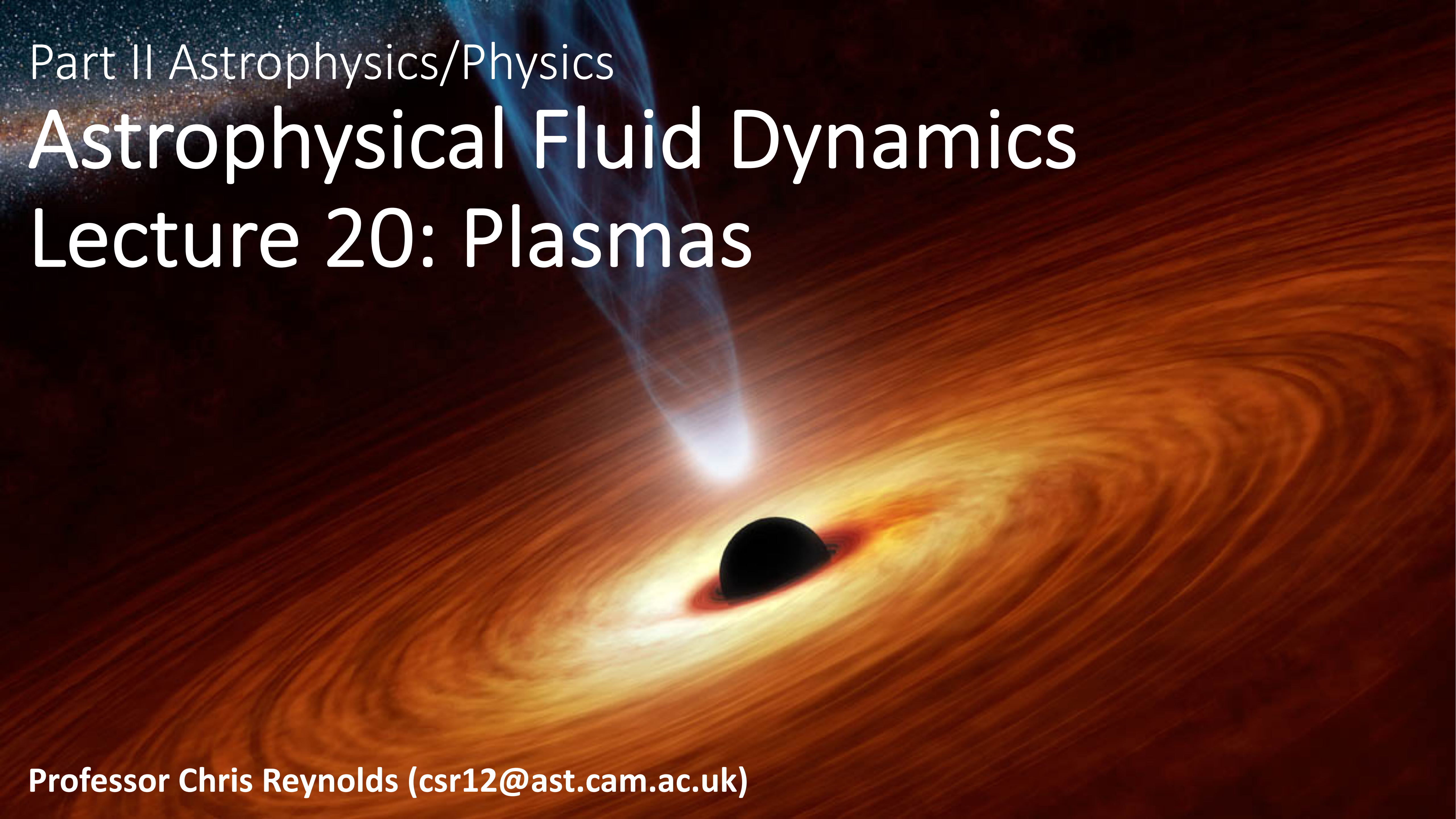


Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 20: Plasmas

Professor Chris Reynolds (csr12@ast.cam.ac.uk)

Taking stock

- Story so far…
- Ideal fluids (high collisionality limit)
  - Developed basic equations of fluid dynamics
  - Hydrostatic equilibria (& relevance to stellar structure)
  - Supersonic flows and shocks (& relevance to supernova)
  - Transonic flows and critical points (rocket nozzles, spherical accretion/winds)
  - Fluid instabilities
- Viscous flows (leading order finite-λ correction)
  - Diffusion of momentum due to velocity gradients
  - Navier-Stokes equations
  - Vorticity, pipes, and turbulence
  - Application to accretion disks

This lecture : Plasmas (Chapter J)

- Major omission so far… have neglected electromagnetic forces!
- Relevance for ionized gases
  - charge separation ⇒ electric fields
  - Relative motion of +/- charges ⇒ currents ⇒ magnetic fields
- Equations of non-relativistic magnetohydrodynamics (Chapter J.1)
- Magnetic pressure and tension (Chapter J.2)

Chapter J.1 : Magnetohydrodynamics (MHD)

For simplicity, consider fully ionized hydrogen. Think of it as being composed of two-cohabiting fluids;

1. Proton fluid (particle mass m^+, number density n^+, velocity $\mathbf{u}^+$)
2. Electron fluid (particle mass m^-, number density n^-, velocity $\mathbf{u}^-$)

Total mass density of combined fluid is $\rho = m^+n^+ + m^-n^-$

Centre-of-mass velocity is $$\mathbf{u} = \frac{m^+n^+\mathbf{u}^+ + m^-n^-\mathbf{u}^-}{m^+n^+ + m^-n^-}$$

Charge density is $q = n^+e^+ + n^-e^-$

Current density is $\mathbf{j} = e^+n^+\mathbf{u}^+ + e^-n^-\mathbf{u}^-$

Conservation of mass (or really particle number) applied to each fluid:

$$\frac{\partial n^+}{\partial t} + \boldsymbol{\nabla} \cdot \left(n^+ \mathbf{u}^+\right) = 0 \qquad (1)$$

$$\frac{\partial n^-}{\partial t} + \boldsymbol{\nabla} \cdot \left(n^- \mathbf{u}^-\right) = 0 \qquad (2)$$

Multiply (1) and (2) by particle masses and add, we get

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \qquad \text{Continuity Equation}$$

Multiple (1) and (2) by particle charges,

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \qquad \text{Charge Conservation}$$

Next to momentum equation: each particle experiences a Lorentz force

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

So, the momentum equation for each fluid will read

$$m^+ n^+ \left(\frac{\partial \mathbf{u}^+}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}^+ \right) = e^+ n^+ (\mathbf{E} + \mathbf{u}^+ \times \mathbf{B}) - f^+ \boldsymbol{\nabla} p$$

$$m^- n^- \left(\frac{\partial \mathbf{u}^-}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}^- \right) = e^- n^- (\mathbf{E} + \mathbf{u}^- \times \mathbf{B}) - f^- \boldsymbol{\nabla} p$$

Summing:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) = q\mathbf{E} + \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} p$$

So, we pick up two new electromagnetic terms in the momentum equation.

What is the current **j** for a given **E** and **B**?

Ohms law connects these quantities in terms of the electrical conductivity:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Pulling together equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = q\mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This system of equations significantly simplifies in the case of a non-relativistic and highly conducting plasma.

Suppose fields are varying over lengthscales l and timescales τ:

- Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \Rightarrow \qquad \frac{E}{B} \sim \frac{l}{\tau} \sim u$$

- Now consider ratio:

$$\left| \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right| \Big/ |\nabla \times \mathbf{B}| \sim \frac{1}{c^2} \left(\frac{l}{\tau} \right)^2 \sim \frac{u^2}{c^2} \ll 1 \qquad \text{(non-relativistic)}$$

So displacement current can be neglected and we have simply $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$

- Next compare two e/m terms from momentum equation:

$$\frac{|q\mathbf{E}|}{|\mathbf{j}\times\mathbf{B}|} \sim \frac{qE}{jB} \sim \frac{\epsilon_0 E/l}{B/l\mu_0}\frac{E}{B} \sim u^2\epsilon_0\mu_0 \sim \left(\frac{u}{c}\right)^2 \ll 1$$

So charge neutrality is preserved to a high approximation… electric fields are so strong that any charge imbalance is rapidly erased.

- Return to current and Ohm's law $\nabla\times\mathbf{B} = \mu_0\mathbf{j} = \mu_0\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B})$

Take curl:

$$\underbrace{\nabla\times(\nabla\times\mathbf{B})}_{=-\nabla^2\mathbf{B}-\nabla(\nabla\cdot\mathbf{B})\nearrow^0} = \mu_0\sigma(\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial\mathbf{B}}{\partial t}}+\nabla\times(\mathbf{u}\times\mathbf{B}))$$

$$\Rightarrow \quad \frac{\partial\mathbf{B}}{\partial t} = \underbrace{\nabla\times(\mathbf{u}\times\mathbf{B})}_{\text{advection of the field by the flow}} + \underbrace{\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B}}_{\text{dissipation of the field through the flow}}$$

c.f. behaviour of vorticity in viscous fluid

For highly conducting fluid, diffusion term is negligible and we can write

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})$$

By exact analogy with Helmholtz equation, this says that the flux of magnetic field threading some surface S moving with the flow is preserved.

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \text{constant}$$

This is the **flux-freezing condition**. Magnetic field lines are advected along in flow (see solar flare rain video).

- Look again at Ohm's law in case of highly conducting plasma:

$$\begin{aligned}
& \mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) && \text{is finite} \\
\Rightarrow \quad & \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 && \text{as } \sigma \to \infty \\
\Rightarrow \quad & \mathbf{E} \cdot \mathbf{B} = 0 && \\
\text{i.e.} \quad & \mathbf{E} \perp \mathbf{B} &&
\end{aligned}$$

So our highly conducting non-relativistic plasma is described by:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) = \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} p$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B})$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

Substituting in for current we get common form of MHD momentum equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

Chapter J.2 : Magnetic Pressure and Tension

So, the electromagnetic force per unit volume of the fluid is

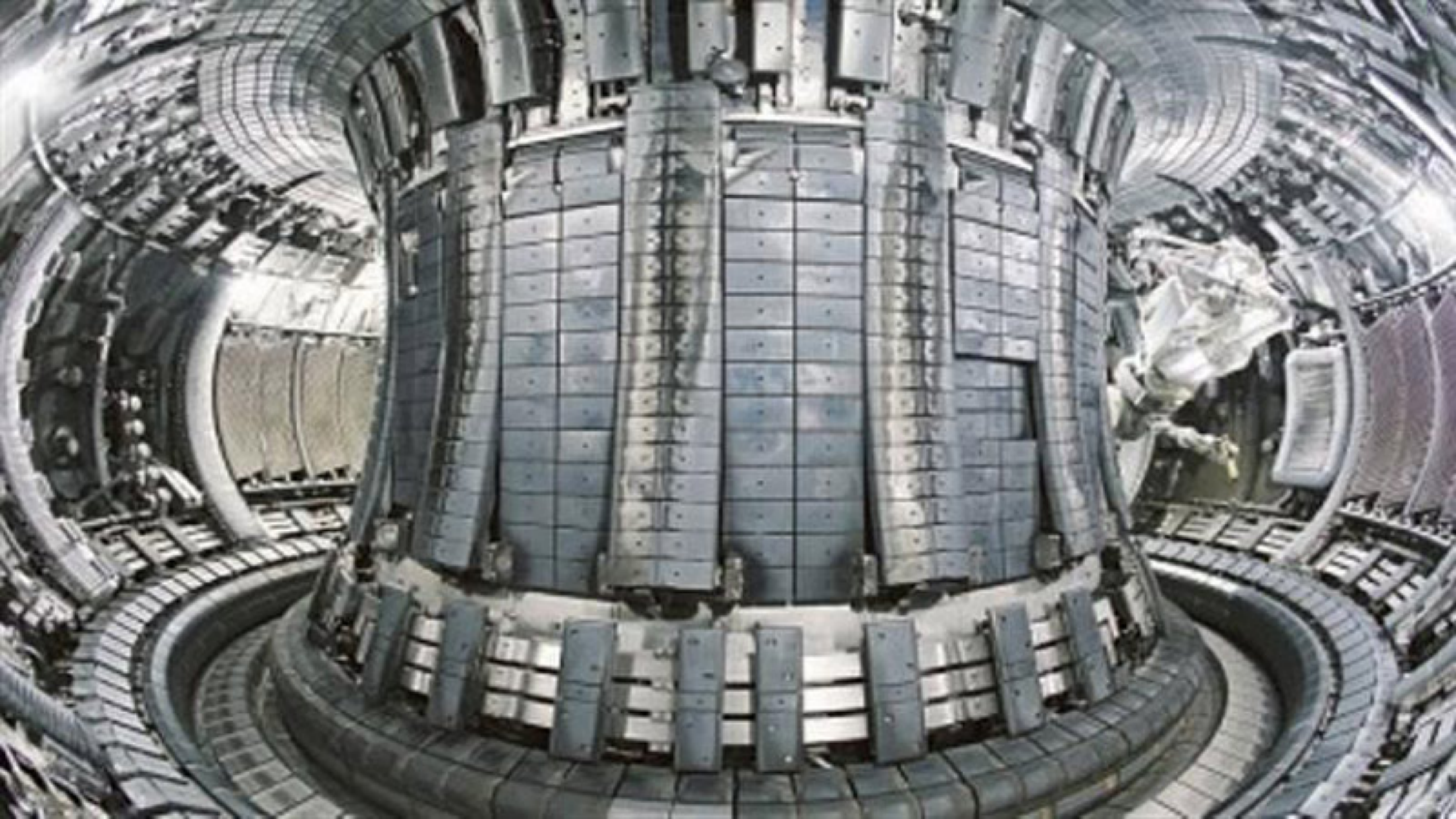
$$\mathbf{f}_{\text{mag}} = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}$$

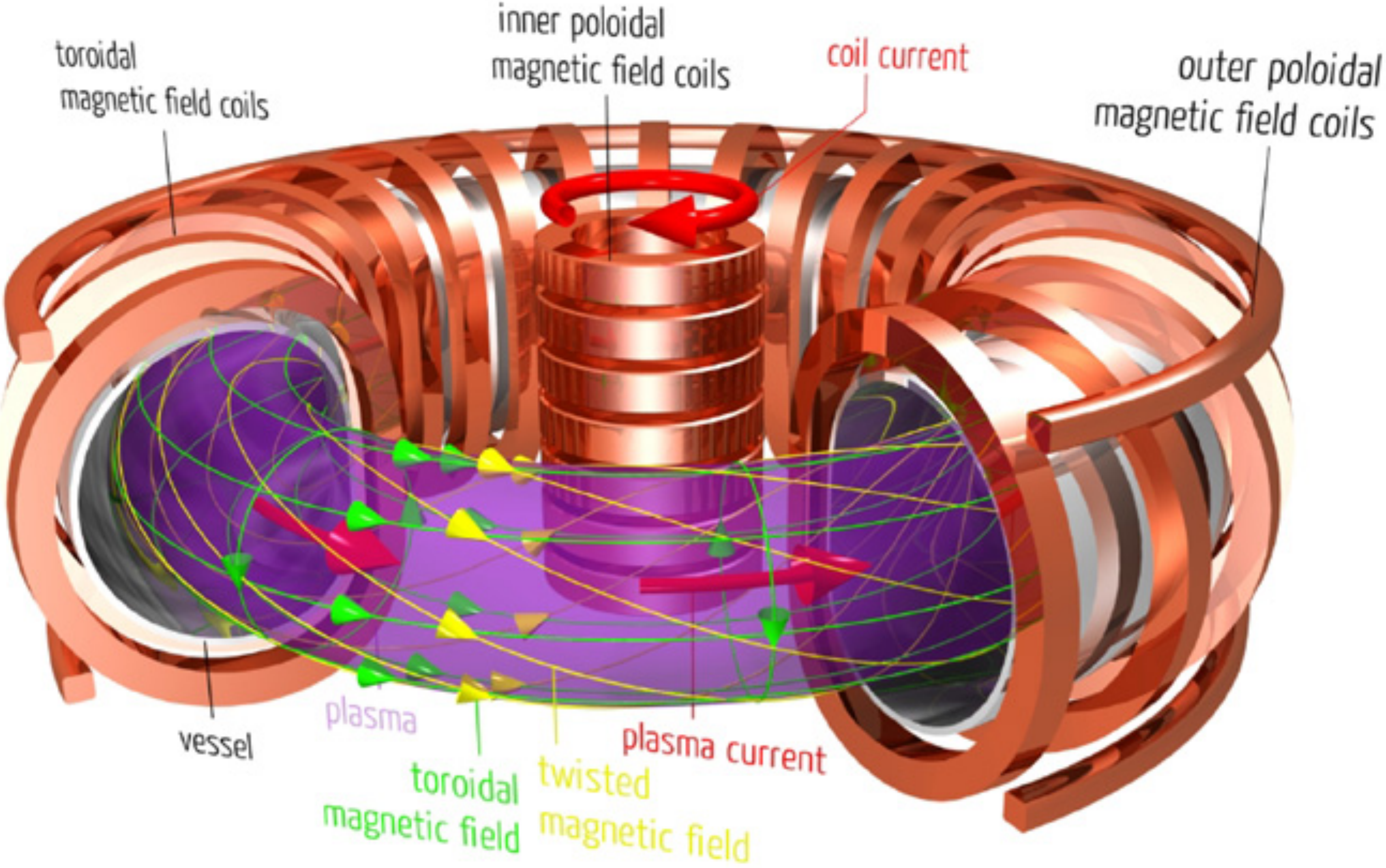
Using vector identity:

$$\mathbf{f}_{\text{mag}} = \frac{1}{\mu_0}\Bigg[\underbrace{-\boldsymbol{\nabla}\left(\frac{B^2}{2}\right)}_{\substack{\text{magnetic pressure} \\ \text{term with} \\ p_{\text{mag}}=B^2/2\mu_0}} + \underbrace{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}_{\substack{\text{magnetic tension} \\ \text{term (vanishes for} \\ \text{straight field lines)}}} \Bigg]$$

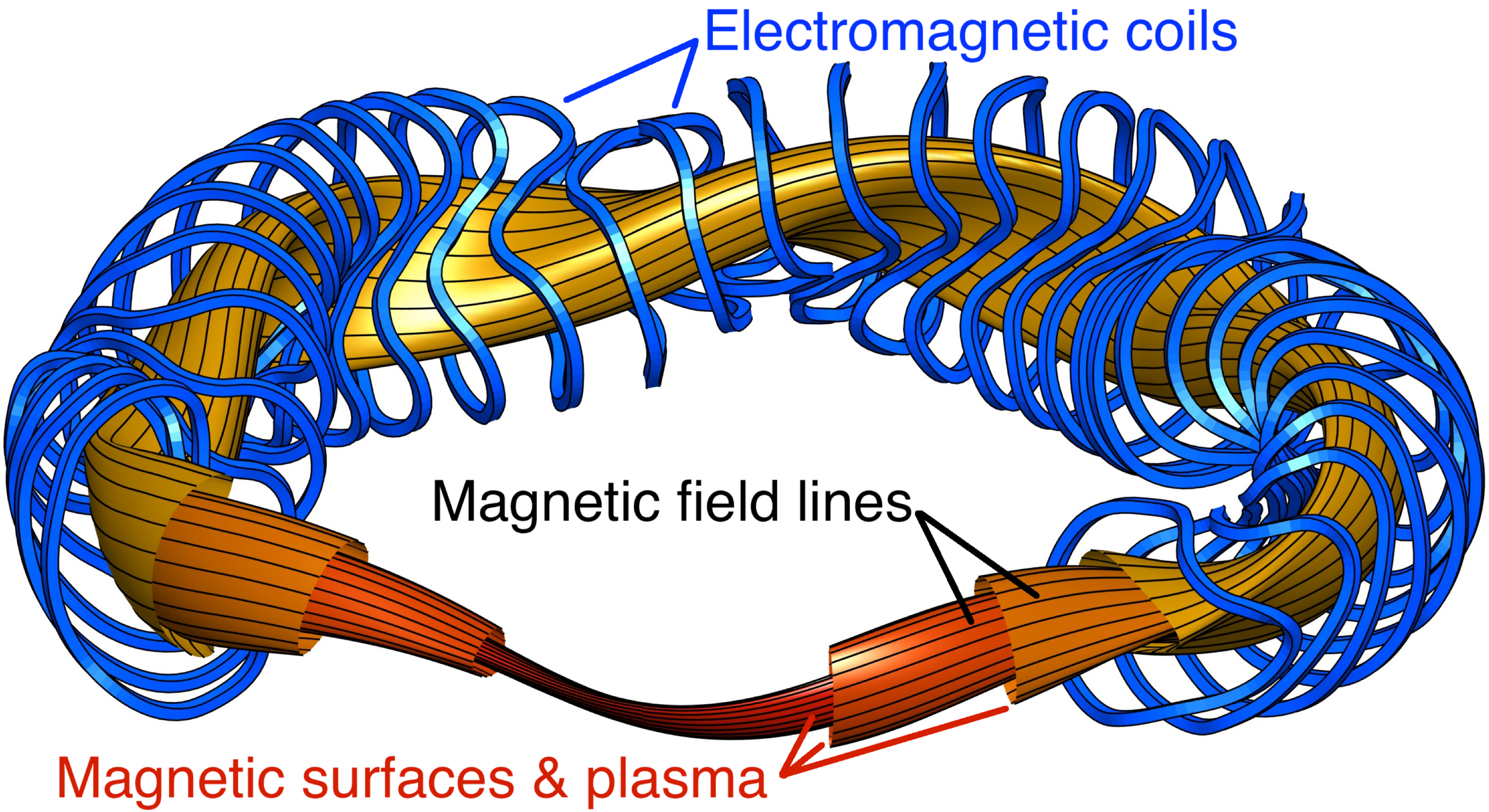
Can rewrite mtm equation:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla p_{\text{tot}} \qquad p_{\text{tot}} = p + \frac{B^2}{2\mu_0}$$

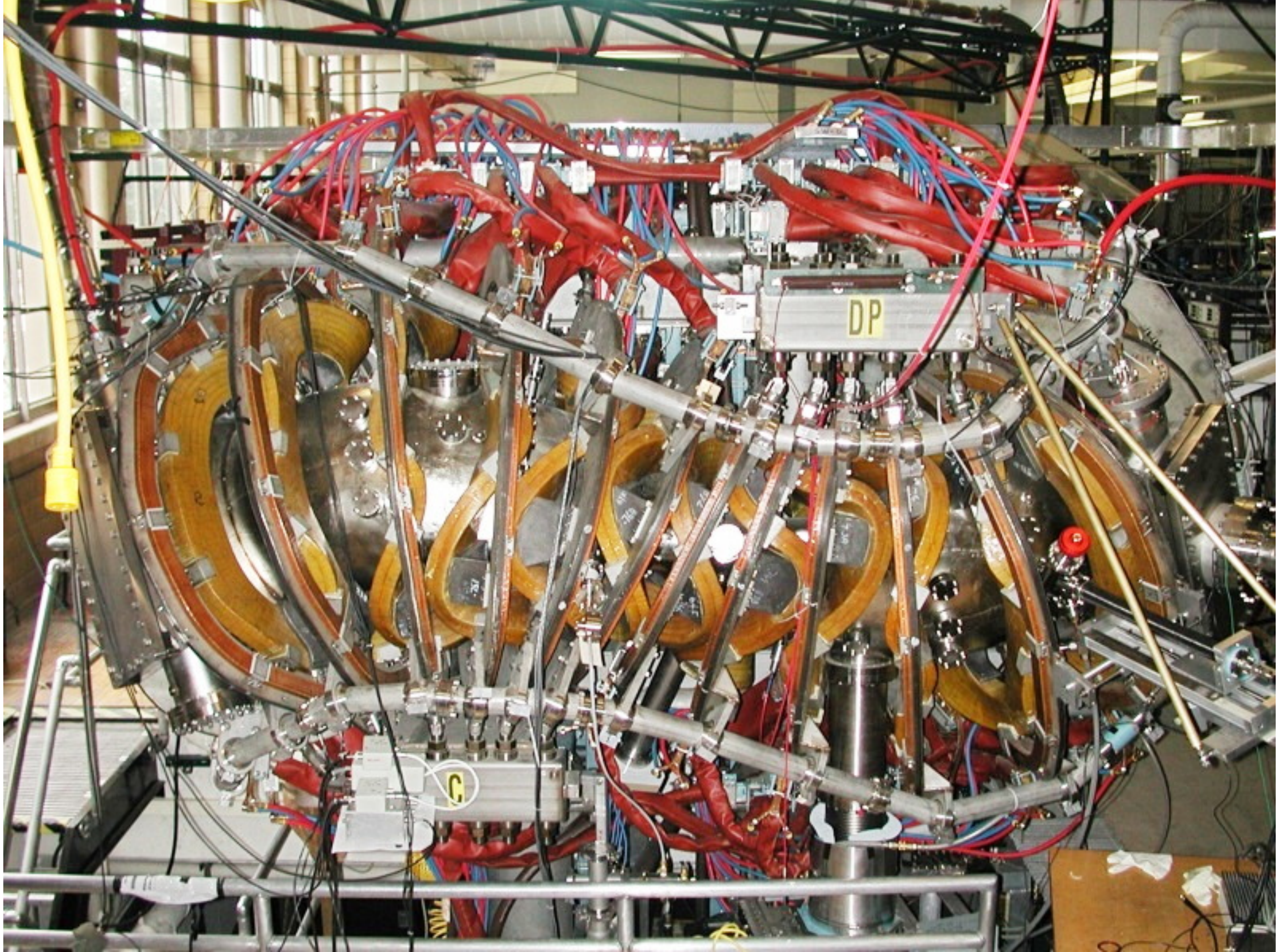




toroidal
magnetic field coils
inner poloidal
magnetic field coils
coil current
outer poloidal
magnetic field coils
vessel
plasma
toroidal
magnetic field
twisted
magnetic field
plasma current



Stellarator


DP
C