Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 21: MHD Waves

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Last lecture

- Started discussion of plasmas (gases of charged particles)
- Electromagnetic forces important
- Developed equations of ideal, non-relativistic magnetohydrodynamics
- Flux freezing

This lecture : MHD Waves

- Understanding starts with analysis of basic waves
- Perturbation analysis MHD equations
- Alfven waves, fast and slow magnetosonic waves

Perturbation analysis of MHD equations

Start with basic MHD equations and assume barotropic equation of state:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} \right) = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} - \boldsymbol{\nabla} p$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$p = p(\rho)$$

Assume that equilibrium is static with uniform density, pressure, and magnetic field

$$\rho = \rho_0 + \delta\rho$$
$$p = p_0 + \delta p$$
$$\mathbf{u} = \delta\mathbf{u}$$
$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$$

Introduce perturbations and linearize:

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\delta\mathbf{u}) = 0$$
$$\rho_0 \frac{\partial \delta\mathbf{u}}{\partial t} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \delta\mathbf{B}) \times \mathbf{B}_0 - c_s^2 \boldsymbol{\nabla} \delta\rho$$
$$\frac{\partial \delta\mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta\mathbf{u} \times \mathbf{B}_0) = -\mathbf{B}_0(\boldsymbol{\nabla} \cdot \delta\mathbf{u}) + (\mathbf{B}_0 \cdot \boldsymbol{\nabla})\delta\mathbf{u}$$
$$\boldsymbol{\nabla} \cdot \delta\mathbf{B} = 0$$

Introduce plane wave solutions

$$\delta\rho = \delta\rho_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$
$$\delta p = \delta p_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$
$$\delta\mathbf{u} = \delta\mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$
$$\delta\mathbf{B} = \delta\mathbf{B}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

Continuity equation:

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\delta \mathbf{u}) = 0$$

$$-i\omega\delta\rho + i\rho_0 \mathbf{k} \cdot \delta\mathbf{u} = 0$$

$$\Rightarrow \qquad \omega\delta\rho = \rho_0 \mathbf{k} \cdot \delta\mathbf{u}$$

Momentum equation:

$$\rho_0 \frac{\partial \delta\mathbf{u}}{\partial t} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \delta\mathbf{B}) \times \mathbf{B}_0 - c_s^2 \boldsymbol{\nabla} \delta\rho$$

$$-i\omega\rho_0\delta\mathbf{u} = \frac{i}{\mu_0}(\mathbf{k} \times \delta\mathbf{B}) \times \mathbf{B}_0 - ic_s^2 \delta\rho\mathbf{k}$$

$$\Rightarrow \qquad \omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}\left((\mathbf{B}_0 \cdot \delta\mathbf{B})\mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k})\delta\mathbf{B}\right) + c_s^2 \delta\rho\, \mathbf{k}$$

Induction (flux freezing) equation

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta \mathbf{u} \times \mathbf{B}_0) = -\mathbf{B}_0(\boldsymbol{\nabla} \cdot \delta \mathbf{u}) + (\mathbf{B}_0 \cdot \boldsymbol{\nabla})\delta \mathbf{u}$$

$$-i\omega \delta \mathbf{B} = -i\mathbf{B}_0(\mathbf{k} \cdot \delta \mathbf{u}) + i(\mathbf{B}_0 \cdot \mathbf{k})\delta \mathbf{u}$$

$$\Rightarrow \qquad \omega \delta \mathbf{B} = \mathbf{B}_0(\mathbf{k} \cdot \delta \mathbf{u}) - (\mathbf{B}_0 \cdot \mathbf{k})\delta \mathbf{u}$$

So, pulling together…

$$\omega \delta \rho = \rho_0 \mathbf{k} \cdot \delta \mathbf{u}$$

$$\omega \rho_0 \delta \mathbf{u} = \frac{1}{\mu_0}\left((\mathbf{B}_0 \cdot \delta \mathbf{B})\mathbf{k} - (\mathbf{B}_0 \cdot \mathbf{k})\delta \mathbf{B}\right) + c_s^2 \delta \rho \, \mathbf{k}$$

$$\omega \delta \mathbf{B} = \mathbf{B}_0(\mathbf{k} \cdot \delta \mathbf{u}) - (\mathbf{B}_0 \cdot \mathbf{k})\delta \mathbf{u}$$

$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\delta\mathbf{u}$$

$$\omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}\left((\mathbf{B}_0\cdot\delta\mathbf{B})\mathbf{k} - (\mathbf{B}_0\cdot\mathbf{k})\delta\mathbf{B}\right) + c_s^2\delta\rho\,\mathbf{k}$$

$$\omega\delta\mathbf{B} = \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) - (\mathbf{B}_0\cdot\mathbf{k})\delta\mathbf{u}$$

Consider two special cases…

Case 1 : perturbation directed perpendicular to field $\mathbf{k}\perp\mathbf{B}_0$

$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\delta\mathbf{u}$$

$$\omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}(\mathbf{B}_0\cdot\delta\mathbf{B})\mathbf{k} + c_s^2\delta\rho\,\mathbf{k}$$

$$\omega\delta\mathbf{B} = \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u})$$

$$\Rightarrow \qquad \omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}B_0^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k} + c_s^2\rho_0(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}$$

$$\Rightarrow \qquad \omega^2\rho_0 = \frac{k^2B_0^2}{\mu_0} + c_s^2\rho_0k^2$$

$$\Rightarrow \qquad \omega^2 = \left(c_s^2 + \frac{B^2}{\mu_0\rho_0}\right)k^2$$

Define Alfven speed:

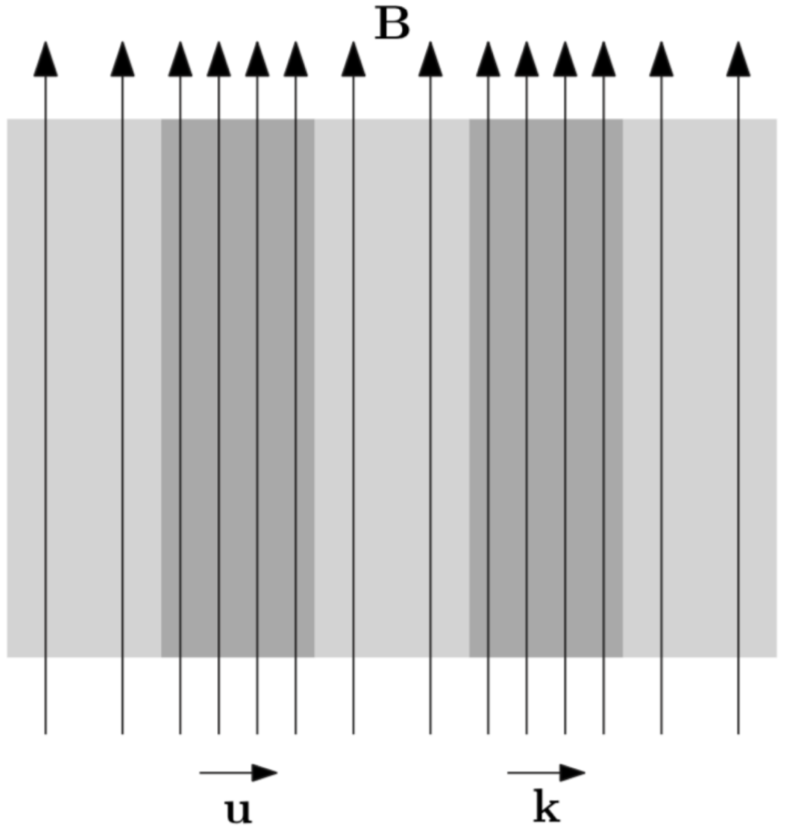
$$v_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}},$$

Then, the dispersion relation for the case $\mathbf{k} \perp \mathbf{B}_0$ is:

$$\omega^2 = \left(c_s^2 + v_A^2\right) k^2$$

This is the fast magnetosonic wave…

- Wave has $\delta\mathbf{u} \parallel \mathbf{k}$
- Wave is compressive
- Gas and magnetic pressure acting in concert



$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\delta\mathbf{u}$$
$$\omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}\left((\mathbf{B}_0\cdot\delta\mathbf{B})\mathbf{k} - (\mathbf{B}_0\cdot\mathbf{k})\delta\mathbf{B}\right) + c_s^2\delta\rho\,\mathbf{k}$$
$$\omega\delta\mathbf{B} = \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) - (\mathbf{B}_0\cdot\mathbf{k})\delta\mathbf{u}$$

Case II : perturbation directed parallel to field $\mathbf{k}\parallel\mathbf{B}_0$

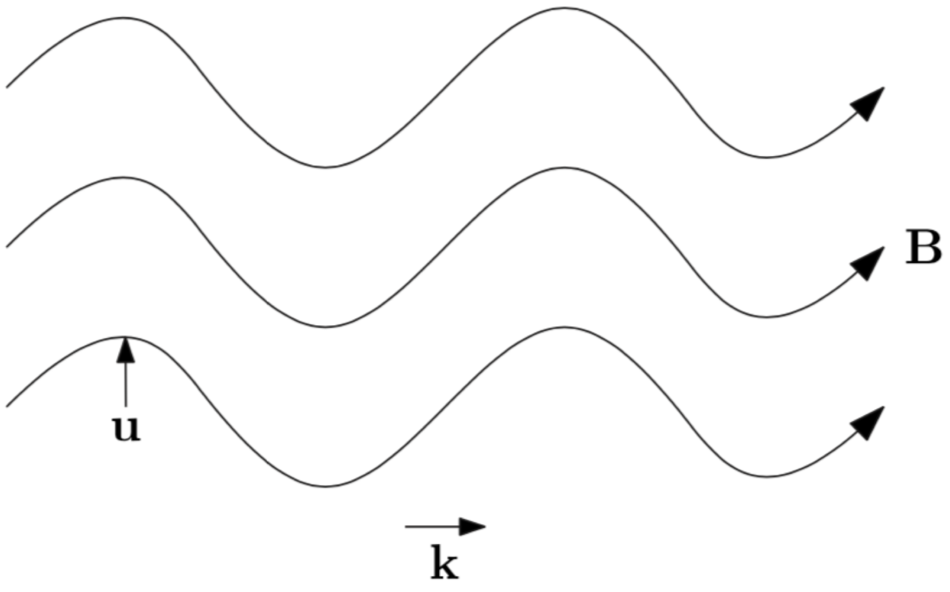
$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\delta\mathbf{u}$$
$$\omega\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}\left((\mathbf{B}_0\cdot\delta\mathbf{B})\mathbf{k} - B_0k\,\delta\mathbf{B}\right) + c_s^2\delta\rho\,\mathbf{k}$$
$$\omega\delta\mathbf{B} = \mathbf{B}_0(\mathbf{k}\cdot\delta\mathbf{u}) - B_0k\delta\mathbf{u}$$

$$\Rightarrow \qquad \omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}(B_0^2k^2\delta\mathbf{u} - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k\mathbf{k}) + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}$$

Cross with **k** $\Rightarrow \quad \omega^2 = \dfrac{B_0^2}{\mu_0\rho_0}k^2 = v_A^2k^2$

These are **Alfven waves**

- Wave is transverse
- Wave is incompressible
- Due to magnetic tension



In this case, there is another branch:

$$\omega^2\rho_0\delta\mathbf{u} = \frac{1}{\mu_0}(B_0^2k^2\delta\mathbf{u} - (\mathbf{B}_0\cdot\delta\mathbf{u})B_0k\mathbf{k}) + c_s^2(\mathbf{k}\cdot\delta\mathbf{u})\mathbf{k}$$

Dot with **k** ⇒ $\omega^2 = c_s^2k^2$

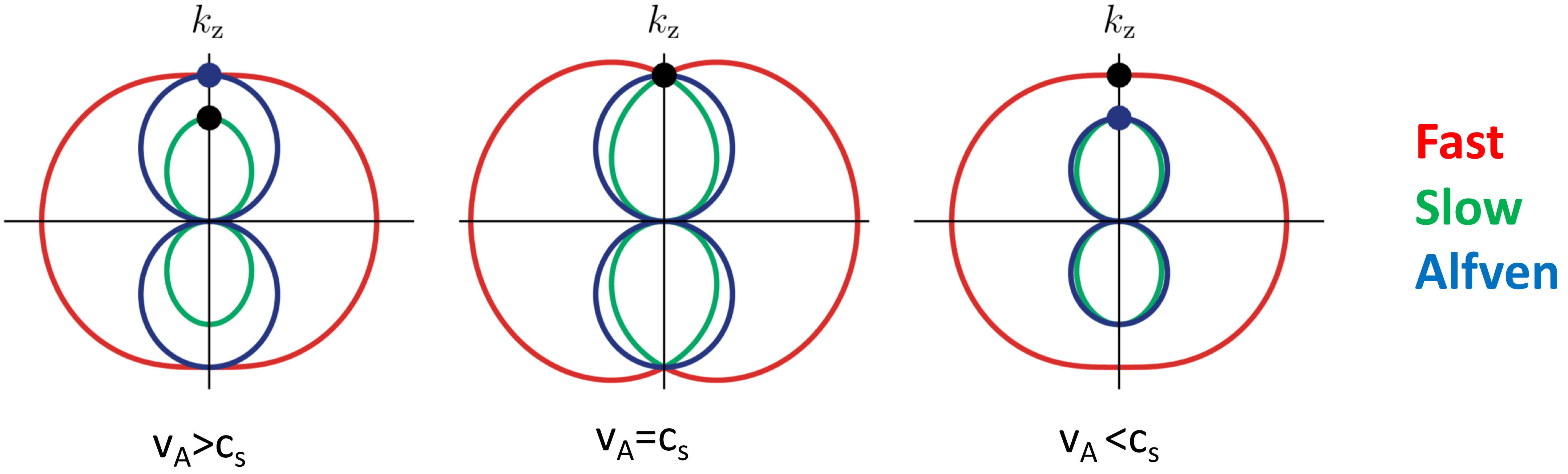
Just the **normal sound wave**... Longitudinal, compressive wave, due to gas pressure (magnetic field not perturbed).

General perturbation (**B** and **k** at some angle θ)… find three modes

- Alfven waves (phase speed goes to $\theta = \pi/2$)
- Fast magnetosonic waves
- Slow magnetosonic waves

Become degenerate when $\theta = 0$

Friedrichs diagrams showing phase speed as function of perturbation direction



Next time… back to accretion disks!