to the initiative of the student to search for them in the literature. A good source of atomic data is:

http://www.nist.gov/pml/data/asd.cfm

Choosing the options LEVELS, and then H I and 'Levels Units' as eV, shows the different energy levels of the hydrogen atoms. From this table, it can be seen that $E_1 = 0$ and $E_3 = 12.09$ eV.

The statistical weights g_n of the levels are not given explicitly in this table. However, in the lectures the students have been told that $g_n = 2J_n + 1$ (lecture 3.2). Thus, for $n = 1$, $J = 1/2$ and $g_1 = 2$.

The $n = 3$ level is divided into three sub-levels according to the orbital angular momentum of the electron: $3p$, $3s$ and $3d$. In turn, the $3p$ level is split into two sublevels depending on whether the electron orbital and spin angular momentum vectors are parallel or anti-parallel, with corresponding $J = 3/2$ and $J = 1/2$. For $3s$, $J = 1/2$. For the $3d$, $J = 5/2$ and $J = 3/2$. Adding together the values of $2J+1$ for all these sublevels, we find $g_3 = 18$.

With these values, eq. 3.79 becomes:

$$\frac{N_1}{N_3} = \frac{2}{18}\, e^{-(0-12.09)/kT} = 1\,. \tag{3.80}$$

or

$$
\begin{aligned}
T &= \frac{12.09}{\ln(9)k} \quad \frac{\text{eV}}{\text{eV deg}^{-1}} \\
&= \frac{12.09}{2.20 \cdot 8.62 \times 10^{-5}} \quad \text{deg} \\
&\simeq 63850\,\text{K}
\end{aligned}
\tag{3.81}
$$

Q6(ii): Calculate the electron density, n_e, of a pure hydrogen gas at $T = 14\,000\,\text{K}$ where 70% of the atoms are ionised. You may assume that the partition function is $Z(T) = 2$ for neutral hydrogen at this temperature.

A6(ii): The relative proportions of ions in two successive stages of ionisation is given by the Saha equation:

$$
\frac{n_e \cdot N_{i+1}}{N_i} = 2\,\frac{Z_{i+1}}{Z_i} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} \tag{3.82}
$$

where Z_i is the partition function of the ith ionisation stage, χ_i is the ionisation potential of the ith ionisation stage (the energy required to ionise ion i to ion $i+1$), and the other symbols have their usual meaning.

Since $N_2 = 0.7(N_2 + N_1)$, $N_1 = 0.43N_2$. Also, since the gas is made up of only H, $n_e = N_2$.

While $Z(T)$ is given for H I, the student is left to work out for him/herself the value of $Z(T)$ for H II. The student should come to realise that, having lost its electron and therefore not having atomic energy levels, ionised hydrogen is a special case. Its partition function is $Z_{\text{H II}} = 1$, i.e. it may be assumed that a hydrogen ion has a single state of energy equal to 0 eV.

So, now we have:

$$\frac{n_e \cdot N_{i+1}}{N_i} = \frac{n_e}{0.43}$$
$$= \frac{2}{2}\left(\frac{2 \cdot 3.14 \cdot 5.11 \times 10^5/c^2 \cdot 8.62 \times 10^{-5} \cdot 14\,000}{(4.14 \times 10^{-15})^2}\right)^{3/2} \exp\left[-\frac{13.6}{8.62 \times 10^{-5} \cdot 14\,000}\right]$$
$$= 4.0 \times 10^{21} \cdot 1.3 \times 10^{-5} \quad \left(\frac{\text{eV cm}^{-2}\text{ s}^2\text{ eV deg}^{-1}\text{ deg}}{\text{eV}^2\text{ s}^2}\right)^{3/2}$$
$$= 5.1 \times 10^{16}\,\text{cm}^{-3} \tag{3.83}$$

or

$$n_e = 2.2 \times 10^{16}\,\text{cm}^{-3} \tag{3.84}$$

Q7: Estimate the time the Sun will spend on the horizontal branch, if 15% of its mass is converted from ^{4}He to ^{12}C via the triple-alpha reaction.

A7: The triple alpha reaction:

$$3\,^4\mathrm{He} \Longrightarrow \; ^{12}\mathrm{C} + 3\,\gamma$$

releases the energy corresponding to the mass difference:
$(3 \times 4.002602 - 12)u = 0.00781u$, i.e.

$$E = 0.00781 \times 931.5 = 7.275\,\mathrm{MeV}\,.$$

Since three ^{4}He nuclei are involved, the *fraction* of mass transformed into energy is:

$$f = \frac{7.275\,\mathrm{MeV}}{3\,m_{^4\mathrm{He}}\,c^2} = \frac{7.275\,\mathrm{MeV}}{3 \times 4.002602 \times 931.5\,\mathrm{MeV/c^2} \cdot c^2} = 6.50 \times 10^{-4}\,.$$

The total energy emitted during the horizontal branch phase is therefore:

$$\begin{aligned} E_{\mathrm{HB}} &= 0.15 f M_\odot c^2 \\ &= 0.15 \cdot 6.50 \times 10^{-4} \cdot 2.0 \times 10^{33}\,\mathrm{g} \cdot \left(3.0 \times 10^{10}\,\mathrm{cm\ s^{-1}}\right)^2 \\ &= 1.76 \times 10^{50}\,\mathrm{erg} \end{aligned}$$

On the horizontal branch, the luminosity of the Sun is $L_{\mathrm{HB}} \simeq 50 L_\odot$ (see, for example, Figure 13.1 of the lecture notes). Thus,

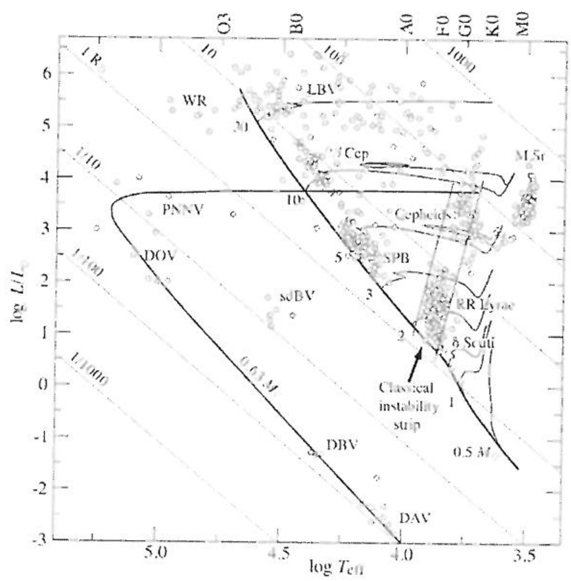
$$t_{\mathrm{HB}} = \frac{E_{\mathrm{HB}}}{L_{\mathrm{HB}}} = \frac{1.76 \times 10^{50}\,\mathrm{erg}}{50 \cdot 3.85 \times 10^{33} \cdot \mathrm{erg\ s^{-1}}} = 9.1 \times 10^{14}\,\mathrm{s} \simeq 3 \times 10^7\,\mathrm{yr}\,.$$

The lifetime on the horizontal branch is therefore much shorter than the lifetime on the main sequence. This is due to two factors: (i) the triple-alpha reaction releases less energy per unit mass than hydrogen burning, and (ii) the *rate* of energy production by He-burning on the horizontal branch is much larger than the main-sequence H-burning energy production rate (i.e. the luminosity of a horizontal branch star is much larger than its luminosity on the main sequence).

Note that the value of t_{HB} deduced here is $\sim 4\times$ less than the value $t_{\mathrm{HB}} \simeq$ 120 Myr given in the lecture notes (section 13.4). One of the reasons is that the hydrogen burning shell also contributes to the stellar luminosity, making the stellar lifetime longer.

EXERCISES: Set 4 of 4

Q1: The figure below shows, among other things, the path followed in the H-R diagram by a cooling white dwarf of mass $M = 0.63M_{\odot}$.



(i) Deduce the slope of the straight-line track followed by the cooling WD. How does it compare with the slope in the Figure?

(ii) What is the radius (in solar units) of the white dwarf when $L/L_{\odot} = 0.01$?

Q2: The following three approximate relations apply to massive stars on the main sequence:

(1) The mass–luminosity relation:

$$\log\left(\frac{L}{L_{\odot}}\right) \approx 0.78 + 2.76\,\log\left(\frac{M_{\rm i}}{M_{\odot}}\right)$$

where $M_{\rm i}$ is the initial mass;

(2) The mass-loss rate–luminosity relation:

$$\log\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right) \approx -12.76 + 1.30 \log\left(\frac{L}{L_\odot}\right)$$

where $\mathrm{d}M/\mathrm{d}t$ is in $M_\odot/\mathrm{yr}$; and

(3) The main-sequence lifetime–mass relation:

$$\log \tau_{\mathrm{MS}} \approx 7.72 - 0.66 \log\left(\frac{M_{\mathrm{i}}}{M_\odot}\right)$$

(i) Use these relations to calculate the fraction of the initial mass that is lost by massive stars with $M_{\mathrm{i}} = 25$, 40, 60, 85, and $120\,M_\odot$ before they evolve off the main sequence.

(ii) A star with $M_{\mathrm{i}} = 85 M_\odot$ has a convective core that contains 83% of the stellar mass. Calculate the time after the star appears on the main sequence at which the products of nuclear burning will appear at the surface. How would such a star be classified at this time?

Q3: (i) Show that for a spherically symmetric star the equations of mass continuity and hydrostatic equilibrium can be combined into the second-order differential equation:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right] = -4\pi G\rho$$

(ii) For a gas with an equation of state, $P = K\rho^\gamma$, where K is a constant, use the above equation to derive a second-order differential equation involving only density and radius. Using the dimensionless variables $r' \equiv r/R_*$ and $\rho' \equiv \rho/\rho_0$, show that the term $K\rho_0^{\gamma-2}/R_*^2$ is a dimensionless constant, and hence that $R_* \propto \rho_0^{\gamma/2-1}$.

(iii) White dwarfs obey the equation of state $P = K\rho^\gamma$, with $\gamma = 5/3$ for non-relativistic conditions and $\gamma = 4/3$ in the relativistic regime. Using the above result, $R \propto \rho_0^{\gamma/2-1}$, show that for non-relativistic white dwarfs $R \propto M^{-1/3}$, while for relativistic white dwarfs $R \neq f(M)$.

Q4: In a $10M_\odot$ star the $1M_\odot$ core collapses to produce a Type II supernova. Assume that 100% of the energy released by the collapsing core is converted to neutrinos and that 1% of the neutrinos are absorbed by the overlying envelope to power the ejection of the supernova remnant.

(i) Estimate the final radius of the stellar remnant if the energy liberated is just enough to eject the remaining $9M_\odot$ to infinity. State clearly any assumptions made.

(ii) What is the typical velocity of the ejecta, if the energy absorbed by the envelope is 10^{51} erg?

(iii) An astronomer announces the discovery of a Type II supernova in a Globular cluster, but her colleagues are skeptical. Why?

Q5: Type Ia supernovae are thought to be the explosion and complete disruption of a white dwarf in a binary system. Carbon and oxygen, the dominant constituents of white dwarfs, are burned to heavy elements, primarily Ni and Fe, during the explosion.

(i) Calculate the nuclear energy released by a Type Ia SN assuming that: (a) the exploding white dwarf has a mass of $1.4\,M_\odot$ consisting of ^{12}C and ^{16}O in equal proportions by number, and (b) all of the C and O are burned to ^{56}Ni in the explosion. The mass of a given nucleus of atomic number Z and mass number A is given by the formula:

$$m(A, Z) = Am_{\rm u} + m_{\rm ex}(A, Z)$$

where one atomic mass unit $m_{\rm u} = 931.5\,{\rm MeV/c^2}$, and the mass excess $m_{\rm ex} = 0$, -4.7, and $-53.9\,{\rm MeV/c^2}$ for ^{12}C, ^{16}O, and ^{56}Ni respectively. [Note: $1\,{\rm MeV} = 1.6 \times 10^{-13}\,{\rm J}$.]

(ii) Given that the white dwarf had a gravitational binding energy $E_{\rm g} = 5 \times 10^{43}\,{\rm J}$, obtain a simple estimate of the average velocity of the ejected matter in the explosion, if all of the energy released is transformed into kinetic energy.

Q6: Assume that a star evolves homologously and that angular momentum is not lost via a wind.

(i) How will the rotation speed of the star depend on its radius?

(ii) The Sun will ultimately evolve into a white dwarf, with radius $R_{\rm WD} = 10^7$ m. Given that the Sun has a rotation period of 28 days, obtain an estimate of the rotation period of the white dwarf it will eventually become. Comment on whether this is likely to be an upper or lower limit to the actual value.

(iii) Neutron stars are believed to be formed in Type II supernovae as the core of a massive star collapses. If the core had an initial radius $R_{\rm c} = 10^7$ m and a rotation period of 28 days, estimate the rotation period of the neutron star, assuming $R_{\rm ns} = 10$ km. Compute the minimum rotation period possible for a neutron star. How do the two numbers compare?

Q7: Consider an accreting High Mass X-ray Binary with a circular orbit in which the donor is a $15.0 M_\odot$ star filling its Roche lobe and the recipient is a neutron star of mass $M_{\rm ns} = 1.4 M_\odot$ and radius $R_{\rm ns} = 10^4$ m.

(i) Comment on how the changing separation between the two stars affects the accretion.

(ii) If 2/3 of the donor mass were to accrete onto the neutron star in a continuous stream over $\sim 10^4$ years, what is the expected luminosity (primarily in the X-ray regime)? How does it compare with the Eddington luminosity? Where do you think that $10 M_\odot$ of donated gas would actually end up?

Q8: An astronomer claims that in a distant galaxy the stellar Initial Mass Function is 'top-heavy'.

(i) Explain what is meant by such a statement.

(ii) Put forward some observational tests that may verify the validity of this claim.

Q9: In a distant galaxy, a burst of star formation forms stars with a Salpeter initial mass function between $M_{\rm min} = 0.1 M_\odot$ and $M_{\rm max} = 60 M_\odot$. For our purposes, the burst can be considered to be instantaneous. At the end of their lives, stars with initial mass $M_{\rm i} \leq 8 M_\odot$ leave a compact remnant with mass $M_{\rm r} = M_{\rm i}/5$, while stars with $M_{\rm i} > 8 M_\odot$ leave a remnant with mass $M_{\rm r} = 1.4 M_\odot$.

(i) Calculate the fraction of the stellar mass that is returned to the interstellar medium 10 Gyr after the burst of star formation.

(ii) Comment briefly on the result.

Q10: At the end of its life, two physical processes remove pressure support from the core of a massive star, precipitating core collapse on the timescale of a few seconds: photodisintegration and neutronisation.

(i) In photodisintegration, each iron nucleus can absorb 124.4 MeV of energy in the process $\gamma + ^{56}_{26}\,{\rm Fe} \longrightarrow 13\,^{4}_{2}{\rm He} + 4{\rm n}$. If 3/4 of the core mass $M_{\rm c} = 1.4 M_\odot$ is dissociated in this way, calculate the total energy absorbed by this process.

(ii) Neutronisation is the inverse process to beta decay: ${\rm p} + {\rm e}^- \longrightarrow {\rm n} + \nu_{\rm e}$. The conversion of protons (in nuclei) to neutrons by electron capture is possible if the gas is sufficiently dense for degenerate electrons to have an energy above the 1.3 MeV mass-energy excess of neutrons compared to protons. If each $\nu_{\rm e}$ produced by the above reaction carries away 10 MeV of energy, how much energy is removed from the core if the entire $M_{\rm c} = 1.4 M_\odot$ undergoes neutronisation?

(iii) Compare the combined energy loss by photodisintegration and neutronisation to the luminosity of a main sequence star with mass $M = 12 M_\odot$, and comment on the result.

EXERCISES, Set 4: Solutions

Q1(i): Deduce the slope of the straight-line track followed by the cooling WD. How does it compare with the slope in the Figure?

A1(i): The question is deliberately vague as to which is the WD track because the student should be able to recognise it from the material covered in the lectures.

We start from the Stefan-Boltzmann law:

$$L = 4\pi R^2 \sigma T_{\rm eff}^4 \tag{4.1}$$

The H-R diagram in the Figure is on a log-log scale; therefore:

$$\log L = \log 4\pi\sigma \; + \; 2\log R \; + \; 4\log T_{\rm eff} \tag{4.2}$$

If the white dwarf cools at constant radius (recall that the radius of a white dwarf is determined uniquely by its mass), the first two quantities on the right-hand side of the above equation are constant, and the slope on a $\log L$–$\log T_{\rm eff}$ plot is -4 ($T_{\rm eff}$ increases to the left).

In the Figure, the track is steeper than the constant radius lines which have slope of -4. Gradual shrinkage of the star as it cools would do this.

Q1(ii): What is the radius (in solar units) of the white dwarf when $L/L_\odot = 0.01$?

A1(ii): From (4.1), we have:

$$\frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{1/2} \left(\frac{T_{\rm eff}}{T_{{\rm eff},\odot}}\right)^{-2} \tag{4.3}$$

From the Figure we can read off $\log T_{\rm eff} \simeq 4.2$ when $\log\left(L/L_\odot\right) = -2$. With $T_{{\rm eff},\odot} = 5777\,$K, $T/T_\odot = 2.8$; hence:

$$\frac{R}{R_\odot} = (0.01)^{1/2}\,(2.8)^{-2} = 0.0128 \tag{4.4}$$

consistent with the lines of constant radius shown in the Figure.

Q2(i): Use the three relations given to calculate the fraction of the initial mass that is lost by massive stars with $M_\mathrm{i} = 25$, 40, 60, 85, and $120\,M_\odot$ before they evolve off the main sequence.

A2(i): The three relations can be combined to give a fourth relation between the fractional mass loss during the main sequence lifetime and the initial mass, as follows:

$$\begin{aligned}\log\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right) &\approx -12.76 + 1.3\left[0.78 + 2.76\log\left(\frac{M_\mathrm{i}}{M_\odot}\right)\right] \\ &\approx -11.75 + 3.59\log\left(\frac{M_\mathrm{i}}{M_\odot}\right)\end{aligned} \tag{4.5}$$

and

$$\frac{\mathrm{d}M}{\mathrm{d}t} \approx 10^{-11.75}\left(\frac{M_\mathrm{i}}{M_\odot}\right)^{3.59} \tag{4.6}$$

or

$$\mathrm{d}M \approx 10^{-11.75}\left(\frac{M_\mathrm{i}}{M_\odot}\right)^{3.59}\mathrm{d}t \tag{4.7}$$

Therefore,

$$\Delta M \approx 10^{-11.75}\left(\frac{M_\mathrm{i}}{M_\odot}\right)^{3.59}\Delta t \tag{4.8}$$

where Δt is the main sequence lifetime, which we are told has a mass dependence:

$$\Delta t \approx 10^{7.72}\left(\frac{M_\mathrm{i}}{M_\odot}\right)^{-0.66} \tag{4.9}$$

Combining the last two equations and dividing by M_i, we arrive at the required relationship:

$$\frac{\Delta M}{M_\mathrm{i}} \approx 10^{-4.03}\left(\frac{M_\mathrm{i}}{M_\odot}\right)^{1.93} \tag{4.10}$$

giving the following values:

M_i ($M_\odot$)	$\Delta M/M_\mathrm{i} \times 100$
25	4.7
40	11.5
60	25.2
85	49.4
120	96.1

Stars with $M_{\rm i} > 85 M_\odot$ lose more than 50% of their mass while on the main sequence!

Q2(ii): A star with $M_{\rm i} = 85 M_\odot$ has a convective core that contains 83% of the stellar mass. Calculate the time after the star appears on the main sequence at which the products of nuclear burning will appear at the surface. How would such a star be classified at this time?

A2(ii): This follows straightforwardly from part (i), eq. 4.8 which can be rearranged to give:

$$\Delta t = 10^{11.75}\, \Delta M \left(\frac{M_{\rm i}}{M_\odot}\right)^{-3.59} . \tag{4.11}$$

With $\Delta M = (1 - 0.83) \cdot 85 = 14.5\, M_\odot$, we have $\Delta t = 0.96 \times 10^6\,$yr.

Such a star would be classified as a WNL star, that is, a late Wolf-Rayet star of the WN sub-type.

Q3(i): Show that for a spherically symmetric star the equations of mass continuity and hydrostatic equilibrium can be combined into the second-order differential equation:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right] = -4\pi G\rho \tag{4.12}$$

A3(i): The mass-continuity equation is:

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2\rho(r) \tag{4.13}$$

and the equation of hydrostatic equilibrium can be written as:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -G\frac{M(r)}{r^2} \tag{4.14}$$

Rearranging (4.14):

$$M(r) = -\frac{1}{G}\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} \tag{4.15}$$

Differentiate (4.15):

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = -\frac{1}{G}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right] \tag{4.16}$$

From the equalities at (4.13) and (4.16), we have:

$$\boxed{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right] = -4\pi G\rho}$$

Q3(ii): For a gas with an equation of state, $P = K\rho^\gamma$, where K is a constant, use the above equation to derive a second-order differential equation involving only density and radius. Using the dimensionless variables $r' \equiv r/R_*$ and $\rho' \equiv \rho/\rho_0$, show that the term $K\rho_0^{\gamma-2}/R_*^2$ is a dimensionless constant, and hence that $R_* \propto \rho_0^{\gamma/2-1}$.

A3(ii): Substituting $P = K\rho^\gamma$ into eq. 4.12, gives:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}K\rho^\gamma}{\mathrm{d}r}\right] = -4\pi G\rho \qquad (4.17)$$

Substituting the dimensionless variables $r' \equiv r/R_*$ and $\rho' \equiv \rho/\rho_0$:

$$\frac{1}{(R_* r')^2}\frac{\mathrm{d}}{\mathrm{d}(R_* r')}\left[\frac{(R_* r')^2}{(\rho_0\rho')}\frac{\mathrm{d}K(\rho_0\rho')^\gamma}{\mathrm{d}(R_* r')}\right] = -4\pi G(\rho_0\rho')\ . \qquad (4.18)$$

Rearranging (4.18):

$$4\pi G\rho'(r')^2 = -\frac{1}{\rho_0 R_*^2}\frac{1}{R_*}\frac{R_*^2}{\rho_0}\frac{\mathrm{d}}{\mathrm{d}r'}\left[\frac{(r')^2}{\rho'}\frac{\mathrm{d}K(\rho_0\rho')^\gamma}{\mathrm{d}(R_* r')}\right] \qquad (4.19)$$

$$4\pi G\rho'(r')^2 = -\frac{1}{\rho_0 R_*^2}\frac{1}{R_*}\frac{R_*^2}{\rho_0}\frac{K\rho_0^\gamma}{R_*}\frac{\mathrm{d}}{\mathrm{d}r'}\left[\frac{(r')^2}{\rho'}\frac{\mathrm{d}(\rho')^\gamma}{\mathrm{d}(r')}\right] \qquad (4.20)$$

Simplifying:

$$4\pi G\rho'(r')^2 = -\left(\frac{K\rho_0^{\gamma-2}}{R_*^2}\right)\frac{\mathrm{d}}{\mathrm{d}r'}\left[\frac{(r')^2}{\rho'}\frac{\mathrm{d}(\rho')^\gamma}{\mathrm{d}(r')}\right] \qquad (4.21)$$

The term in the curved brackets in (4.21) must be dimensionless (as the rest of the equation is) and constant, given that all stars must have the same solution. This is because the boundary conditions of $\rho' = 1$ at $r' = 0$, $\rho' = 0$ at $r' = 1$, and $\mathrm{d}\rho'/\mathrm{d}r' = 0$ at $r' = 0$ (by spherical symmetry, since the pressure gradient vanishes there) apply to all stars.

Hence, $\rho_0^{\gamma-2}/R_*^2 =$ constant, $R_*^2 \propto \rho_0^{\gamma-2}$, or $R_* \propto \rho_0^{\gamma/2-1}$.

Q3(iii): White dwarfs obey the equation of state $P = K\rho^\gamma$, with $\gamma = 5/3$ for non-relativistic conditions and $\gamma = 4/3$ in the relativistic regime.

Using the above result, $R \propto \rho_0^{\gamma/2-1}$, show that for non-relativistic white dwarfs $R \propto M^{-1/3}$, while for relativistic white dwarfs $R \neq f(M)$.

A3(iii): Non relativistic case: $R \propto \rho_0^{-1/6} \to \rho_0 \propto R^{-6}$.
Recalling that $M \propto \rho_0 R_*^3$, we have: $M \propto R_*^{-3} \to R_* \propto M^{-1/3}$.

Relativistic case: $R \propto \rho_0^{-1/3} \to \rho_0 \propto R^{-3}$.
Hence, $M \propto R^{-3} R^3$, i.e. $M \neq f(R)$.

Q4(i): Estimate the final radius of the stellar remnant if the energy liberated is just enough to eject the remaining $9M_\odot$ to infinity. State clearly any assumptions made.

A4(i): The energy that goes into ejecting the remaining mass is (see Lecture 7, eq. 7.8 and following):

$$\Delta E_\nu = \frac{3}{10} f G\, 1 M_\odot^2 \left(\frac{1}{r_{\mathrm{c,f}}} - \frac{1}{r_{\mathrm{c,i}}} \right) \tag{4.22}$$

where $f = 0.01$ and $r_{\mathrm{c,i}}$ and $r_{\mathrm{c,f}}$ are, respectively, the core radius at the start of the collapse and the core's final radius.

The energy needed to eject $9\,M_\odot$ of material out of the potential well created by the remaining $1\,M_\odot$ is:

$$E_{\mathrm{ej}} = \frac{G\, 9 M_\odot^2}{\langle r_9 \rangle} \tag{4.23}$$

where $\langle r_9 \rangle$ is an average radius of the $9\,M_\odot$ before the core collapses.

Equating the two expressions leads to:

$$r_{\mathrm{c,f}} = \left(\frac{30}{f \langle r_9 \rangle} + \frac{1}{r_{\mathrm{c,i}}} \right)^{-1} \tag{4.24}$$

Let's consider some possibilities for $\langle r_9 \rangle$. One extreme is to take the radius of a $10\,M_\odot$ supergiant, i.e. $\langle r_9 \rangle \simeq 100 R_\odot$ (see, for example, Lecture 2, Figure 2.9). With $r_{\mathrm{c,i}} \sim r_\oplus \sim 6000\,\mathrm{km}$, we have:

$$r_{\mathrm{c,f}} = \left(\frac{30}{0.01 \cdot 100 \times 7 \times 10^5\,\mathrm{km}} + \frac{1}{6000\,\mathrm{km}} \right)^{-1} \simeq 4770\,\mathrm{km}\,. \tag{4.25}$$

Another extreme would be to assume that all of the $9\,M_\odot$ is located in a thin shell just above the collapsing core, in which case $\langle r_9 \rangle \simeq R_\oplus$, and:

$$r_{\mathrm{c,f}} = \left(\frac{30}{0.01 \cdot 6000\,\mathrm{km}} + \frac{1}{6000\,\mathrm{km}} \right)^{-1} \simeq 2\,\mathrm{km}\,. \tag{4.26}$$

Using an intermediate and crude "guesstimate" of $\langle r_9 \rangle \simeq 1 R_\odot$ for a centrally condensed supergiant, we have:

$$r_{\mathrm{c,f}} = \left(\frac{30}{0.01 \cdot 1 \times 7 \times 10^5\,\mathrm{km}} + \frac{1}{6000\,\mathrm{km}} \right)^{-1} \simeq 225\,\mathrm{km}\,. \tag{4.27}$$

Therefore, there appears to be plenty of energy available in the gravitational collapse of the core to eject the $9M_\odot$ of overlying material (recall that the final core radius quoted in Lecture 16 is roughly 20 km).

Q4(ii): What is the typical velocity of the ejecta, if the energy absorbed by the envelope is 10^{51} erg?

A4(ii): We have:

$$E_{\rm ej} = 1 \times 10^{51}\,{\rm erg} = \frac{1}{2} M_{\rm ej} \langle v \rangle^2 \tag{4.28}$$

and therefore:

$$\begin{aligned}\langle v \rangle = \left(\frac{2E_{\rm ej}}{M_{\rm ej}}\right)^{1/2} &= \left(\frac{2 \times 10^{51}\,{\rm g\ cm^2\ s^{-2}}}{9 \times 2 \times 10^{33}\,{\rm g}}\right)^{1/2} \\ &= 3.3 \times 10^8\,{\rm cm\ s^{-1}} \equiv 3300\,{\rm km\ s^{-1}}\,.\end{aligned} \tag{4.29}$$

Q4(iii): An astronomer announces the discovery of a Type II supernova in a Globular cluster, but her colleagues are skeptical. Why?

A4(iii): Type II supernovae are generally believed to be the end points of the evolution of stars with masses $M \gtrsim 10 M_\odot$ (Lecture 16.3.1).

Such stars have lifetimes $\tau \lesssim 3 \times 10^7$ yr.

Globular clusters have ages $\tau \gtrsim 3 \times 10^9$ yr.

Thus, all massive stars originally in a globular cluster have long ago exploded as Type II supernovae.

Q5(i): Calculate the nuclear energy released by a Type Ia SN assuming that: (a) the exploding white dwarf has a mass of $1.4\,M_\odot$ consisting of ^{12}C and ^{16}O in equal proportions by number, and (b) all of the C and O are burned to ^{56}Ni in the explosion. The mass of a given nucleus of atomic number Z and mass number A is given by the formula:

$$m(A, Z) = Am_{\rm u} + m_{\rm ex}(A, Z)$$

where one atomic mass unit $m_{\rm u} = 931.5\,{\rm MeV/c^2}$, and the mass excess $m_{\rm ex} = 0$, -4.7, and $-53.9\,{\rm MeV/c^2}$ for ^{12}C, ^{16}O, and ^{56}Ni respectively. [Note: $1\,{\rm MeV} = 1.6 \times 10^{-13}\,{\rm J}$.]

A5(i): The conversion of ^{12}C and ^{16}O to ^{56}Ni involves:

$$2\left(^{12}{\rm C} + {}^{16}{\rm O}\right) \longrightarrow {}^{56}{\rm Ni} + \ldots \tag{4.30}$$

The total energy released in the explosion is the mass difference between the left- and right-hand sides of eq. 4.30 $\times$ the number of such reactions required to convert $1.4M_\odot$ of ^{12}C and ^{16}O into ^{56}Ni.

The mass on the L.H.S. of 4.30 is:

$$\begin{aligned} m' &= 2 \times [12 \times 931.5 + 16 \times 931.5 - 4.7]\,\frac{\rm MeV}{c^2} \\ &= 52155\,\frac{\rm MeV}{c^2} \\ &= 9.3 \times 10^{-26}\,{\rm kg}, \end{aligned} \tag{4.31}$$

since $1m_{\rm u} = 1.66053886 \times 10^{-27}\,{\rm kg}$.

Thus, to burn $1.4M_\odot$ of a CO white dwarf we require:

$$\mathcal{N}_{\rm reac} = \frac{M_{\rm WD}}{m'} = \frac{1.4 \times 2.0 \times 10^{30}\,{\rm kg}}{9.3 \times 10^{-26}\,{\rm kg}} = 3.0 \times 10^{55} \tag{4.32}$$

The mass on the R.H.S. of 4.30 is:

$$m'' = 56 \times 931.5 - 53.9 = 52110\,\frac{\rm MeV}{c^2} \tag{4.33}$$

ignoring the mass of electrons and neutrinos generated in the nuclear reactions.

Thus, the energy released per reaction ($E = mc^2$) is:

$$\Delta E = m' - m'' = 52155 - 52110 = 45\,{\rm MeV} \tag{4.34}$$

and the total energy released in the explosion is:

$$\begin{aligned} E = \Delta E \times \mathcal{N}_{\rm reac} &= 45 \times 3.0 \times 10^{55}\,\text{MeV} \\ &= 1.4 \times 10^{57} \times 1.6 \times 10^{-13}\,\text{J} \\ &= 2.2 \times 10^{44}\,\text{J}\,. \end{aligned} \tag{4.35}$$

Q5(ii): Given that the white dwarf had a gravitational binding energy $U_{\rm g} = 5 \times 10^{43}$ J, obtain a simple estimate of the average velocity of the ejected matter in the explosion, if all of the energy released is transformed into kinetic energy.

A5(ii): With the rather extreme assumption that all the energy liberated by the explosion of the white dwarf is turned into kinetic energy of the ejecta (and thus ignoring radiation losses), we have:

$$K_{\rm ej} = E - U_{\rm g} = 2.2 \times 10^{44} - 5 \times 10^{43} = 1.7 \times 10^{44}\,\text{J} \tag{4.36}$$

Since $K = \frac{1}{2}mv^2$, we have:

$$\langle v \rangle = \left(\frac{2K}{M_{\rm WD}}\right)^{1/2} = \left(\frac{3.4 \times 10^{44}\,\text{kg m}^2\,\text{s}^{-2}}{1.4 \times 2.0 \times 10^{30}\,\text{kg}}\right)^{1/2} \simeq 11\,000\,\text{km s}^{-1} \sim 0.04c!$$

Q6(i): How will the rotation speed of the star depend on its radius?

A6(i): The standard expression for angular momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{4.37}$$

can be applied to a star of mass M and radius R (dropping the vector notation):

$$L = M\, v_{\rm rot}\, R \equiv M\omega R^2 \tag{4.38}$$

where the rotational velocity and the angular frequency ω are related by: $v_{\rm rot} = \omega R$.

With L and M constant, it is straightforward to see that as the star evolves:

$$v_{\rm rot} \propto \frac{1}{R} \tag{4.39}$$

Q6(ii): The Sun will ultimately evolve into a white dwarf, with radius $R_{\rm WD} = 10^7$ m. Given that the Sun has a rotation period of 28 days, obtain an estimate of the rotation period of the white dwarf it will eventually become. Comment on whether this is likely to be an upper or lower limit to the actual value.

A6(ii): If $v_{\rm rot} \propto 1/R$ and $\omega = v_{\rm rot}/R$, it follows that $\omega \propto 1/R^2$. Furthermore, since the period is just:

$$P = \frac{2\pi}{\omega} \tag{4.40}$$

we have:

$$\left(\frac{P_{\rm WD}}{P_\odot}\right) = \left(\frac{R_{\rm WD}}{R_\odot}\right)^2 \tag{4.41}$$

Thus:

$$P_{\rm WD} = \left(\frac{10^7\,{\rm m}}{7 \times 10^8\,{\rm m}}\right)^2 \cdot 28\,{\rm days} = \frac{28}{4900} = 5.7 \times 10^{-3}\,{\rm days} = 8.3\,{\rm minutes} \tag{4.42}$$

In reality, this is a lower limit to the rotation period of the WD. As we have discussed in the lectures, before becoming a white dwarf the Sun will lose a substantial fraction of its mass during the RGB, AGB, and PN evolutionary stages. This will result in angular momentum loss; consequently the WD remnant will not spin as fast.

Q6(iii): Neutron stars are believed to be formed in Type II supernovae as the core of a massive star collapses. If the core had an initial radius $R_c = 10^7$ m and a rotation period of 28 days, estimate the rotation period of the neutron star, assuming $R_{NS} = 10$ km. Compute the minimum rotation period possible for a neutron star. How do the two numbers compare?

A6(iii): Again, assuming homologous contraction and constant angular momentum, we have:

$$\frac{P_{NS}}{P_c} = \left(\frac{R_{NS}}{R_c}\right)^2 \tag{4.43}$$

and

$$P_{NS} = 28\,\text{days}\left(\frac{10^4}{10^7}\right)^2 = 28 \times 24 \times 3600 \times 10^{-6} = 2.4\,\text{s} \tag{4.44}$$

The minimum rotation period possible is determined by the ability of gravity to provide the centripetal force necessary to hold the star together, i.e.

$$\omega_{\text{max}}^2 R = \frac{GM}{R^2} \tag{4.45}$$

Since $P = 2\pi/\omega$, we have:

$$P_{\text{min}} = 2\pi\left(\frac{R^3}{GM}\right)^{1/2} \tag{4.46}$$

For a neutron star of $R = 10^4$ m and $M = 1.4M_\odot$, we have:

$$P_{\text{min}} = 6.28\left(\frac{10^{12}\,\text{m}^3}{6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2} \times 1.4 \times 2 \times 10^{30}\,\text{kg}}\right)^{1/2} = 4.6\times10^{-4}\,\text{s} \tag{4.47}$$

From the above result we see that P_{min} is some four orders of magnitude smaller than the initial rotational period of the neutron star in our example.

Q7(i): Comment on how the changing separation between the two stars affects the accretion.

A7(i): In lecture 18 we derived the expression that relates the binary separation a to the mass transfer rate $\dot{M}$:

$$\frac{1}{a}\frac{da}{dt} = 2\dot{M}_1 \frac{M_1 - M_2}{M_1 M_2} \tag{4.48}$$

Since the mass transfer is from the more massive star to the less massive one, $M_1 < M_2$ and therefore da/dt is −ve: the stars get closer together (assuming conservation of mass and angular momentum).

We also saw that the distance from the inner Lagrangian point L1 to the donor star, depends on the binary separation:

$$\ell_1^2 = a\left[0.500 + 0.227\log_{10}\left(\frac{M_2}{M_1}\right)\right]. \tag{4.49}$$

Thus, as a shrinks, L1 will move closer to, or into, the donor, greatly enhancing the accretion. *This is unstable mass transfer!.*

Q7(ii): If 2/3 of the donor mass were to accrete onto the neutron star in a continuous stream over $\sim 10^4$ years, what is the expected luminosity (primarily in the X-ray regime)? How does it compare with the Eddington luminosity? Where do you think that $10M_\odot$ of donated gas would actually end up?

A7(ii): The potential energy lost by an element of mass dM as it infalls from "infinite" radius to radius R is:

$$\mathrm{d}U = \frac{GM\mathrm{d}m}{R} \tag{4.50}$$

Division by the time interval dt yields the rate at which potential energy is given up. The maximum luminosity that one would expect under the assumption of spherically symmetric, steady-state radial infall onto the surface of the neutron star, with all the energy being reradiated in the waveband observed, is:

$$L \approx \frac{GM_{\rm ns}}{R_{\rm ns}}\frac{\mathrm{d}m}{\mathrm{d}t}. \tag{4.51}$$

In our case, we have:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{10 M_\odot}{10^4\,\mathrm{yr}} = \frac{10 \times 2 \times 10^{30}\,\mathrm{kg}}{10^4 \times 3.2 \times 10^7\,\mathrm{s}} = 6.3 \times 10^{19}\,\mathrm{kg\ s^{-1}}\,. \tag{4.52}$$

Entering $M_{\mathrm{ns}} = 1.4 M_\odot$, $R_{\mathrm{ns}} = 10^4\,\mathrm{m}$, and the above value of $\mathrm{d}m/\mathrm{d}t$ into eq. 4.51, we have:

$$\begin{aligned} L_{\mathrm{X}} &\approx \frac{6.7 \times 10^{-11}\,\mathrm{m^3\ kg^{-1}\ s^{-2}} \times 1.4 \times 2 \times 10^{30}\,\mathrm{kg}}{10^4\,\mathrm{m}} \times 6.3 \times 10^{19}\,\mathrm{kg\ s^{-1}} \\ &\approx 1.2 \times 10^{36}\,\mathrm{kg\ m^2\ s^{-2}\ s^{-1}} \equiv 1.2 \times 10^{36}\,\mathrm{W} \quad . \end{aligned} \tag{4.53}$$

In Lecture 10 (eq. 10.41), we saw that the Eddington luminosity is:

$$L_{\mathrm{edd}} \simeq 3.8 \times 10^4 \frac{M}{M_\odot} L_\odot\,; \tag{4.54}$$

hence:

$$L_{\mathrm{edd,ns}} \simeq 3.8 \times 10^4 \times 1.4 \times 3.9 \times 10^{26}\,\mathrm{W} = 2.1 \times 10^{31}\,\mathrm{W}\,. \tag{4.55}$$

Comparing (4.53) and (4.55), we see that the Eddington luminosity is a factor of 10^5 smaller than the luminosity that would be expected from all the matter being lost from the donor. In such circumstances, radiation pressure would prevent almost all of the matter from being accreted, so it must be lost to the system—possibly aided by turbulence created during an in-spiralling phase.

High mass X-ray binaries do radiate close to the Eddington limit.

Q8(i): Explain what is meant by a 'top-heavy' IMF.

A8(i): These two questions are meant to stimulate the curiosity of the student; the relevant material has only been touched upon in the lecture notes (Lecture 11.9).

A 'top-heavy' IMF simply implies a stellar initial mass function in which the balance between high- and low-mass stars is 'tilted' in favour of high-mass stars compared to a standard reference, normally taken to be the Salpeter IMF:

$$N(M)\,dM \propto M^{-2.35}\,dM \tag{4.56}$$

where $N(M)\,dM$ is the number of stars per unit volume with mass between M and $M + dM$. In a 'top-heavy' IMF the exponent of the power law is greater (i.e. less negative), giving a higher proportion of high-mass stars.

Q8(ii): Put forward some observational tests that may verify the validity of this claim.

A8(ii): To answer this question, the student has to draw on material that she/he has encountered at several places in the lecture course. Possible tests of a top-heavy IMF would include:

(1) A lower than 'normal' mass-to-light ratio. We saw in Lecture 4.5 that $L \propto M^{3.5}$. Consequently, $M/L \propto M^{-2.5}$. Thus, a stellar population with a top-heavy IMF would by overluminous for its mass.

(2) Unusually strong P-Cygni lines. In Lecture 15.4.4 we considered the wind-momentum luminosity relation for mass-losing stars:

$$\dot{M} v_\infty \left(\frac{R_*}{R\odot}\right)^{1/2} \propto L^{1.46}\,. \tag{4.57}$$

which shows that the product of the mass-loss rate and the wind terminal velocity is proportional to the luminosity of a star. P-Cygni lines are diagnostic of powerful, optically thick stellar winds; in a galaxy with an unusually high proportion of massive stars, the contrast of the P-Cygni lines over the UV continuum would be increased (and viceversa)—see Figure 4.1.

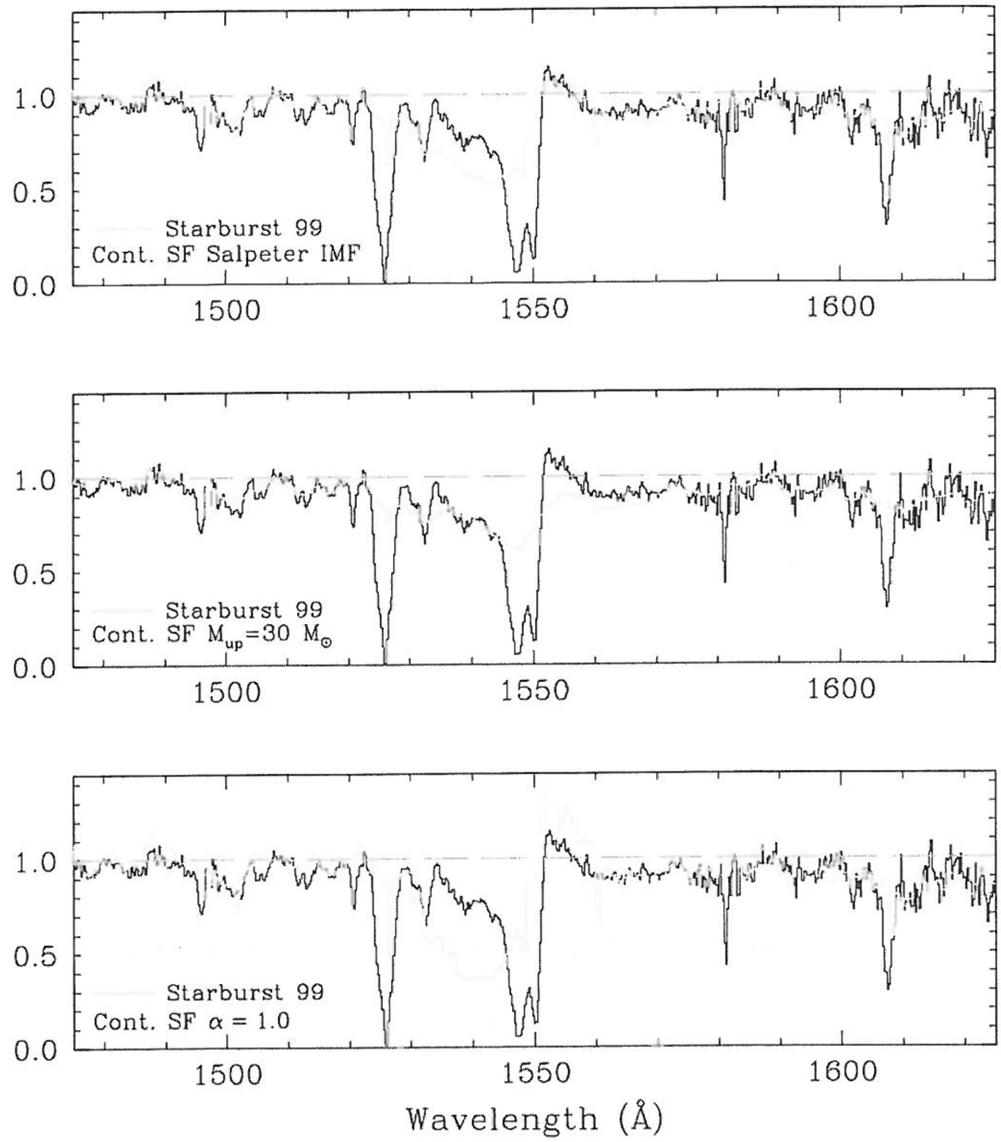


Figure 4.1: Comparisons between *Starburst99* (Leitherer et al. 1999) population synthesis models with different IMFs (green lines) and the Keck spectrum of MS 1512-cB58 analysed by Pettini et al. (2000b) in the region near the C IV doublet (black histogram). *Top panel:* 'Standard' Salpeter IMF, with $\alpha = 2.35$ in the mass range 1–120$M_\odot$. *Middle panel:* 'Standard' Salpeter IMF, with $\alpha = 2.35$, but with the upper mass limit truncated at $M_{\rm up} = 30 M_\odot$. This IMF *lacks* the most massive stars; consequently the strength of the C IV P-Cygni line is reduced significantly compared to the top panel. *Bottom panel:* A 'top-heavy' IMF, with a much flatter power-law slope ($\alpha = 1.0$) results in a much stronger P-Cygni line. The y-axis is residual intensity.

(3) Chemical abundance anomalies. In Lecture 11.9 we briefly mentioned that stars of different masses synthesise (and expel) different elements in different proportions. For example, massive stars are thought to be the main producers of O, while intermediate- and low-mass stars make most of the Fe. Thus, a top-heavy IMF would lead higher O/Fe ratios than

observed, for example, in the Milky Way.

(4) Qualitatively, the students should at least surmise that a top-heavy IMF will lead to a higher density/frequency of the end products of massive stars: core-collapse supernovae, neutron stars and pulsars, gamma-ray bursts, and black holes.

Q9(i): Calculate the fraction of the stellar mass that is returned to the interstellar medium 10 Gyr after the burst of star formation.

A9(i): In Lecture 11.9 we saw that the Salpeter initial mass function describes the number of stars (per unit volume) in the mass interval between M and $M + dM$ and has the functional form:

$$\xi(M)dM \propto M^{-2.35}dM\,. \tag{4.58}$$

Let the IMF have some normalisation a, i.e. $\xi(M) = aM^{-2.35}$. Then, the total initial mass of the stellar population is:

$$\begin{aligned} M_{\rm tot,i} &= \int_{0.1M_\odot}^{60M_\odot} M_{\rm i}\,\xi(M_{\rm i})dM_{\rm i} \\ &= a\int_{0.1M_\odot}^{60M_\odot} M_{\rm i}^{-1.35}dM_{\rm i} \\ &= 5.71a \end{aligned} \tag{4.59}$$

Note that the IMF needs to be defined between $M_{\rm min}$ and $M_{\rm max}$, or the integral will be ill-behaved.

In the lectures, it was also stated that the lifetime of a star with $M_{\rm i} = 1M_\odot$ is 10 Gyr (e.g. lecture 4.5 and Figure 4.10). The mass locked-up in stellar remnants and in low mass stars (with $M_{\rm i} < 1M_\odot$) 10 Gyr after the star formation episode is therefore:

$$
\begin{aligned}
M_{\rm tot,r} &= \int_{0.1M_\odot}^{1M_\odot} M_{\rm i}\,\xi(M_{\rm i})dM_{\rm i} + \int_{1M_\odot}^{8M_\odot} \frac{M_{\rm i}}{5}\,\xi(M_{\rm i})dM_{\rm i} + \int_{8M_\odot}^{60M_\odot} 1.4\,\xi(M_{\rm i})dM_{\rm i} \\
&= a\int_{0.1M_\odot}^{1M_\odot} M_{\rm i}^{-1.35}dM_{\rm i} + \frac{a}{5}\int_{1M_\odot}^{8M_\odot} M_{\rm i}^{-1.35}dM_{\rm i} + 1.4a\int_{8M_\odot}^{60M_\odot} M_{\rm i}^{-2.35}dM_{\rm i} \\
&= 3.89a
\end{aligned}
\tag{4.60}
$$

The amount returned to the ISM is therefore:

$$M_{\rm ret} = M_{\rm i} - M_{\rm r} = 5.71a - 3.89a = 1.82a \tag{4.61}$$

and the returned *fraction* is:

$$\frac{M_{\rm ret}}{M_{\rm i}} = \frac{1.82}{5.71} = 0.32\,. \tag{4.62}$$

Q9(ii): Comment briefly on the result.

A9(ii): Even after 10 Gyr, most of the initial mass is still locked up in low mass stars (still on the main sequence) and compact stellar remnants.

In reality, the returned fraction is somewhat larger than the above estimate, because the IMF does not follow a Salpeter slope all the way down to $0.1M_\odot$, but turns over near $M \simeq 1M_\odot$.

Q10(i): In photodisintegration, each iron nucleus can absorb 124.4 MeV of energy in the process $\gamma +^{56}_{26}\text{Fe} \longrightarrow 13\,^4_2\text{He} + 4\text{n}$. If 3/4 of the core mass $M_\text{c} = 1.4M_\odot$ is dissociated in this way, calculate the total energy absorbed by this process.

A10(i): The number of ^{56}Fe nuclei is:

$$\mathcal{N}_\text{Fe} = \frac{3}{4}1.4M_\odot \frac{1}{56\text{u}} \qquad (4.63)$$

where u is the unified atomic mass unit. Thus:

$$\mathcal{N}_\text{Fe} = \frac{3}{4} \cdot 1.4 \times 1.99 \times 10^{33}\text{g} \frac{1}{56 \times 1.661 \times 10^{-24}\,\text{g}} = 2.25 \times 10^{55}\,. \qquad (4.64)$$

If each nucleus absorbs 124.4 MeV, the total energy loss is:

$$\Delta E = -2.25 \times 10^{55} \times 124.4 \times 10^6 \times 1.60 \times 10^{-12} = -4.5 \times 10^{51}\,\text{erg}\,. \qquad (4.65)$$

Q10(ii): If each ν_e produced by the inverse beta-decay carries away 10 MeV of energy, how much energy is removed from the core if the entire $M_\text{c} = 1.4M_\odot$ undergoes neutronisation?

A10(ii): The number of protons contained in a stellar core of mass $M_\text{c} = 1.4M_\odot$ is approximately:

$$\mathcal{N}_\text{p} \simeq \frac{1}{2}\frac{1.4M_\odot}{\text{u}} = 8.4 \times 10^{56}\,. \qquad (4.66)$$

(the other half being neutrons). Assuming charge neutrality, there will also be 8.4×10^{56} electrons. Since each neutronisation produces one electron neutrino, and each ν_e carries away 10 MeV, the total energy lost to the system is

$$\Delta E = -8.4 \times 10^{56} \times 10 \times 10^6 \times 1.60 \times 10^{-12} = -1.3 \times 10^{52}\,\text{erg}\,. \qquad (4.67)$$

Q10(iii): Compare the combined energy loss by photodisintegration and neutronisation to the luminosity of a main sequence star with mass $M = 12M_\odot$, and comment on the result.

A10(iii): In Lecture 4.5, we saw that for stars on the main sequence, $L \propto M^{3.5}$. Thus, the luminosity of a $M = 12M_\odot$ star is:

$$L = 12^{3.5}L_\odot \simeq 6000L_\odot = 6000 \times 3.839 \times 10^{33}\,\text{erg s}^{-1} = 2.3 \times 10^{37}\,\text{erg s}^{-1}.$$

We also saw that the main sequence lifetime of stars with masses $M > 8M_\odot$ can be approximated by the relation:

$$\log(t_{\rm MS}/\text{yr}) = 9.01 - 1.57\log(M/M_\odot) \quad (4.68)$$

(Figure 4.10). Thus, a star with $M = 12M_\odot$ will spend

$$t_{\rm MS} = 20.7\,\text{Myr} = 20.7 \times 10^6 \times 31556926\,\text{s} = 6.5 \times 10^{14}\,\text{s}$$

on the main sequence (ignoring mass loss).

During this time, the star will radiate an energy:

$$\Delta E = -2.3 \times 10^{37}\,\text{erg s}^{-1} \times 6.5 \times 10^{14}\,\text{s} = -1.5 \times 10^{52}\,\text{erg}$$

comparable to the sum of the energy loss due to photodisintegration and neutronisation, $\Delta E = -(0.45 + 1.3) \times 10^{52} = -1.75 \times 10^{52}\,\text{erg}$ (eqs. 4.65 and 4.67).

Thus, in just a few seconds at the end of its life, a massive star loses as much energy as that radiated throughout its entire life on the main sequence.