Stellar Dynamics and Structure of Galaxies

Jeans Theorem

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Outline I

1. Jeans Theorem
   Integrals of Motion

2. Application of Jeans theorem
   Obtaining self-consistent models
   Eddington Formula
   Harmonic oscillator potential
   Spherically symmetric solutions of the collisionless Boltzmann equation
   Plummer potential
   Isothermal sphere
Jeans Theorem

If we go back the the **Collisionless Boltzmann Equation** and look for a steady state solution (so $\frac{\partial}{\partial t} = 0$)

$$\mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

where $f(x, v, t)$ is the stellar distribution function in phase space $(x, v)$. Recall that each star follows a **path in phase space** given by $(x(t), v(t))$ where

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\nabla \Phi$$

(6.1)

Define an **integral of the motion** as a function of the phase space coordinates $I(x, v)$ which is constant along the path.
Constants of Motion: any function of the phase-space coordinates and time $C(x, v, t)$ that is constant along every orbit where $x(t)$ and $v(t)$ are a solution to the equations of motion

$$C[x(t_1), v(t_1); t_1] = C[x(t_2), v(t_2); t_2]$$

(6.2)

for any $t_1$ and $t_2$

Any orbit in any force field has six independent constants of motion. For example, the initial phase-space coordinates $(x_0, v_0) \equiv [x(0), v(0)]$ can always be obtained from the equations of motion and can be regarded as six constants of motion.

The above procedure reminds us that physics is invariant to time translations i.e., the time at which we pick our initial conditions does not hold any information regarding the dynamical system.
Integrals of Motion: any function $I(x, v)$ of the phase-space coordinates alone that is constant along any orbit

$$I[x(t_1), v(t_1)] = I[x(t_2), v(t_2)] \quad (6.3)$$

Every integral is a constant of motion, but every constant of motion is not an integral.

For example, on a circular orbit in a spherical potential, the azimuthal speed

$$\psi = \Omega t + \psi_0$$

Hence, $C(\psi, t) \equiv t - \psi/\Omega$ will be constant of motion, but is not an integral of motion because it depends on time.
Integrals of Motion

**Integrals of Motion:** any function $I(x, v)$ of the phase-space coordinates alone that is constant along any orbit

$$I[x(t_1), v(t_1)] = I[x(t_2), v(t_2)]$$  \hspace{1cm} (6.3)

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Hence, $C(\psi, t) \equiv t - \psi/\Omega$ will be constant of motion, but is not an integral of motion because it depends on time.
Integrals of motion come in two flavors:

- **Isolating Integrals of Motion** reduce the dimensionality of the orbit by one, i.e. with energy $E$ or angular momentum $L$ in hand, the motion is restricted to 5D manifold in 6D dimensional phase-space. These are of great practical and theoretical importance in Dynamics.

- **Non-Isolating Integrals of Motion** do not affect the phase-space distribution of an orbit, i.e. do not reduce the dimensionality of the motion. These carry no practical value.

And, finally,

> Energy is always an isolating integral of motion
Integrals of Motion

Integrals of motion come in two flavors:

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**Energy is always an isolating integral of motion**
Integrals of Motion

Isolating Integral
Integrals of Motion

The figure above illustrates the application of Jeans' theorem in the context of galactic dynamics. The graph demonstrates the behavior of integrals of motion, which are crucial for understanding the stability and structure of galaxies.
For example, in a static potential $\Phi(x)$, the energy

$$E = \frac{1}{2} v^2 + \Phi(x) \quad (6.4)$$

is an integral of the motion because

$$\frac{dE}{dt} = v \cdot \frac{dv}{dt} + \nabla \Phi \cdot \frac{dx}{dt} = v \cdot (-\nabla \Phi) + \nabla \Phi \cdot v = 0$$
Thus, for an integral of the motion $I$, we require

$$\frac{d}{dt} \{ I [x(t), v(t)] \} = 0 \quad (6.5)$$

$$\Rightarrow$$

$$\frac{dI}{dt} = ∇I \cdot \frac{dx}{dt} + \frac{∂I}{∂v} \cdot \frac{dv}{dt} = 0$$

i.e.

$$v \cdot ∇I - ∇Φ \cdot \frac{∂I}{∂v} = 0 \quad (6.6)$$

Recall the steady state collisionless Boltzmann equation

$$v \cdot ∇f - ∇Φ \cdot \frac{∂f}{∂v} = 0$$

i.e. $f$ and $I$ obey the same equation.
Thus, for an integral of the motion $I$, we require

$$\frac{d}{dt} \{ I [x(t), v(t)] \} = 0 \quad (6.5)$$

$$\Rightarrow \quad \frac{dI}{dt} = \nabla I \cdot \frac{dx}{dt} + \frac{\partial I}{\partial v} \cdot \frac{dv}{dt} = 0$$

i.e.

$$v \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial v} = 0 \quad (6.6)$$

Recall the steady state collisionless Boltzmann equation

$$v \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0$$

i.e. $f$ and $I$ obey the same equation.
Jeans Theorem

Theorem (Jeans Theorem)

i) Any steady state solution of the Collisionless Boltzmann Equation depends on the phase-space coordinates \((x,v)\) only through integrals of the motion in a static potential, and ii) any function of the integrals yields a steady state solution of the collisionless Boltzmann equation.

Proof.

Suppose \( f \) is a steady state solution of the collisionless Boltzmann equation. Then we have just shown \( \frac{df}{dt} = 0 \), and so \( f \) is an integral of the motion i.e. \( f \) can depend only on integrals of the motion.

Conversely if there are \( n \) integrals of the motion \( I_1, I_2, \ldots, I_n \), and if \( f \) is any function of these then

\[
\frac{d}{dt} \left[ f(I_1(x,v), I_2(x,v), \ldots, I_n(x,v)) \right] = \sum_{m=1}^{n} \frac{\partial f}{\partial I_m} \frac{dl_m}{dt} = 0
\]

and so \( f \) satisfies the collisionless Boltzmann equation.
The value of Jeans theorem is that it gives us a way of closing the loop for solving the Collisionless Boltzmann Equation.

- Taking moments gave us insight about the properties of the solutions but not the actual solutions.
- The Jeans equation approach gave us more models, but no guarantee that they were physical.
Given $\Phi(x)$ we know that any function

$$f(E) = f\left(\frac{1}{2}v^2 + \Phi(x)\right) \quad (6.7)$$

is a solution of the collisionless Boltzmann equation. Now assume that all stars have the same mass $m$, then

$$\rho(x) = m \int \int \int f d^3v = m\nu(x)$$

or, without loss of generality, redefine $f$ as the mass distribution function (rather than the number). Then

$$\nabla^2\Phi = 4\pi G \rho = 4\pi G \int \int \int f d^3v \quad (6.8)$$

If we can find a function $f(E)$ which satisfies both (6.7) and (6.8) then we have a self-consistent solution in which the stars all obey Newton’s laws in the potential $\Phi(x)$, and the potential $\Phi(x)$ is due to the stars.
Application of Jeans theorem

Obtaining self-consistent models

**Notation:** To make things easier we redefine the potential and the energy by adjusting the arbitrary constant and changing the sign.

Let \( \Psi = -\Phi + \Phi_0 \). This is **relative potential**.

and \( \mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2 \). This is **relative energy**

Then we choose \( \Phi_0 \) such that

\[
\begin{align*}
  f &> 0 \quad \text{for } \mathcal{E} > 0 \\
  f &= 0 \quad \text{for } \mathcal{E} \leq 0
\end{align*}
\]

Then, the relative potential satisfies the Poisson’s equation

\[
\nabla^2 \Psi = -4\pi G \rho
\]

and \( \Psi \to \Phi_0 \) as \(|x| \to \infty\).
Application of Jeans theorem

If we have **spherical symmetry**, so $\Phi$ depends only on $r$, then

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Psi}{dr} \right) = -4\pi G \rho
$$

$$
= -4\pi G \int \int \int f d^3v
$$

$$
= -4\pi G \int_0^{\sqrt{2}\Psi} f(E)4\pi v^2 dv, \text{ since } f \text{ depends on } v \text{ and not on } v
$$

the upper limit comes from $f \neq 0$ only if $E = \Psi - \frac{1}{2}v^2 > 0$

$$
= -16\pi^2 G \int_0^{\sqrt{2}\Psi} f(\Psi - \frac{1}{2}v^2)v^2 dv
$$

Now $dE = -v dv$, with limits $v = 0$ $E = \Psi$ and $v = \sqrt{2\Psi} \ E = 0$, so

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Psi}{dr} \right) = -16\pi^2 G \int_0^{\Psi} f(E)\sqrt{2(\Psi(r) - E)} \ dE
$$
Application of Jeans theorem

Eddington Formula

So, how to get from $\rho$ to $f$?
We start by going the other way round:

\[ \nu(r) = \nu(\Psi) = \int d^3v f = 4\pi \int dv v^2 f(\Psi - \frac{1}{2}v^2) = 4\pi \int_{\Psi}^\Psi d\mathcal{E} f(\mathcal{E}) \sqrt{2(\Psi - \mathcal{E})} \quad (6.9) \]

Noting that potential \( \Psi \) is a monotonic function of \( r \) in any spherical system. Differentiating both sides with respect to \( \Psi \)

\[ \frac{1}{\sqrt{8\pi}} \frac{d\nu}{d\Psi} = \int_0^\Psi d\mathcal{E} \frac{f(\mathcal{E})}{\sqrt{\Psi - \mathcal{E}}} \quad (6.10) \]

This is an Abel integral equation having solution:

\[ f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\mathcal{E}} \int_0^\mathcal{E} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} \frac{d\nu}{d\Psi} \quad (6.11) \]

\[ f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \left[ \int_0^\mathcal{E} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} d^2\nu + \frac{1}{\sqrt{\mathcal{E}}} \left( \frac{d\nu}{d\Psi} \right)_{\Psi=0} \right] \quad \text{Eddington’s formula} \quad (6.12) \]
Sir Arthur Stanley Eddington
To summarize: Given a spherically symmetric density distribution, which can be written as $\rho = \rho(\Psi)$ (may not always be possible), Eddington’s formula yields the corresponding distribution function $f = f(\mathcal{E})$

Because we require $f(\mathcal{E}) \geq 0$ everywhere, Eddington’s formula

$$f(\mathcal{E}) = \frac{1}{\sqrt{8\pi^2}} \frac{d}{d\mathcal{E}} \int_{0}^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} d\mathcal{E}$$

demands that the function $\int_{0}^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} d\mathcal{E} d\mathcal{E}$ is an increasing function of $\mathcal{E}$. 
The problem (for the spherical case) is to find a pair of functions $f, \Psi$ which satisfy this equation.

What does this problem amount to? Instead of looking at the 6-D case, let us illustrate the main ideas by taking a simple example - a 1-D harmonic oscillator potential.

[Part of the motivation for this is that inside a $\rho =$constant sphere]

$$
\Phi = \frac{2}{3} \pi G \rho_0 (r^2 - 3r_0^2) = \frac{1}{2} \omega_0^2 (x^2 + y^2 + z^2) + C \quad (6.13)
$$

where $\omega_0$ and $C$ are constants, and this is a 3-D harmonic oscillator.

So we take $E = \frac{1}{2} m v^2 + \frac{1}{2} \omega_0^2 x^2$ (from $\Phi = \frac{1}{2} \omega_0^2 x^2$), and then from Poissons equation

$$
\rho(x) = \frac{1}{4 \pi G} \frac{d^2 \Phi}{dx^2} = \frac{\omega_0^2}{4 \pi G} \quad (6.14)
$$
Phase space \((x, v)\) orbit is an ellipse determined entirely by \(E\), so all orbits with the same \(E\) lie on top of each other.

Then \(f(E)\) just determines how many orbits there are of a given amplitude.

Note though that the contribution to the density at \(x = 0\) is different for each \(E\), since \(v\) there increases with \(E\), so ones with higher \(E\) spend less time there.

The question is now: can we find \(f(E)\) which gives \(\rho = \rho_0 = \text{constant}\) out to some \(x_0\)?
Application of Jeans theorem

Harmonic oscillator potential

Let

\[ \psi = -\Phi + \Phi_0 = C - \frac{1}{2} \omega_0^2 x^2 \]

\[ \mathcal{E} = -E + \Phi_0 = C - \frac{1}{2} \omega_0^2 x^2 - \frac{1}{2} v^2 \]

At \( x = x_0 \) need \( v = 0 \), so choose \( C = \frac{1}{2} \omega_0^2 x_0^2 \)

\[ \mathcal{E} = \frac{1}{2} \omega_0^2 x_0^2 - \frac{1}{2} \omega_0^2 x^2 - \frac{1}{2} v^2 = \psi - \frac{1}{2} v^2 \quad (6.15) \]

Then

\[ f > 0 \text{ for } \mathcal{E} > 0 \]

\[ f = 0 \text{ for } \mathcal{E} \leq 0 \]

\[ \rho(x) = \int_0^\infty f dv = \int_0^\infty \sqrt{2\psi(x)} \]

\[ fdv \]
Application of Jeans theorem

Harmonic oscillator potential

In terms of $\mathcal{E}$ we use $-vdv = d\mathcal{E}$, with limits $v = 0 \leftrightarrow \mathcal{E} = \Psi$ and $v = \sqrt{2\Psi} \leftrightarrow \mathcal{E} = 0$

to obtain

$$\rho(x) = \int_{0}^{\Psi(x)} \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{2(\Psi(x) - \mathcal{E})}}$$

where

$$\Psi(x) = \frac{1}{2} \omega_{0}^{2}(x_{0}^{2} - x^{2}).$$

[Note that 1-D differs from 3-D for this]

In fact it is easier to use the $v$ equation, i.e.

$$\rho(x) = \int_{0}^{\sqrt{\omega_{0}^{2}(x_{0}^{2} - x^{2})}} f \left( \frac{1}{2} \omega_{0}^{2}(x_{0}^{2} - x^{2}) - \frac{1}{2} v^{2} \right) \, dv \quad (6.16)$$

Now need to find a function $f$ which gives us constant $\rho$. We can do this by trial and error, or inspired guesswork...
Try $f = \text{constant} = f_0$. Then

$$\rho(x) = [f_0 v]_0 \sqrt{\frac{\omega^2_0 (x_0^2 - x^2)}{\omega^2_0 (x_0^2 - x^2) - v^2}} = f_0 \sqrt{\frac{\omega^2_0 (x_0^2 - x^2)}{\omega^2_0 (x_0^2 - x^2) - v^2}}$$

which is not constant, so we have chosen the wrong $f$.

So try $f = \frac{k}{\sqrt{\varepsilon}}$, where $k$ is a constant.

$$\rho(x) = \int_0^{\sqrt{\frac{\omega^2_0 (x_0^2 - x^2)}{\omega^2_0 (x_0^2 - x^2) - v^2}}} \frac{\sqrt{2} k \, dv}{\sqrt{\omega^2_0 (x_0^2 - x^2) - v^2}}$$

$$= \left[ \sqrt{2} k \sin^{-1}\left( \frac{v}{\sqrt{\omega^2_0 (x_0^2 - x^2)}} \right) \right]_0^{\frac{\omega^2_0 (x_0^2 - x^2)}}$$

$$= \frac{k \pi}{\sqrt{2}} = \text{constant as required}$$
Ellipticals either have “cores” or “extra light”

Lauer et al 2007, HST data
Application of Jeans theorem

Surface brightness profiles of elliptical galaxies

"Extra light" = shells of accreted material
Application of Jeans theorem

Dark Matter only N-body simulations

Universal DM radial density profile discovered

Moore et al, 1999
Application of Jeans theorem

Baryonic physics affects Dark Matter

Cusps are turned into cores with supernova feedback

Pontzen & Governato, 2011
Application of Jeans theorem

Two-Power Law Density Models

The two-power law models motivated by the measurements of the light profile of elliptical galaxies and by the results of dark matter N-body simulations.

\[
\rho(r) = \frac{\rho_0}{(r/a)^\alpha(1 + r/a)^{\beta-\alpha}} \tag{6.17}
\]

For several $\alpha$ and $\beta$ there are models with particularly simple analytic properties. For example

- $\beta = 4$ Dehnen (Dehnen 1993)
- $\alpha = 1, \beta = 4$ Hernquist (Hernquist 1990)
- $\alpha = 2, \beta = 4$ Jaffe (Jaffe 1983)
- $\alpha = 1, \beta = 3$ NFW (Navarro, Frenk & White 1993)
- $1 < \alpha < 1.5, \beta \simeq 3$ for dark haloes
Application of Jeans theorem

Two-Power Law Density Models

Circular speed versus radius

- **Jaffe**
- **NFW**
- **Hernquist**

Graph showing circular speed $v_c$/$(4\pi G \rho_0 a^3)$ versus radius $r/a$ for different density models.
Application of Jeans theorem

Two-Power Law Density Models

Distribution functions for simple two-power law models

\[ \log(GM/\rho)^{3/2} \]

\[ \delta/(GM/a) \]
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

These still have one spatial coordinate, but note that the orbits are not just radial.

A simple form of the distribution function is

\[
f = \begin{cases} 
  F \epsilon^{-n-\frac{3}{2}} & \epsilon > 0 \\
  0 & \epsilon \leq 0 
\end{cases} \quad (6.18)
\]

where \( F \) is a constant.
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

\[ f = \begin{cases} 
  F E^{n-\frac{3}{2}} & \mathcal{E} > 0 \\
  0 & \mathcal{E} \leq 0 
\end{cases} \]

Then

\[ \rho(r) = 4\pi \int_{0}^{\infty} f(\Psi - \frac{1}{2}v^2)v^2 \, dv \]

with \( \Psi = \Psi(r) \). So

\[ \rho(r) = 4\pi F \int_{0}^{\sqrt{2\Psi}} (\Psi - \frac{1}{2}v^2)^{n-\frac{3}{2}}v^2 \, dv \quad (6.19) \]
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

\[
\rho(r) = 4\pi F \int_0^{\sqrt{2\Psi}} \left(\Psi - \frac{1}{2} v^2\right)^{n-\frac{3}{2}} v^2 \, dv
\]

Let

\[v^2 = 2\Psi \cos^2 \theta\]

so

\[v \, dv = -2\Psi \cos \theta \sin \theta \, d\theta\]
\[v^2 \, dv = -(2\Psi)^{\frac{3}{2}} \cos^2 \theta \sin \theta \, d\theta\]

Limits are \(v = 0 \leftrightarrow \theta = \frac{\pi}{2}\) and \(v = \sqrt{2\Psi} \leftrightarrow \theta = 0\)

\[\Rightarrow \rho(r) = 4\pi F \int_{0}^{\frac{\pi}{2}} \Psi^{n-\frac{3}{2}} \sin^{2n-3} \theta (2\Psi)^{\frac{3}{2}} \cos^2 \theta \sin \theta \, d\theta \quad (6.20)\]
### Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

\[
\rho(r) = 4\pi F \int_0^{\frac{\pi}{2}} \psi^{n-\frac{3}{2}} \sin^{2n-3} \theta (2\psi)^{\frac{3}{2}} \cos^2 \theta \sin \theta \, d\theta
\]

\[
= 2^{\frac{7}{2}} \pi F \psi^n \left[ \int_0^{\frac{\pi}{2}} \sin^{2n-2} \theta \, d\theta - \int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, d\theta \right]
\]

\[
= C_n \psi^n \quad \text{where } \psi > 0 \quad \text{(otherwise 0)} \quad (6.21)
\]

where

\[
C_n = \frac{(2\pi)^{\frac{3}{2}} (n - \frac{3}{2})! F}{n!} \quad (6.22)
\]

Note that for \( C_n \) to be finite we need \( n - \frac{3}{2} > -1 \Rightarrow n > \frac{1}{2} \) since \((n - \frac{3}{2})! = \Gamma(n - \frac{1}{2})\), and \( \Gamma(x) \) is finite for \( x > 0 \).
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

Gamma function:

\[ \Gamma(z + 1) = \int_0^\infty t^z e^{-t} \, dt, \quad \Gamma(1) = \Gamma(2) = 1 \]

Integration by parts gives

\[ \Gamma(z + 1) = z\Gamma(z) \Rightarrow \Gamma(z + 1) = z! \text{ for integer } z. \]

Also have (Euler's reflection formula)

\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} = \int_0^\infty \frac{t^{z-1}}{1 + t} \, dt \]

\[ \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \]

\[ \Gamma(z) \text{ has simple poles at } z = 0, -1, -2 \ldots \]

\[ \Rightarrow C_n \text{ finite requires } n > \frac{1}{2} \text{ for} \]

\[ C_n = \frac{(2\pi)^{3/2} (n - \frac{3}{2})! F}{n!} \]
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

\[ \rho(r) = C_n \Psi^n \text{ where } \Psi > 0 \text{ (otherwise 0)} \]

Now we can substitute the expression for \( \rho \) into Poisson’s equation, so

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Psi}{dr} \right) = -4\pi G C_n \Psi^n \quad (6.23) \]

We can rescale this, so \( s = \frac{r}{b} \), where

\[ b = \left( 4\pi G \Psi_0^{n-1} C_n \right)^{-\frac{1}{2}} \quad (6.24) \]

\( \psi = \Psi / \Psi_0 \) with \( \Psi_0 = \Psi(0) \) and then

\[ \frac{1}{s^2} \frac{d}{ds} \left( s^2 \frac{d\psi}{ds} \right) = \left\{ \begin{array}{ll} -\psi^n & \psi > 0 \\ 0 & \psi \leq 0 \end{array} \right\} \quad (6.25) \]

\( (\Psi \leq 0 \Rightarrow \mathcal{E} \leq 0 \Rightarrow f = 0 \Rightarrow \rho = 0) \)
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

$$\frac{1}{s^2} \frac{d}{ds} \left( s^2 \frac{d\psi}{ds} \right) = \begin{cases} -\psi^n & \psi > 0 \\ 0 & \psi \leq 0 \end{cases}$$

This is the **Lane-Emden equation**, which you are familiar with from the fluids course.

The boundary conditions are: at $s = 0$ $\psi = 1$ by definition, and $\frac{d\psi}{ds} = 0$ because there is no gravitational force at $s = 0$.

The equation for $\psi(r)$ is the same as the equation for $\rho(r)$ for a star with an equation of state $p = K\rho^{1+\frac{1}{n}}$. And we know there are analytic solutions for $n = 0, 1, 5$, and that the one with $n = 5$ has infinite radius. Here we need $n > \frac{1}{2}$.

What we have done here is chosen $f(\mathcal{E})$, and then obtained the differential equation to solve for $\Psi$ and hence $\rho$. 
This is the model with $n = 5$. Solution is

$$\psi = \frac{1}{\sqrt{1 + \frac{1}{3}s^2}}$$  \hspace{1cm} (6.26)

It satisfies the boundary conditions, and you can check it satisfies

$$\frac{1}{s^2} \frac{d}{ds} \left( s^2 \frac{d\psi}{ds} \right) = -\psi^5$$  \hspace{1cm} (6.27)

$$\Rightarrow$$

$$\rho = C_5 \psi^5 = \frac{c_5 \psi_0^5}{(1 + \frac{1}{3}s^2)^{\frac{5}{2}}}$$  \hspace{1cm} (6.28)

so the density extends to $\infty$. 
But the mass

\[
M = \int_0^\infty 4\pi \rho r^2 \, dr
\]

\[
= -\int_0^\infty \frac{1}{G} \frac{d}{dr} \left( r^2 \frac{d\Psi}{dr} \right) \, dr
\]

\[
= \frac{1}{G} \left[ r^2 \frac{d\Psi}{dr} \right]_0^\infty
\]

\[
= \lim_{r \to \infty} -\frac{1}{G} \left( r^2 \frac{d\Psi}{dr} \right)
\]

\[
= -\frac{b}{G} \left( s^2 \frac{d\Psi}{ds} \right)_{s \to \infty}
\]

\[
= \frac{b \Psi_0}{G} \text{ which is finite}
\]
This is quite a good model of most globular clusters, and (for the light profiles) of dwarf spheroidal galaxies. But not so good for E0 galaxies because $\rho \sim r^{-5}$ at large radii.
Application of Jeans theorem

Stellar density in dwarf spheroidals

![Graph showing stellar density in dwarf spheroidals](image-url)
Application of Jeans theorem

Constant velocity dispersion in dwarfs

Application of Jeans theorem

- Obtaining self-consistent models
- Eddington Formula
- Harmonic oscillator potential
- Spherically symmetric solutions of the collisionless Boltzmann equation
- Plummer potential
- Isothermal sphere

Walker et al
Application of Jeans theorem

Isothermal sphere

\[ i.e. \quad \sigma^2(r) = \text{constant}. \]

This is the limit \( n \to \infty \) (as in fluids, where \( p = k \rho^{1 + \frac{1}{n}} \) with \( n \to \infty \Rightarrow p = K \rho \)), but it is easier to start again.

Assume that the distribution function is Maxwellian with constant velocity dispersion, so guess

\[
\begin{align*}
    f(\mathcal{E}) &= \frac{\rho_1}{(2\pi \sigma^2)^{\frac{3}{2}}} \exp \left( \frac{\mathcal{E}}{\sigma^2} \right) \\
    &= \frac{\rho_1}{(2\pi \sigma^2)^{\frac{3}{2}}} \exp \left( \frac{\Psi(r) - \frac{1}{2}v^2}{\sigma^2} \right)
\end{align*}
\]

where \( \rho_1 \) is a constant.

\[ \Rightarrow \]

\[
\rho(r) = \int_0^\infty 4\pi v^2 f(v) dv = \rho_1 \exp \left( \frac{\Psi}{\sigma^2} \right)
\]

(6.29)
Application of Jeans theorem

Isothermal sphere

\[ \rho(r) = \int_0^\infty 4\pi v^2 f(v) dv = \rho_1 \exp \left( \frac{\psi}{\sigma^2} \right) \]

which means

\[ \psi = \sigma^2 (\ln \rho - \ln \rho_1) \]

Poisson’s equation

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) = -4\pi G \rho_1 \exp \left( \frac{\psi}{\sigma^2} \right) \]

is then

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \ln \rho \right) = - \frac{4\pi G}{\sigma^2} \rho \quad \text{(6.30)} \]

One solution to this equation is

\[ \rho(r) = \frac{\sigma^2}{2\pi G r^2} \quad \text{(6.31)} \]

(which you can easily check).
Application of Jeans theorem

Isothermal sphere

\[ \rho(r) = \frac{\sigma^2}{2\pi G r^2} \]

This is called the **Singular Isothermal Sphere**.

- \( \rho \to \infty \) as \( r \to 0 \) (singular)
- \( M(r) = \frac{2\sigma^2 r}{G} \to \infty \) as \( r \to \infty \) (awkward)
- \( \Sigma(R) = \frac{\sigma^2}{2GR} \)
- \( \Phi(r) = 2\sigma^2 \ln(r) + \text{constant} \)
Application of Jeans theorem

We’d prefer a solution which is well behaved at the origin, so $\Psi \to \text{constant and } \frac{d\Psi}{dr} \to 0$ there. It is convenient to rescale the variables first, so

$$\tilde{\rho} = \rho / \rho_0$$

and

$$\tilde{r} = r / r_0$$

where

$$r_0 = \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}}$$

Then in terms of the new variables the Poisson’s equation (6.30) becomes

$$\frac{1}{\tilde{r}^2} \frac{d}{d\tilde{r}} \left( \tilde{r}^2 \frac{d}{d\tilde{r}} \ln \tilde{\rho} \right) = -9\tilde{\rho} \quad (6.32)$$

with boundary conditions $\tilde{\rho}(0) = 1$ and $\frac{d\tilde{\rho}}{d\tilde{r}}\bigg|_{\tilde{r}=0} = 0$. 

Isothermal sphere
Application of Jeans theorem

Isothermal sphere

\[ \frac{1}{\tilde{r}^2} \frac{d}{d\tilde{r}} \left( \tilde{r}^2 \frac{d}{d\tilde{r}} \ln \tilde{\rho} \right) = -9\tilde{\rho} \]

This is a numerical problem (see Fig 4-7 from Binney & Tremaine).

At large radii \( r \gg r_0 \) have \( \rho \propto r^{-2} \) and \( M(r) \approx \frac{2\sigma^2}{G} r \) so \( M \to \infty \) and \( v_{\text{escape}} = \infty \).

It is of interest to calculate the mean square speed of the stars:

\[
\overline{v^2} = \frac{\int_0^\infty f(\mathcal{E}) v^2 4\pi v^2 \, dv}{\int_0^\infty f(\mathcal{E}) 4\pi v^2 \, dv} = \frac{\int_0^\infty \exp \left( \frac{-1}{2} \frac{v^2}{\sigma^2} \right) v^2 4\pi v^2 \, dv}{\int_0^\infty \exp \left( \frac{-1}{2} \frac{v^2}{\sigma^2} \right) 4\pi v^2 \, dv}
\]

Let \( x^2 = \frac{v^2}{2\sigma^2} \), and noting that \( \exp \Psi \) terms cancel

\[
= 2\sigma^2 \frac{\int_0^\infty e^{-x^2} x^4 \, dx}{\int_0^\infty e^{-x^2} x^2 \, dx}
\]
Application of Jeans theorem

Isotropical sphere

[These are fairly standard:

\[
\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}
\]

\[
\frac{d}{d\alpha} : -\int_0^\infty x^2 e^{-\alpha x^2} dx = -\frac{\sqrt{\pi}}{4} \alpha^{-\frac{3}{2}}
\]

\[
\frac{d}{d\alpha} : \int_0^\infty x^4 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{4} \frac{3}{2} \alpha^{-\frac{5}{2}}
\]

Hence

\[
\overline{v^2} = 2\sigma^2 \times \frac{3}{2} = 3\sigma^2
\]

So \(\sigma\) is the one-dimensional velocity dispersion.
Application of Jeans theorem

Isothermal sphere

**Figure 4.6** Volume ($\rho/\rho_0$) and projected ($\Sigma/\rho_0 r_0$) mass densities of the isothermal sphere. The dotted lines show the volume- and surface-density profiles of the singular isothermal sphere. The dashed curve shows the surface density of the modified Hubble model (4.109a).