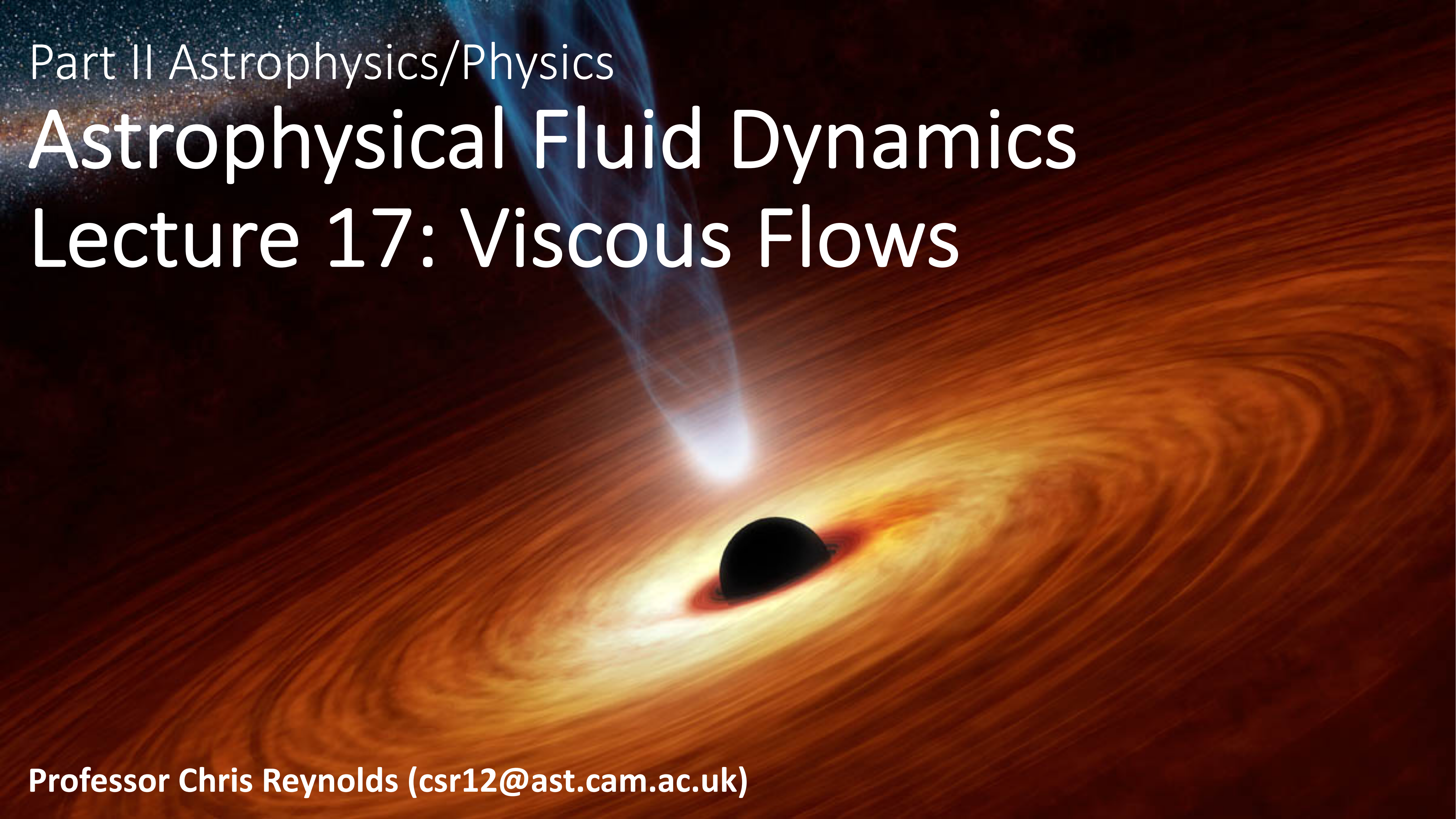

Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 17: Viscous Flows

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Taking stock

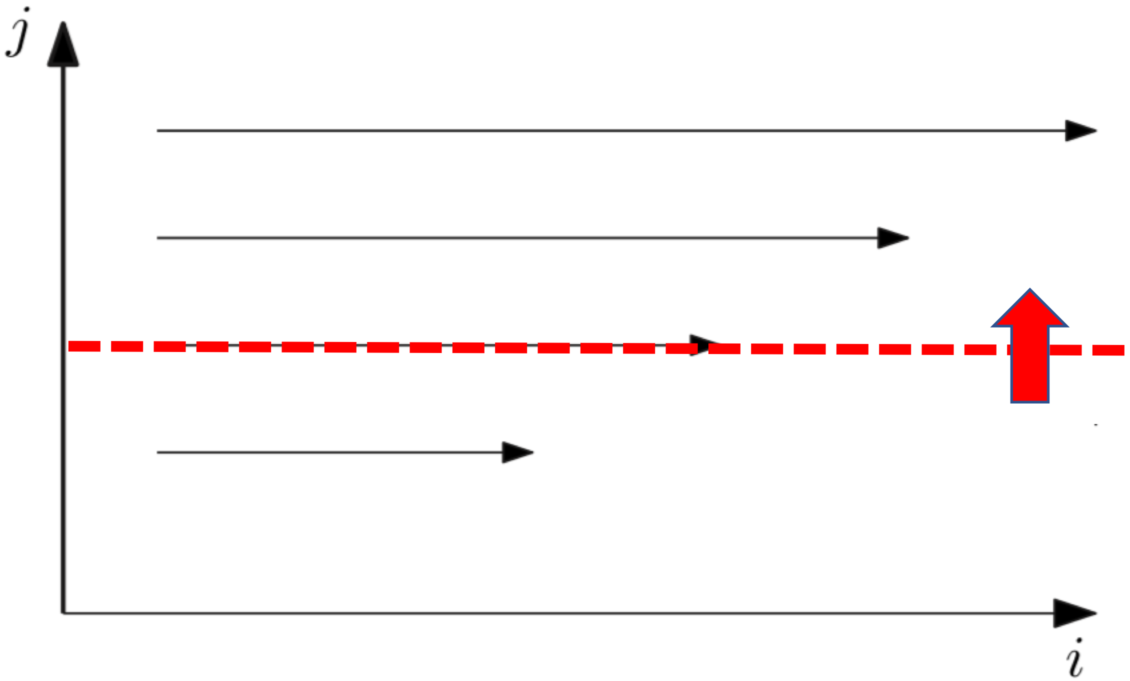
- Story so far…
  - Developed basic equations of fluid dynamics
  - Hydrostatic equilibria (& relevance to stellar structure)
  - Supersonic flows and shocks (& relevance to supernova)
  - Transonic flows and critical points (rocket nozzles, spherical accretion/winds)
  - Fluid instabilities
- Current focus has been on ideal, high-collisionality gas
- Formally, our current treatment deals with the zero MFP ($\lambda \rightarrow 0$) limit.

This Lecture

- Viscous flows (Chapter I)
  - Viscosity as the diffusion of momentum (lowest order finite-λ correction)
  - Viscous stress tensor (I.1)
  - Navier-Stokes equation (I.2)
  - Vorticity in viscous flows (I.3)
  - Energy dissipation in viscous flows (I.4)

Chapter I : Viscous Flows

Consider a simple linear shear flow…



The flux of the *i*-th component of momentum in the *j*-th direction (from below) is

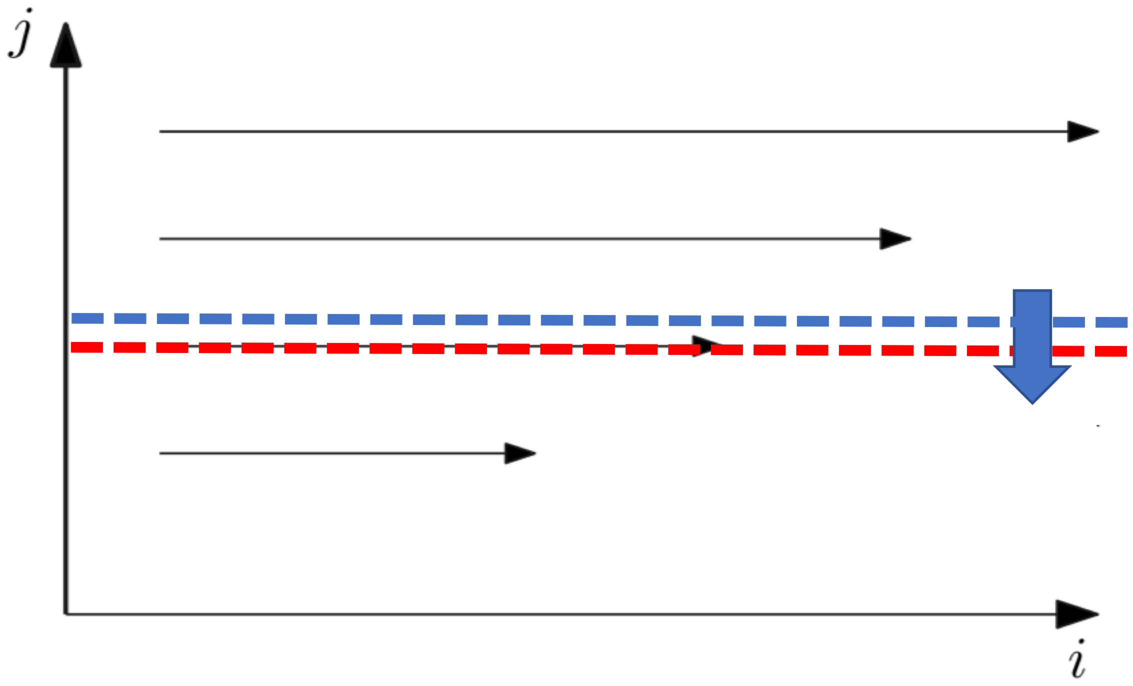
$$\langle \rho v_i v_j \rangle = \alpha \, \rho u_i \sqrt{\frac{k_B T}{m}}$$

particle velocities

order unity

bulk flow

thermal (random) velocities



The flux of the *i*-th component of momentum in the *j*-th direction (from above) is

$$-\rho u_i^* \alpha \sqrt{\frac{kT}{m}}$$

The <u>net</u> flux of the *i*-th component of momentum in the *j*-th direction is

$$\alpha\, \rho(u_i - u_i^*)\sqrt{\frac{k_B T}{m}}$$

Repeating: net flux of the *i*-th component of momentum in the *j*-th direction is

$$\alpha\, \rho(u_i - u_i^*)\sqrt{\frac{k_B T}{m}}$$

This is non-zero due to the existence of the velocity gradient and the finite length scale δl over which the particle momenta get mixed. Writing

$$u_i^* = u_i + \delta l(\partial_j u_i)$$

So, net flux is

$$-\rho(\partial_j u_i)\delta l\sqrt{\frac{kT}{m}}$$

This ”mixing length” is naturally identified with the mean free path

$$\delta l \sim \lambda = \frac{1}{n\sigma};$$

Number density (n)

Collision cross-section (σ)

For a neutral gas, we can use a “hard sphere” model for the cross section.

$$\sigma = \pi a^2$$

So,

$$\text{net momentum flux} = -\rho(\partial_j u_i)\frac{m}{\rho \pi a^2}\alpha\sqrt{\frac{kT}{m}}$$

This term goes into the conservative form of the momentum equation

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \partial_j\Bigg[\underbrace{\frac{\alpha}{\pi a^2}\sqrt{mkT}}_{\equiv \eta,\ \text{shear viscosity}}\ \partial_j u_i\Bigg] + \rho g_i$$

A more rigorous derivation from kinetic theory shows that $\alpha = 5\sqrt{\pi}/64$.

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \partial_j(\eta\,\partial_j u_i) + \rho g_i$$

Note about shear viscosity coefficient η :

- η is independent of density (denser gas has more particles to transport momentum but mean-free-path is commensurably smaller)
- η increases with T; an isothermal system has constant η
- Functional dependence on T depends on collision model…
  - Hard sphere model gives $\eta \propto T^{1/2}$
  - Coulomb collisions (relevant for fully ionized plasma)

$$\lambda \propto T^2, \qquad v_{\rm th} \propto \sqrt{T} \qquad \Rightarrow \qquad \eta \propto T^{5/2}$$

More generally, we write

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij} - \sigma'_{ij}) + \rho g_i$$

viscous stress tensor

Our approach so far has not been general wrt geometry (i.e. assumed no velocity gradients in the i-th direction). Let's ask… what is most general form of σ'_{ij} that is

- Galilean invariant (shouldn't introduce viscous stresses via frame transformation)
- Linear in velocity gradients
- Isotropic (basic response of fluid same in all directions)

<u>Answer</u>:

$$\sigma'_{ij} = \eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k$$

Bulk viscosity

where η and ζ are independent of velocity

So, momentum equation now reads…

$$\frac{\partial(\rho u_i)}{\partial t} = -\,\partial_j(\rho u_i u_j) - \partial_j p\delta_{ij} + \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i$$

Combine with continuity equation to get

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\partial_j u_i\right) = -\,\partial_j p\delta_{ij} + \underbrace{\partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right]}_{\text{Viscous force}} + \rho g_i$$

This is the general form of the Navier-Stokes equation.

Outside of shocks, bulk viscosity usually not important.

If flow is also isothermal (so that η=const), then we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Psi + \underbrace{\frac{\eta}{\rho}}_{\substack{\equiv \nu \\ \text{kinematic} \\ \text{viscosity}}} \left[\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right]$$

This is the more commonly used form of the Navier-Stokes equation.

The importance of viscosity in a flow is characterized via the **Reynolds number**

$$Re = \frac{|\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}|}{|\nu \nabla^2 \mathbf{u}|} \quad \Rightarrow \quad Re = \frac{UL}{\nu}$$

U = characteristic velocity of system
L = characteristic lengthscale of system

Consequences of viscosity

- Shear leads to transmission of momentum through flow (layers rub)
- Vorticity
  - Can introduce vorticity into initially irrotational flows from the boundaries
  - Vorticity diffuses through the flow (advection/diffusion = Re)
- Generally has stabilizing effect on various fluid instabilities
- Dissipates kinetic energy into heat...

Vorticity in viscous flows

Let's re-examine the evolution of vorticity once the effects of viscosity are included.

Start with new momentum equation for η=const and ζ=0:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Psi + \nu \left(\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right)$$

Take curl:

$$\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} + \boldsymbol{\nabla} \times (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) \\ &\quad = \boldsymbol{\nabla} \times \left(-\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla} \Psi + \frac{\eta}{\rho} \left[\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) \right] \right) \end{aligned}$$

Then tidy up terms…

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \frac{1}{2}\boldsymbol{\nabla}u^2 - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

$$\Rightarrow \qquad \boldsymbol{\nabla}\times(\mathbf{u}\cdot\boldsymbol{\nabla}) = -\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{w}).$$

$$\frac{\partial \mathbf{w}}{\partial t} + \boldsymbol{\nabla}\times(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u})$$

$$= \boldsymbol{\nabla}\times\left(-\frac{1}{\rho}\boldsymbol{\nabla}p - \boldsymbol{\nabla}\Psi + \frac{\eta}{\rho}\left[\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})\right]\right)$$

Ignore gradients of this quantity (strictly, isothermal and uniform density)

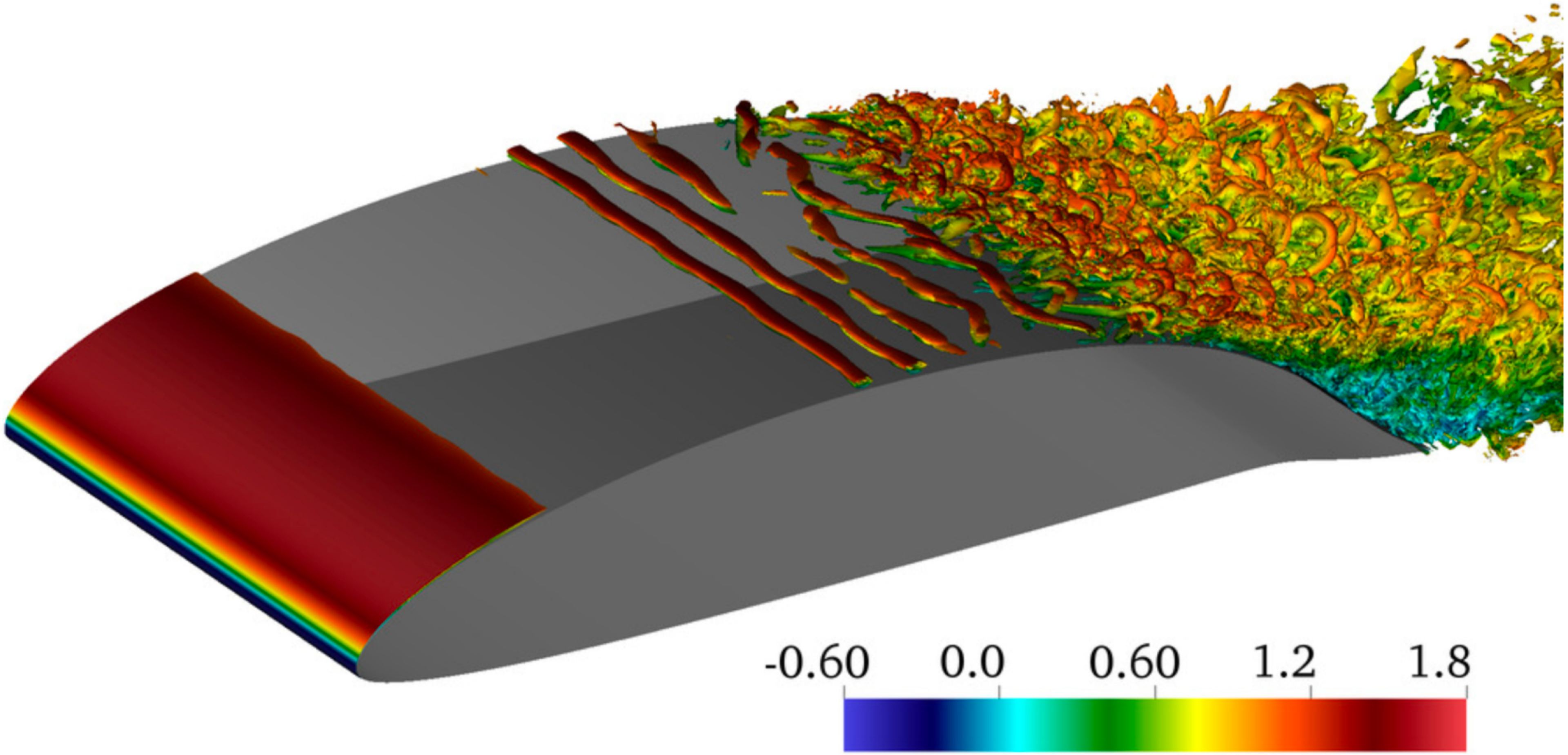
$$\boldsymbol{\nabla}\times\left(\frac{1}{\rho}\boldsymbol{\nabla}p\right) = \boldsymbol{\nabla}\left(\frac{1}{\rho}\right)\times\boldsymbol{\nabla}p + \frac{1}{\rho}\boldsymbol{\nabla}\times\boldsymbol{\nabla}p$$

$$= -\frac{1}{\rho^2}\underbrace{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}_{\substack{=0 \text{ since surfaces of} \\ \text{constant } \rho \text{ and } p \text{ align}}}$$

$$\Rightarrow \qquad \boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{w}) + \frac{\eta}{\rho}\nabla^2\mathbf{w}}$$

Lines of vorticity are advected in the flow AND diffuse through the flow due to viscosity. Viscous term gives a way for vorticity to enter a previously irrotational flow due to boundary interactions. Relative importance of advection and diffusion given by Reynolds number.

(a) $\alpha = 6.7°$



Negi et al. (2019)

I.4 : Energy Dissipation in a Viscous Flow

Viscosity is a dissipative process. It can lead to the irreversible conversion of kinetic energy into heat.

To gain insight into this, we restrict (purely for convenience) to incompressible flows so that we don't need to worry about *p dV* work. Kinetic energy of fluid is

$$E_{\mathrm{kin}} = \frac{1}{2}\int \rho u^2 \,\mathrm{d}V$$

Consider rate of change of kinetic energy density:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) &= u_i \frac{\partial}{\partial t}(\rho u_i) \\ &= -u_i \partial_j(\rho u_i u_j) - u_i \partial_j \delta_{ij} p + u_i \partial_j \sigma'_{ij} \\ &= -u_i \partial_j(\rho u_i u_j) - u_i \partial_i p + \partial_j(u_i \sigma'_{ij}) - \sigma'_{ij}\partial_j u_i\end{aligned}$$

$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$

Look at term:

$$u_i \partial_j (\rho u_i u_j) = u_i \left(u_j \partial_j (\rho u_i) + \rho u_i \cancelto{0}{\partial_j u_j} \right)$$

also note that

$$\partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i \right) = \frac{1}{2} \rho u_i u_i \cancelto{0}{\partial_j u_j} + u_j \partial_j \left(\frac{1}{2} \rho u_i u_i \right)$$

$$= u_j u_i \partial_j (\rho u_i)$$

$$\therefore \qquad u_i \partial_j (\rho u_i u_j) = \partial_j \left(\rho u_j \cdot \frac{1}{2} u_i u_i \right)$$

So,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\partial_j\left(\rho u_j \cdot \frac{1}{2}u_i u_i\right) - \partial_i(u_i p) + p\partial_i u_i + \partial_j(u_i \sigma'_{ij}) - \sigma'_{ij}\partial_j u_i$$
$$= -\partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j \sigma'_{ij}\right) - \sigma'_{ij}\partial_j u_i.$$

Integrating over volume:

$$\begin{aligned}\frac{\partial E_{\mathrm{kin}}}{\partial t} &= \frac{\partial}{\partial t}\int_V \frac{1}{2}\rho u^2 \,\mathrm{d}V \\ &= -\int \partial_i\left(\rho u_i\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - u_j\sigma'_{ij}\right)\mathrm{d}V - \int \sigma'_{ij}\partial_j u_i \,\mathrm{d}V \\ &= \underbrace{-\oint\left(\rho \mathbf{u}\left[\frac{1}{2}u^2 + \frac{p}{\rho}\right] - \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}'\right)\cdot \mathrm{d}\mathbf{S}}_{\text{Energy flux into volume including work done by viscous forces } \mathbf{u}\cdot\underline{\underline{\boldsymbol{\sigma}}}} \quad \underbrace{-\int \sigma'_{ij}\partial_j u_i \,\mathrm{d}V}_{\text{Rate of change of } E_{\mathrm{kin}} \text{ due viscous dissipation}}\end{aligned}$$

If volume V is whole fluid, then surface integral is zero. So,

$$\begin{aligned}\frac{\partial E_{\mathrm{kin}}}{\partial t} &= -\int \sigma'_{ij}\partial_j u_i \,\mathrm{d}V \\ &= -\frac{1}{2}\int \sigma'_{ij}(\partial_j u_i + \partial_i u_j)\,\mathrm{d}V \qquad \text{since } \sigma' \text{ is symmetric}\end{aligned}$$

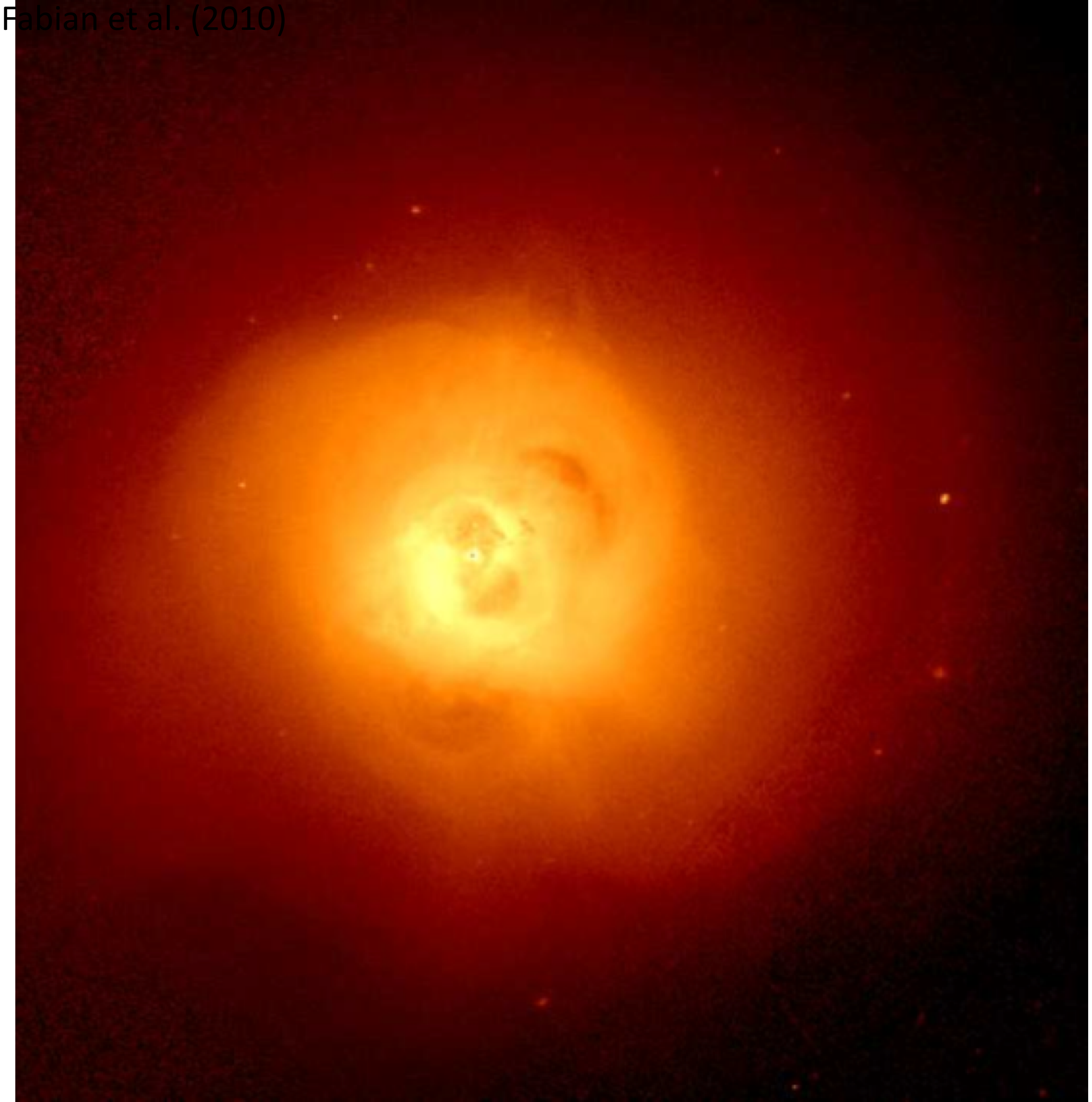
But for an incompressible flow we have $\sigma'_{ij} = \eta(\partial_j u_i + \partial_i u_j)$

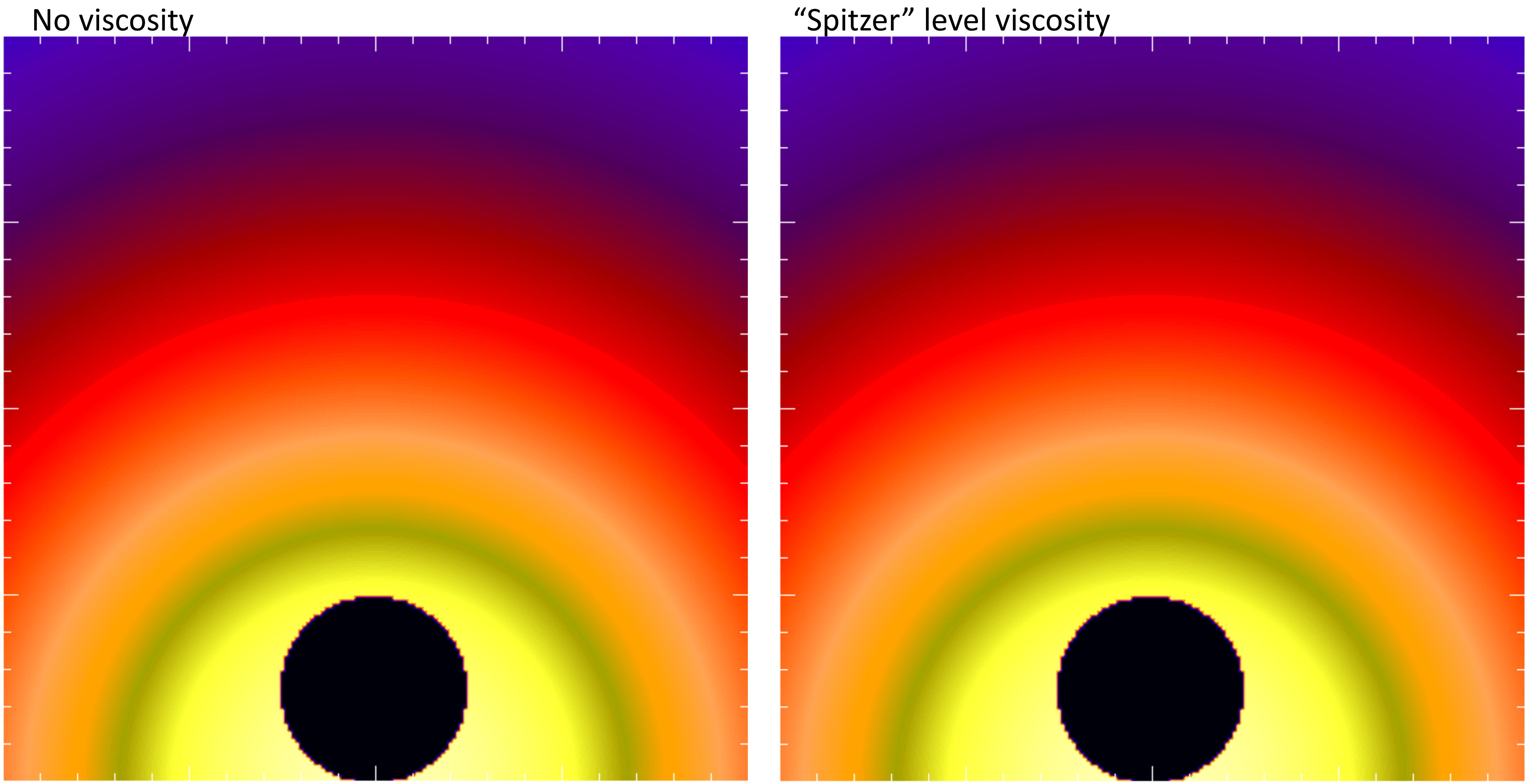
So,

$$\frac{\partial E_{\mathrm{kin}}}{\partial t} = -\frac{1}{2}\int \eta(\partial_j u_i + \partial_i u_j)^2\,\mathrm{d}V$$

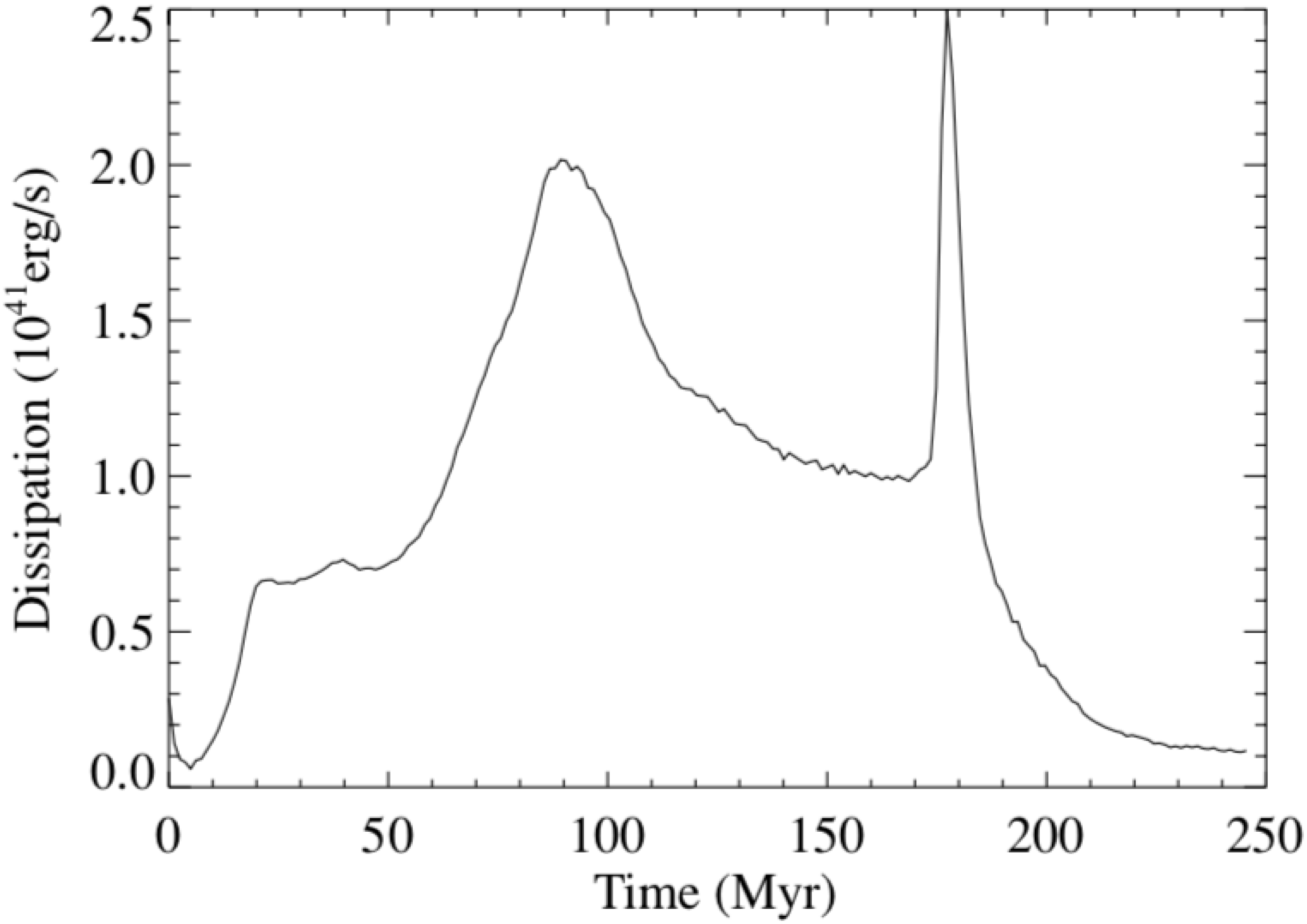
2^{nd} law of thermodynamics dictates that kinetic energy must "grind down" to heat rather than reverse. So we can see that the 2^{nd} law dictates that $\eta \geq 0$.

Fabian et al. (2010)





Reynolds et al. (2005)



2.5
2.0
1.5
1.0
0.5
0.0
0
50
100
150
200
250
Dissipation (10^41 erg/s)
Time (Myr)