Stellar Dynamics and Structure of Galaxies

Orbits in a given potential

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Before we start modelling stellar systems

Collisions
Model requirements

Basics

Newton’s law
Orbits
Orbits in spherical potentials
Equation of motion in two dimensions
Path of the orbit
Energy per unit mass
Kepler’s Laws
Unbound orbits
Escape velocity

Binary star orbits

General orbit under radial force law

Orbital periods
Example
Collisions

Do we have to worry about collisions?

Globular clusters look densest, so obtain a rough estimate of collision timescale for them
Collisions in globular clusters

The case of NGC 2808

\[ \rho_0 \sim 8 \times 10^4 \, \text{M}_\odot \, \text{pc}^{-3} \]
\[ M_* \sim 0.8 \, \text{M}_\odot. \]
\[ \Rightarrow n_0 \sim 10^5 \, \text{pc}^{-3} \] is the star number density.
We have \( \sigma_r \sim 13 \, \text{km s}^{-1} \) as the typical 1D speed of a star, so the 3D speed is
\[ \sim \sqrt{3} \times \sigma_r \left( = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \right) \sim 20 \, \text{km s}^{-1}. \]
Since \( M_* \propto R_* \) (see Fluids, or Stars, course notes), have \( R_* \sim 0.8R_\odot. \)
Collisions in globular clusters
The case of NGC 2808

For a collision, need the volume $\pi(2R_*)^2 \sigma t_{coll}$ to contain one star, i.e.

$$n_0 = 1/(\pi(2R_*)^2 \sigma t_{coll})$$ \hspace{1cm} (1.1)$$

or

$$t_{coll} = 1/(4\pi R_*^2 \sigma n_0)$$ \hspace{1cm} (1.2)$$

Putting in the numbers gives $t_{coll} \sim 10^{14}$ yr.

So direct collisions between stars are rare, but if you have $\sim 10^6$ stars then there is a collision every $\sim 10^8$ years, so they do happen.

Note that NGC 2808 is 10 times denser than typical

So, for now, ignore collisions, and we are left with stars orbiting in the potential from all the other stars in the system.
Model requirements

Model (e.g., a globular cluster) just as a self-gravitating collection of objects. Have a gravitational potential well $\Phi(r)$, approximately smooth if the number of particles $\gg 1$. Conventionally take $\Phi(\infty) = 0$.

Stars orbit in the potential well, with time per orbit (for a globular cluster)

$\sim 2R_h/\sigma \sim 10^6$ years $\ll$ age.

Stars give rise to $\Phi(r)$ by their mass, so for this potential in a steady state could average each star over its orbit to get $\rho(r)$.

The key problem is therefore self-consistently building a model which fills in the terms:

$$\Phi(r) \rightarrow \text{stellar orbits} \rightarrow \rho(r) \rightarrow \Phi(r)$$

Note that in most observed cases we only have $v_{\text{line of sight}}(R)$, so it is even harder to model real systems.

Remember how to measure age for globular clusters?

Self-consistent $=$ orbits & stellar mass give $\rho$, which leads to $\Phi$, which supports the orbits used to construct $\rho$. 
The law of attraction

Newton’s laws of motion and Newtonian gravity

GR not needed, since

- $10 < v < 10^3 \text{ km/s} \ll c = 3 \times 10^5 \text{ km/s}$
- $\frac{GM}{rc^2} \sim ???

The gravitational force per unit mass acting on a body due to a mass $M$ at the origin is

$$f = -\frac{GM}{r^2} \hat{r} = -\frac{GM}{r^3} r$$

(1.4)

We can write this in terms of a potential $\Phi$, using

$$\nabla \left( \frac{1}{r} \right) = -\frac{1}{r^2} \hat{r}$$

(1.5)
The corresponding potential

So

\[ \mathbf{f} = - \nabla \Phi \] (1.6)

where \( \Phi \) is a scalar,

\[ \Phi = \Phi(r) = -\frac{GM}{r} \] (1.7)

Hence the potential due to a point mass \( M \) at \( r = r_1 \) is

\[ \Phi(r) = -\frac{GM}{|r - r_1|} \] (1.8)
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**Binary star orbits**

- General orbit under radial force law

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**Density vs Potential**

From Hayashi et al, “The shape of the gravitational potential in cold dark matter haloes”
Orbits

The law of motion
Orbits

Particle of constant mass $m$ at position $\mathbf{r}$ subject to a force $\mathbf{F}$. Newton’s law:

$$\frac{d}{dt} (m\dot{\mathbf{r}}) = \mathbf{F} \quad (1.9)$$

i.e.

$$m\ddot{\mathbf{r}} = \mathbf{F} \quad (1.10)$$

If $\mathbf{F}$ is due to a gravitational potential $\Phi(\mathbf{r})$, then

$$\mathbf{F} = m\mathbf{f} = -m\nabla \Phi \quad (1.11)$$

The angular momentum about the origin is $\mathbf{H} = \mathbf{r} \times (m\dot{\mathbf{r}})$. Then

$$\frac{d\mathbf{H}}{dt} = \mathbf{r} \times (m\ddot{\mathbf{r}}) + m\dot{\mathbf{r}} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{F} \equiv \mathbf{G} \quad (1.12)$$

where $\mathbf{G}$ is the torque about the origin.
Orbits

Particle of constant mass \( m \) at position \( r \) subject to a force \( F \). Newton’s law:

\[
\frac{d}{dt} (m \dot{r}) = F \tag{1.9}
\]

i.e.

\[
m \ddot{r} = F \tag{1.10}
\]

If \( F \) is due to a gravitational potential \( \Phi(r) \), then

\[
F = mf = -m \nabla \Phi \tag{1.11}
\]

The angular momentum about the origin is \( H = r \times (m \dot{r}) \). Then

\[
\frac{dH}{dt} = r \times (m \ddot{r}) + m \dot{r} \times \dot{r} = r \times F \equiv G \tag{1.12}
\]

where \( G \) is the torque about the origin.
The kinetic energy

\[ T = \frac{1}{2} m \dot{r} \cdot \dot{r} \]  

(1.13)

\[ \frac{dT}{dt} = m \dot{r} \cdot \ddot{r} = F \cdot \dot{r} \]  

(1.14)

If \( F = -m \nabla \Phi \), then

\[ \frac{dT}{dt} = -m \dot{r} \cdot \nabla \Phi(r) \]  

(1.15)

But if \( \Phi \) is independent of \( t \), the rate of change of \( \Phi \) along an orbit is

\[ \frac{d}{dt} \Phi(r) = \nabla \Phi \cdot \dot{r} \]  

(1.16)

from the chain rule
The kinetic energy

\[ T = \frac{1}{2} m \dot{r} \cdot \dot{r} \quad (1.13) \]

\[ \frac{dT}{dt} = m \dot{r} \cdot \ddot{r} = F \cdot \dot{r} \quad (1.14) \]

If \( F = -m \nabla \Phi \), then

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But if \( \Phi \) is independent of \( t \), the rate of change of \( \Phi \) along an orbit is

\[ \frac{d}{dt} \Phi(r) = \nabla \Phi \cdot \dot{r} \quad (1.16) \]
Hence
\[
\frac{dT}{dt} = -m \frac{d}{dt} \Phi(r) 
\] (1.17)
\[
\Rightarrow m \frac{d}{dt} \left( \frac{1}{2} \dot{r} \cdot \dot{r} + \Phi(r) \right) = 0 
\] (1.18)
\[
\Rightarrow E = \frac{1}{2} \dot{r} \cdot \dot{r} + \Phi(r) 
\] (1.19)

The total energy is constant for a given orbit.
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Hence

$$\frac{dT}{dt} = -m \frac{d}{dt} \Phi(r) \quad (1.17)$$

$$\Rightarrow m \frac{d}{dt} \left( \frac{1}{2} \mathbf{r} \cdot \mathbf{r} + \Phi(r) \right) = 0 \quad (1.18)$$

$$\Rightarrow E = \frac{1}{2} \mathbf{r} \cdot \mathbf{r} + \Phi(r) \quad (1.19)$$

The total energy is constant for a given orbit
Orbits in spherical potentials

\[ \Phi(r) = \Phi(|r|) = \Phi(r), \] so \( \mathbf{f} = -\nabla \Phi = -\hat{r} \frac{d\Phi}{dr}. \)

The orbital angular momentum \( \mathbf{H} = m\mathbf{r} \times \dot{\mathbf{r}}, \) and

\[ \frac{d\mathbf{H}}{dt} = \mathbf{r} \times m\mathbf{f} = -m \frac{d\Phi}{dr} \mathbf{r} \times \hat{r} = 0. \] (1.20)

So the angular momentum per unit mass \( \mathbf{h} = \mathbf{H}/m = \mathbf{r} \times \dot{\mathbf{r}} \) is a constant vector, and is perpendicular to \( \mathbf{r} \) and \( \dot{\mathbf{r}} \)

\[ \Rightarrow \text{the particle stays in a plane through the origin which is perpendicular to } \mathbf{h} \]

Check: \( \mathbf{r} \perp \mathbf{h}, \mathbf{r} + \delta\mathbf{r} = \mathbf{r} + \dot{\mathbf{r}} \delta t \perp \mathbf{h} \) since both \( \mathbf{r} \) and \( \dot{\mathbf{r}} \perp \mathbf{h}, \) so particle remains in the plane.

Thus the problem becomes a two-dimensional one to calculate the orbit use 2-D cylindrical coordinates \((R, \phi, z)\) at \( z = 0, \) or spherical polars \((r, \theta, \phi)\) with \( \theta = \frac{\pi}{2}. \)

So, in 2D, use \((R, \phi)\) and \((r, \phi)\) interchangeably.
Equation of motion in two dimensions

The equation of motion in two dimensions can be written in radial angular terms, using \( r = r \hat{r} = r \hat{e}_r + 0 \hat{e}_\phi \), so \( r = (r, 0) \).

We know that

\[
\frac{d}{dt} \hat{e}_r = \dot{\phi} \hat{e}_\phi
\]  \( (1.21) \)

and

\[
\frac{d}{dt} \hat{e}_\phi = -\dot{\phi} \hat{e}_r
\]  \( (1.22) \)

\[
\hat{e}_r = \cos(\phi) \hat{e}_x + \sin(\phi) \hat{e}_y
\]

\[
\hat{e}_\phi = -\sin(\phi) \hat{e}_x + \cos(\phi) \hat{e}_y
\]

\[
\frac{d}{dt} \hat{e}_r = -\sin(\phi) \dot{\phi} \hat{e}_x + \cos(\phi) \dot{\phi} \hat{e}_y
\]

\[
\frac{d}{dt} \hat{e}_\phi = -\cos(\phi) \dot{\phi} \hat{e}_x - \sin(\phi) \dot{\phi} \hat{e}_y
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Equation of motion in two dimensions

The equation of motion in two dimensions can be written in radial angular terms, using 
\[ r = r \hat{r} = r \hat{e}_r + 0 \hat{e}_\phi, \] 
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We know that

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and

\[
\frac{d}{dt} \hat{e}_\phi = -\dot{\phi} \hat{e}_r \tag{1.22}
\]

\[
\hat{e}_r = \cos(\phi) \hat{e}_x + \sin(\phi) \hat{e}_y
\]

\[
\hat{e}_\phi = -\sin(\phi) \hat{e}_x + \cos(\phi) \hat{e}_y
\]

\[
\frac{d}{dt} \hat{e}_r = -\sin(\phi) \dot{\phi} \hat{e}_x + \cos(\phi) \dot{\phi} \hat{e}_y
\]

\[
\frac{d}{dt} \hat{e}_\phi = -\cos(\phi) \dot{\phi} \hat{e}_x - \sin(\phi) \dot{\phi} \hat{e}_y
\]
Equation of motion in two dimensions

Hence

\[ \dot{r} = \dot{r} \hat{e}_r + r \dot{\phi} \hat{e}_\phi \]  

(1.23)

or \( \dot{r} = \mathbf{v} = (\dot{r}, \dot{\phi}) \)

and so

\[ \ddot{r} = \ddot{r} \hat{e}_r + \dot{r} \dot{\phi} \hat{e}_\phi + r \ddot{\phi} \hat{e}_\phi + r \dot{\phi}^2 \hat{e}_r \]

\[ = (\ddot{r} - r \dot{\phi}^2) \hat{e}_r + \frac{1}{r} \frac{d}{dt} \left( r^2 \dot{\phi} \right) \hat{e}_\phi \]

\[ = \mathbf{a} = [\ddot{r} - r \dot{\phi}^2, \frac{1}{r} \frac{d}{dt} \left( r^2 \dot{\phi} \right)] \]

(1.24)

In general \( \mathbf{f} = (f_r, f_\phi) \), and then \( f_r = \ddot{r} - r \dot{\phi}^2 \), where the second term is the centrifugal force, since we are in a rotating frame, and the torque \( rf_\phi = \frac{d}{dt} \left( r^2 \dot{\phi} \right) \) (\( = \mathbf{r} \times \mathbf{f} \)).

In a spherical potential \( f_\phi = 0 \), so \( r^2 \dot{\phi} \) is constant.
Path of the orbit

To determine the shape of the orbit we need to remove $t$ from the equations and find $r(\phi)$. It is simplest to set $u = 1/r$, and then from $r^2 \dot{\phi} = h$ obtain

$$\dot{\phi} = hu^2 \tag{1.25}$$

Then

$$\dot{r} = -\frac{1}{u^2} \dot{u} = -\frac{1}{u^2} \frac{du}{d\phi} \dot{\phi} = -h \frac{du}{d\phi} \tag{1.26}$$

and

$$\ddot{r} = -h \frac{d^2 u}{d\phi^2} \dot{\phi} = -h^2 u^2 \frac{d^2 u}{d\phi^2}. \tag{1.27}$$
Path of the orbit

So the radial equation of motion

\[ \ddot{r} - r \dot{\phi}^2 = f_r \]

becomes

\[ -h^2 u^2 \frac{d^2 u}{d\phi^2} - \frac{1}{u} h^2 u^4 = f_r \]  \hspace{1cm} (1.28)

\[ \Rightarrow \frac{d^2 u}{d\phi^2} + u = -\frac{f_r}{h^2 u^2} \]  \hspace{1cm} (1.29)

The orbit equation in spherical potential
Since $f_r$ is just a function of $r$ (or $u$) this is an equation for $u(\phi)$, i.e. $r(\phi)$ - the path of the orbit. Note that it does not give $r(t)$, or $\phi(t)$ - you need one of the other equations for those.

If we take $f_r = -\frac{GM}{r^2} = -GMu^2$, then

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} \quad (1.30)$$

(which is something you will have seen in the Relativity course).
Kepler orbits
Solution to the equation of motion

The solution to this equation is

\[ \frac{\ell}{r} = \ell u = 1 + e \cos(\phi - \phi_0) \]  

which you can verify simply by putting it in the differential equation. Then

\[ -\frac{e \cos(\phi - \phi_0)}{\ell} + \frac{1 + e \cos(\phi - \phi_0)}{\ell} = \frac{GM}{h^2} \]

so \( \ell = \frac{h^2}{GM} \) and \( e \) and \( \phi_0 \) are constants of integration.
Galaxies Part II

Before we start modelling stellar systems

Basics

Newton’s law
Orbits
Orbits in spherical potentials
Equation of motion in two dimensions

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Energy per unit mass
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Escape velocity

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Bound orbits

\[ \frac{1}{r} = \frac{1 + e \cos(\phi - \phi_0)}{\ell} \]

Note that if \( e < 1 \) then \( 1/r \) is never zero, so \( r \) is bounded in the range \( \frac{\ell}{1+e} < r < \frac{\ell}{1-e} \).

Also, in all cases the orbit is symmetric about \( \phi = \phi_0 \), so we take \( \phi_0 = 0 \) as defining the reference line for the angle \( \phi \). \( \ell \) is the distance from the origin for \( \phi = \pm \frac{\pi}{2} \) (with \( \phi \) measured relative to \( \phi_0 \)).
Kepler orbits

Bound orbits

We can use different parameters. Knowing that the point of closest approach (perihelion for a planet in orbit around the Sun, periastron for something about a star) is at $\ell/(1 + e)$ when $\phi = 0$ and the aphelion (or whatever) is at $\ell/(1 - e)$ when $\phi = \pi$, we can set the distance between these two points (= major axis of the orbit) = $2a$. Then

$$\frac{\ell}{1 + e} + \frac{\ell}{1 - e} = 2a \Rightarrow \ell(1 - e) + \ell(1 + e) = 2a(1 - e^2) \tag{1.32}$$

$$\Rightarrow \ell = a(1 - e^2) \tag{1.33}$$

$\Rightarrow r_P = a(1 - e)$ is the perihelion distance from the gravitating mass at the origin, and $r_a = a(1 + e)$ is the aphelion distance. The distance of the Sun from the midpoint is $ae$, and the angular momentum $h^2 = GM\ell = GMa(1 - e^2)$. 
Energy per unit mass

The energy per unit mass

\[ E = \frac{1}{2} \dot{r} \cdot \dot{r} + \Phi(r) = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\phi}^2 - \frac{GM}{r} \]  

(1.34)

This is constant along the orbit, so we can evaluate it anywhere convenient - e.g. at perihelion where \( \dot{r} = 0 \). Then \( \dot{\phi} = \frac{h}{r_p} \) and so

\[
E = \frac{1}{2} \frac{GM(a(1-e^2))}{a^2(1-e)^2} - \frac{GM}{a(1-e)} \\
= \frac{GM}{a} \left[ \frac{1}{2} \left( \frac{1+e}{1-e} \right) - \frac{1}{1-e} \right] \\
= -\frac{GM}{2a}
\]

(1.35)

This is \(< 0\) for a bound orbit, and depends only on the semi-major axis \(a\) (and not \(e\)).
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Orbits in spherical potentials
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Kepler’s Laws

... deduced from observations, and explained by Newtonian theory of gravity.
Kepler’s Laws

1 Orbits are ellipses with the Sun at a focus.
2 Planets sweep out equal areas in equal time

\[ \delta A = \frac{1}{2} r^2 \delta \phi \quad [= \frac{1}{2} r (r \delta \phi)] \quad (1.36) \]

\[ \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{h}{2} = \text{constant} \quad (1.37) \]

⇒ Kepler’s second law is a consequence of a central force, since this is why $h$ is a constant.
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**Kepler’s Laws**

1. Orbits are ellipses with the Sun at a focus.
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\delta A = \frac{1}{2} r^2 \delta \phi \quad [= \frac{1}{2} r(r \delta \phi)]
\]  \hspace{1cm} (1.36)

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\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{h}{2} = \text{constant}
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Before we start modelling stellar systems

Basics
Newton’s law
Orbits
Orbits in spherical potentials
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Kepler’s Laws

3rd Law

(Period)$^2 \propto$ (size of orbit)$^3$

In one period $T$, the area swept out is $A = \frac{1}{2} hT = \left( \int_0^T \frac{dA}{dt} dt \right)$

But $A = \text{area of ellipse} = \pi ab = \pi a^2 \sqrt{1 - e^2}$

\[
A = \int_0^{2\pi} d\phi \int_0^r r dr = \int_0^{2\pi} \frac{1}{2} r^2 d\phi
\]

\[
\frac{\ell}{r} = \ell u = 1 + e \cos(\phi - \phi_0)
\]

Have

\[
\int_0^{\pi} \frac{dx}{(a + b \cos x)^2} = \frac{\pi}{a^2 - b^2} \frac{a}{\sqrt{a^2 - b^2}}
\]
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Basics

Newton’s law
Orbits
Orbits in spherical potentials
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3rd Law

so

\[ A = 2 \frac{\ell^2}{2} \frac{\pi}{1 - e^2} \frac{1}{\sqrt{1 - e^2}} \]

Since \( \ell = a(1 - e^2) \) this implies

\[ A = \pi a^2 \sqrt{1 - e^2} \]

And since \( b = a\sqrt{1 - e^2} \),

\[ A = \pi ab \]
Therefore

\[ T = \frac{2A}{h} \]

\[ = \frac{2\pi a^2 \sqrt{1 - e^2}}{h} \]

\[ = \frac{2\pi a^2 \sqrt{1 - e^2}}{\sqrt{GMa(1 - e^2)}} \]

since \( h^2 = GMa(1 - e^2) \)

\[ T = 2\pi \sqrt{\frac{a^3}{GM}} \]

\[ \Rightarrow T^2 \propto a^3 \] (1.38)

where in this case \( M \) is the mass of the Sun.

Note: Since \( E = -\frac{GM}{2a} \), the period \( T = \frac{2\pi GM}{(-2E)^{\frac{3}{2}}} \).
Unbound orbits

What happens to $\frac{\ell}{r} = 1 + e \cos \phi$ when $e \geq 1$?

- If $e > 1$ then $1 + e \cos \phi = 0$ has solutions $\phi_\infty$ where $r = \infty \rightarrow \cos \phi_\infty = -1/e$
  Then $-\phi_\infty \leq \phi \leq \phi_\infty$, and, since $\cos \phi_\infty$ is negative, $\frac{\pi}{2} < \phi_\infty < \pi$. The orbit is a hyperbola.
- If $e = 1$ then the particle just gets to infinity at $\phi = \pm \pi$ - it is a parabola.
Unbound orbits

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  Then $-\phi_\infty \leq \phi \leq \phi_\infty$, and, since $\cos \phi_\infty$ is negative, $\frac{\pi}{2} < \phi_\infty < \pi$. The orbit is a hyperbola.
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Basics
Newton’s law
Orbits
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Kepler orbits

\[ e = 0.5 \]

\[ e = 1 \]

\[ e = 2 \]

\[ e = \infty \]
Energies for these unbound orbits:

\[ E = \frac{1}{2} \dot{r}^2 + \frac{1}{2} \frac{h^2}{r^2} - \frac{GM}{r} \]

So, as \( r \to \infty \), \( E \to \frac{1}{2} \dot{r}^2 \)
Unbound orbits

Recall

\[ \frac{\ell}{r} = 1 + e \cos \phi \]

\[ \frac{d}{dt} \text{ of this } \Rightarrow \]

\[ -\frac{\ell}{r^2} \ddot{r} = -e \sin \phi \dot{\phi} \]

and since \( h = r^2 \dot{\phi} \)

\[ \dot{r} = \frac{eh}{\ell} \sin \phi \]

As \( r \to \infty \cos \phi \to -1/e \)

\[ E \to \frac{1}{2} \dot{r}^2 = \frac{1}{2} \frac{e^2 h^2}{\ell^2} \left( 1 - \frac{1}{e^2} \right) = \frac{GM}{2\ell} (e^2 - 1) \]

(recalling that \( h^2 = GM\ell \)) Thus \( E > 0 \) if \( e > 1 \) and for parabolic orbits (\( e = 1 \)) \( E = 0 \).
We have seen that in a fixed potential $\Phi(r)$ a particle has constant energy $E = \frac{1}{2}r^2 + \Phi(r)$ along an orbit. If we adopt the usual convention and take $\Phi(r) \to 0$ as $|r| \to \infty$, then if at some point $r_0$ the particle has velocity $v_0$ such that

$$\frac{1}{2}v_0^2 + \Phi(r_0) > 0$$

then it is able to reach infinity. So at each point $r_0$ we can define an escape velocity $v_{\text{esc}}$ such that

$$v_{\text{esc}} = \sqrt{-2\Phi(r_0)}$$
The escape velocity from the Sun

\[ v_{\text{esc}} = \left( \frac{2GM_\odot}{r_0} \right)^{\frac{1}{2}} = 42.2 \left( \frac{r_0}{\text{a.u.}} \right)^{-\frac{1}{2}} \text{ km s}^{-1} \]

Note: The circular velocity \( v_{\text{circ}} \) is such that \(-r\dot{\phi}^2 = -\frac{GM}{r^2}\)

\[ r\dot{\phi} = v_{\text{circ}} = \sqrt{\frac{GM_\odot}{r_0}} = 29.8 \left( \frac{r_0}{\text{a.u.}} \right)^{-\frac{1}{2}} \text{ km s}^{-1} \]

(= 2\(\pi\) a.u./yr).

\( v_{\text{esc}} = \sqrt{2}v_{\text{circ}} \) for a point mass source of the gravitational potential.
Before we start modelling stellar systems

Basics
Newton’s law
Orbits
Orbits in spherical potentials
Equation of motion in two dimensions
Path of the orbit
Energy per unit mass
Kepler’s Laws
Unbound orbits
Escape velocity

Binary star orbits

General orbit under radial force law

**Escape velocity**

From the Galaxy

**Triple-star System Passes Near Milky Way's Central Black Hole**

1. Triple-star system moves near black hole at center of Milky Way galaxy.
2. One star falls toward black hole; binary pair recoils and is ejected.
3. Binary system leaves galaxy.
4. Binary merges to form blue straggler.
5. Blue straggler travels away from galaxy.
Kepler orbits
Binary star orbits

- What we have done so far is assume a potential due to a fixed point mass which we take as being at the origin of our polar coordinates. We now wish to consider a situation in which we have two point masses, $M_1$ and $M_2$ both moving under the gravitational attraction of the other.
- This is a cluster of $N$ stars where $N = 2$ and we can solve it exactly! Hooray!
- The potential is no longer fixed at origin

$$\Phi(r) = -\frac{GM_1}{|r - r_1|} - \frac{GM_2}{|r - r_2|}$$
Binary star orbits

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Binary star orbits

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\[
\Phi(r) = -\frac{GM_1}{|r-r_1|} - \frac{GM_2}{|r-r_2|}
\]
Binary star orbits

Or the force acting on star 1, due to star 2 is

\[ F_1 = \frac{GM_1 M_2}{|r_1 - r_2|^2} \]

in the direction of \( r_2 - r_1 \)

\[ \Rightarrow F_1 = \frac{GM_1 M_2}{|r_1 - r_2|^3} (r_2 - r_1) \]

And by symmetry (or Newton’s 3rd law)

\[ F_2 = \frac{GM_1 M_2}{|r_1 - r_2|^3} (r_1 - r_2) \]
Then we know

\[ \ddot{r}_1 = -\frac{GM_1 M_2}{d^2} \hat{d} \] (1.39)

and

\[ \ddot{r}_2 = -\frac{GM_1 M_2}{d^2} \left(-\hat{d}\right) \] (1.40)

where

\[ d = r_1 - r_2 \] (1.41)

is the vector from \( M_2 \) to \( M_1 \).

Using these two we can write for \( \ddot{d} = \ddot{r}_1 - \ddot{r}_2 \)

\[ \ddot{d} = -\frac{G(M_1 + M_2)}{d^2} \hat{d} \] (1.42)
Before we start modelling stellar systems

Basics

Binary star orbits

General orbit under radial force law

**Binary star orbits**

\[
\ddot{d} = -\frac{G(M_1 + M_2)}{d^2} \dot{d}
\]

which is identical to the equation of motion of a particle subject to a fixed mass \(M_1 + M_2\) at the origin.

So we know that the period

\[
T = 2\pi \sqrt{\frac{a^3}{G(M_1 + M_2)}}
\]

(1.43)

where the size (maximum separation) of the relative orbit is \(2a\).
Galaxies Part II

Before we start modelling stellar systems

Basics

Binary star orbits

General orbit under radial force law

Binary star orbits

\[ \ddot{d} = - \frac{G(M_1 + M_2)}{d^2} \dot{d} \]

which is identical to the equation of motion of a particle subject to a fixed mass \( M_1 + M_2 \) at the origin.

So we know that the period

\[ T = 2\pi \sqrt{\frac{a^3}{G(M_1 + M_2)}} \quad (1.43) \]

where the size (maximum separation) of the relative orbit is \( 2a \).
If we take the coordinates for the centre of mass

\[ \mathbf{r}_{CM} = \frac{M_1}{M_1 + M_2} \mathbf{r}_1 + \frac{M_2}{M_1 + M_2} \mathbf{r}_2 \]  

(1.44)

From equations (1.39) and (1.40) we know that

\[ M_1 \ddot{r}_1 + M_2 \ddot{r}_2 = 0 \]  

(1.45)

and so

\[ \frac{d}{dt} (M_1 \dot{r}_1 + M_2 \dot{r}_2) = 0 \]  

(1.46)

or

\[ (M_1 \dot{r}_1 + M_2 \dot{r}_2) = \text{constant} \]  

(1.47)

i.e. \( \dot{\mathbf{r}}_{CM} = \text{constant} \).

We can choose an inertial frame in which the centre of mass has zero velocity, so might as well do so.
Binary star orbits

If we take the coordinates for the centre of mass

\[ r_{\text{CM}} = \frac{M_1}{M_1 + M_2} \mathbf{r}_1 + \frac{M_2}{M_1 + M_2} \mathbf{r}_2 \]  \hspace{1cm} (1.44)

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We can choose an inertial frame in which the centre of mass has zero velocity, so might as well do so
Binary star orbits

Note that choosing $r_{CM} = 0 \Rightarrow M_1 r_1 = -M_2 r_2$, and so $r_1 = d + r_2 = d - \frac{M_1}{M_2} r_1$

This $\Rightarrow r_1 = \frac{M_2}{M_1 + M_2} d$ and similarly $r_2 = -\frac{M_1}{M_1 + M_2} d$.

The angular momentum $J$ (or $H$ if you want) is

$$ J = M_1 r_1 \times \dot{r}_1 + M_2 r_2 \times \dot{r}_2 $$

$$ = \frac{M_1 M_2}{(M_1 + M_2)^2} d \times \dot{d} + \frac{M_2 M_1}{(M_1 + M_2)^2} d \times \dot{d} $$

$$ = \frac{M_1 M_2}{M_1 + M_2} d \times \dot{d} $$

(1.48)

So

$$ J = \mu h $$

(1.49)

where $\mu$ is the reduced mass, and $h$ is the specific angular momentum.
Note that choosing $r_{CM} = 0 \Rightarrow M_1 r_1 = -M_2 r_2$, and so $r_1 = d + r_2 = d - \frac{M_1}{M_2} r_1$

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The angular momentum $J$ (or $H$ if you want) is

$$J = M_1 r_1 \times \dot{r}_1 + M_2 r_2 \times \dot{r}_2$$

$$= \frac{M_1 M_2^2}{(M_1 + M_2)^2} d \times \dot{d} + \frac{M_2 M_1^2}{(M_1 + M_2)^2} d \times \dot{d}$$

$$= \frac{M_1 M_2}{M_1 + M_2} d \times \dot{d}$$

(1.48)

So

$$J = \mu h$$

(1.49)

where $\mu$ is the reduced mass, and $h$ is the specific angular momentum.
Binary star orbits

Momentum loss due to mass loss

Galaxies Part II

Before we start modelling stellar systems

Basics

Binary star orbits

General orbit under radial force law
Binary star orbits

Momentum loss due to Gravitational Radiation
Before we start modelling stellar systems

Basics

Binary star orbits

General orbit under radial force law

Binary star orbits

Momentum loss due to Gravitational Radiation

Russell A. Hulse

Joseph H. Taylor Jr.

The Nobel Prize in Physics 1993 was awarded jointly to Russell A. Hulse and Joseph H. Taylor Jr. "for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation"

Photos: Copyright © The Nobel Foundation

Question: predict the evolution of the pulsar’s orbit.
Before we start modelling stellar systems, let's discuss the basics of binary star orbits. The general orbit under radial force law is crucial in understanding the dynamics of such systems.

Momentum loss due to Gravitational Radiation

**Figure 2.** Orbital decay caused by the loss of energy by gravitational radiation. The parabola depicts the expected shift of periastron time relative to an unchanging orbit, according to general relativity. Data points represent our measurements, with error bars mostly too small to see.

Weisberg and Taylor 2010.
Galaxies Part II

Before we start modelling stellar systems

Basics

Binary star orbits

General orbit under radial force law

Binary star orbits

Binary Super-massive Black holes
General orbit under radial force law

Remember the orbit equation?

\[
\frac{d^2 u}{d\phi^2} + u = -\frac{f\left(\frac{1}{u}\right)}{h^2 u^2}
\]  

(1.50)

where \( u \equiv \frac{1}{r} \) and \( f_r = f \) for a spherical potential.

For \( f \) from a gravitational potential, we have

\[
f\left(\frac{1}{u}\right) = -\frac{d\Phi}{dr} = u^2 \frac{d\Phi}{du}
\]  

(1.51)

since gravity is conservative.

There are two types of orbit:

- **Unbound**: \( r \to \infty \), \( u \geq 0 \) as \( \phi \to \phi_\infty \)
- **Bound**: \( r \) (and \( u \)) oscillate between finite limits.
General orbit under radial force law

Energy

If we take (1.50) \( \times \frac{du}{d\phi} \):

\[
\frac{du}{d\phi} \frac{d^2 u}{d\phi^2} + u \frac{du}{d\phi} + \frac{u^2}{h^2 u^2} \frac{d\Phi}{du} \frac{du}{d\phi} = 0 \quad (1.52)
\]

\[
\Rightarrow \frac{d}{d\phi} \left[ \frac{1}{2} \left( \frac{du}{d\phi} \right)^2 + \frac{1}{2} u^2 + \frac{\Phi}{h^2} \right] = 0 \quad (1.53)
\]

and integrating over \( \phi \) we have

\[
\Rightarrow \frac{1}{2} \left( \frac{du}{d\phi} \right)^2 + \frac{1}{2} u^2 + \frac{\Phi}{h^2} = \text{constant} = \frac{E}{h^2} \quad (1.54)
\]
and using \( h = r^2 \dot{\phi} \)

\[
\frac{E}{h^2} = \frac{1}{2} \left( \frac{du}{d\phi} \right)^2 + \frac{1}{2} u^2 + \frac{\Phi}{h^2}
\]

\[
E = \frac{r^4 \dot{\phi}^2}{2} \left( \frac{du}{d\phi} \right)^2 + \frac{1}{2} r^2 \dot{\phi}^2 + \Phi(r)
\]

\[
= \frac{r^4}{2} \left( \frac{du}{dt} \right)^2 + \frac{1}{2} r^2 \dot{\phi}^2 + \Phi(r)
\]

\[
= \frac{r^4}{2} \left( \frac{du}{dr} \right)^2 + \frac{1}{2} r^2 \dot{\phi}^2 + \Phi(r)
\]

\[
= \frac{1}{2} i^2 + \frac{1}{2} r^2 \dot{\phi}^2 + \Phi(r)
\]

i.e. we can show that the constant \( E \) we introduced is the energy per unit mass.
Galaxies Part II

Before we start modelling stellar systems

Basics

Binary star orbits

General orbit under radial force law

Orbital periods

Example

**General orbit under radial force law**

Peri and Apo

\[
\frac{E}{h^2} = \frac{1}{2} \frac{d}{d\phi} \left( \frac{du}{d\phi} \right)^2 + \frac{1}{2} u^2 + \frac{\Phi}{h^2}
\]

For bound orbits, the limiting values of \( u \) (or \( r \)) occur where \( \frac{du}{d\phi} = 0 \), i.e. where

\[
u^2 = \frac{2E - 2\Phi(u)}{h^2}
\]  \hspace{1cm} (1.56)

from (1.54).

This has two roots, \( u_1 = \frac{1}{r_1} \) and \( u_2 = \frac{1}{r_2} \)

This is not obvious, since \( \Phi \) is not defined, but it can be proved - it is an Example!

For \( r_1 < r_2 \), where \( r_1 \) is the pericentre, \( r_2 \) the apocentre
The radial period $T_r$ is defined as the time to go from $r_2 \rightarrow r_1 \rightarrow r_2$.

Now take (1.55) and re-write:

\[
\left( \frac{dr}{dt} \right)^2 = 2(E - \Phi(r)) - \frac{h^2}{r^2} \tag{1.57}
\]

where we used $h = r^2 \dot{\phi}$ to eliminate $\dot{\phi}$

So

\[
\frac{dr}{dt} = \pm \sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}} \tag{1.58}
\]

(two signs - $\dot{r}$ can be either $> 0$ or $< 0$, and $\dot{r} = 0$ at $r_1$ & $r_2$.

Then

\[
T_r = \phi dt = 2 \int_{r_1}^{r_2} \frac{dt}{dr} \, dr = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}} \tag{1.59}
\]
The radial period $T_r$ is defined as the time to go from $r_2 \rightarrow r_1 \rightarrow r_2$.

Now take (1.55) and re-write:

$$\left( \frac{dr}{dt} \right)^2 = 2(E - \Phi(r)) - \frac{h^2}{r^2}$$

(1.57)

where we used $h = r^2 \dot{\phi}$ to eliminate $\dot{\phi}$

So

$$\frac{dr}{dt} = \pm \sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}$$

(1.58)

(two signs - $\dot{r}$ can be either $> 0$ or $< 0$, and $\dot{r} = 0$ at $r_1$ & $r_2$.

Then

$$T_r = \oint_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}}$$

(1.59)
If travelling from $r_2 \rightarrow r_1 \rightarrow r_2$ $\phi$ is increased by an amount

$$\Delta \phi = \phi d\phi = 2 \int_{r_1}^{r_2} \frac{d\phi}{dr} dr = 2 \int_{r_1}^{r_2} \frac{d\phi}{dt} \frac{dt}{dr} dr$$

(1.60)

so

$$\Delta \phi = 2h \int_{r_1}^{r_2} \frac{dr}{r^2 \sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}}$$

(1.61)
Precession of the orbit

For a given orbit, the time taken to go around once (i.e. $0 \rightarrow 2\pi$) depends in general on where you start, so the azimuthal period is not well defined. Instead use the mean angular velocity $\bar{\omega} = \Delta \phi / T_r$ to obtain a mean azimuthal period $T_\phi$, so

$$T_\phi = 2\pi / \bar{\omega} \Rightarrow T_\phi = \frac{2\pi}{\Delta \phi} T_r$$

is the mean time to go around once.

Note that unless $\Delta \phi / 2\pi$ is a rational number the orbit is not closed.
Precession of the orbit

For Keplerian orbit $\Delta \phi = 2\pi \Rightarrow T_r = T_\phi$.
In one period $T_r$ the apocentre (or pericentre) advances by an angle $\Delta \phi - 2\pi$. i.e. the orbit shifts in azimuth at an average rate given by the mean precession rate

$$\Omega_p = \frac{\Delta \phi - 2\pi}{T_r} \text{ rad s}^{-1} \quad (1.62)$$

Thus the precession period is

$$T_p = \frac{2\pi}{\Omega_p} = \frac{T_r}{\frac{\Delta \phi}{2\pi} - 1} \quad (1.63)$$

This precession is in the sense opposite to the rotation of the star

In the special case of a Keplerian orbit $\Delta \phi = 2\pi \Rightarrow T_\phi = T_r$ and $\Omega_p = 0$, i.e. orbits are closed and do not precess. Otherwise general orbit is a rosette between $r_1$ & $r_2$. This allows us to visualize how we can build a galaxy out of stars on different orbits.
Precession of the orbit

Figure 3.1 A typical orbit in a spherical potential (the isochrone, eq. 2.47) forms a rosette.
Before we start modelling stellar systems

Binary star orbits

General orbit under radial force law

Orbital periods

Example

**Example**

\(T_r\) for the Keplerian case \(\Phi(r) = -\frac{GM}{r}\)

We have equation (1.59)

\[
T_r = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}} \tag{1.59}
\]

Now \(r_1\) & \(r_2\) are determined from \(\dot{r} = 0\), i.e.

\[
2(E - \Phi(r)) - \frac{h^2}{r^2} = 0 \tag{1.64}
\]

\[
2E + \frac{2GM}{r} - \frac{h^2}{r^2} = 0 \tag{1.65}
\]

\[
r^2 + \frac{GM}{r} r - \frac{h^2}{2E} = 0 \tag{1.66}
\]

\[
\Leftrightarrow (r - r_1)(r - r_2) = 0 \tag{1.67}
\]

\[
\Rightarrow r_1r_2 = -\frac{h^2}{2E}; \quad r_1 + r_2 = -\frac{GM}{E} \tag{1.68}
\]

(remember \(E < 0\) for a bound orbit).
Before we start modelling stellar systems

**Basics**

Binary star orbits

General orbit under radial force law

Orbital periods

Example

---

\[ T_r \text{ for the Keplerian case } \Phi(r) = -\frac{GM}{r} \]

We have equation (1.59)

\[
T_r = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}}
\]

Now \( r_1 \) & \( r_2 \) are determined from \( \dot{r} = 0 \), i.e.

\[
2(E - \Phi(r)) - \frac{h^2}{r^2} = 0 \quad (1.64)
\]

\[
2E + \frac{2GM}{r} - \frac{h^2}{r^2} = 0 \quad (1.65)
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\[
r^2 + \frac{GM}{E} r - \frac{h^2}{2E} = 0 \quad (1.66)
\]

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\Leftrightarrow (r - r_1)(r - r_2) = 0 \quad (1.67)
\]

\[
\Rightarrow r_1r_2 = -\frac{h^2}{2E}; \quad r_1 + r_2 = -\frac{GM}{E} \quad (1.68)
\]

(remember \( E < 0 \) for a bound orbit).
Rewrite (1.59) as

\[
T_r = 2 \int_{r_1}^{r_2} \frac{rdr}{\sqrt{2E(r - r_1)(r - r_2)}} = \frac{2}{\sqrt{2|E|}} \int_{r_1}^{r_2} \frac{rdr}{\sqrt{(r_2 - r)(r - r_1)}}
\] (1.69)

if \( r_1 < r < r_2 \).

This is another of those integrals. If \( R = a + bx + cx^2 = -r^2 + (r_1 + r_2)r - r_1 r_2 \) and \( \Delta = 4ac - b^2 \) which becomes, using the variables here, \( \Delta = -(r_1 - r_2)^2 \) then

\[
\int \frac{x\,dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c} \frac{1}{\sqrt{-c}} \sin^{-1} \left( \frac{2cx + b}{\sqrt{-\Delta}} \right)
\]

for \( c < 0 \) and \( \Delta < 0 \) (See G&R 2.261 and 2.264).
The first term is 0 at $r_1$ and $r_2$ ($R = 0$ there), so

$$T_r = \frac{2}{\sqrt{2|E|}} \left[ \frac{r_1 + r_2}{2} \right] \left[ \sin^{-1} \left( \frac{-2r_2 + r_1 + r_2}{r_1 - r_2} \right) - \sin^{-1} \left( \frac{-2r_1 + r_1 + r_2}{r_1 - r_2} \right) \right]$$

$$= \frac{2}{\sqrt{2|E|}} \left[ \frac{r_1 + r_2}{2} \right] \left[ \sin^{-1}(1) - \sin^{-1}(-1) \right]$$

$$= \frac{2}{\sqrt{2|E|}} \frac{GM}{2(-E)} \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right]$$

$$= \frac{2\pi GM}{(-2E)^{\frac{3}{2}}}$$

(1.70)