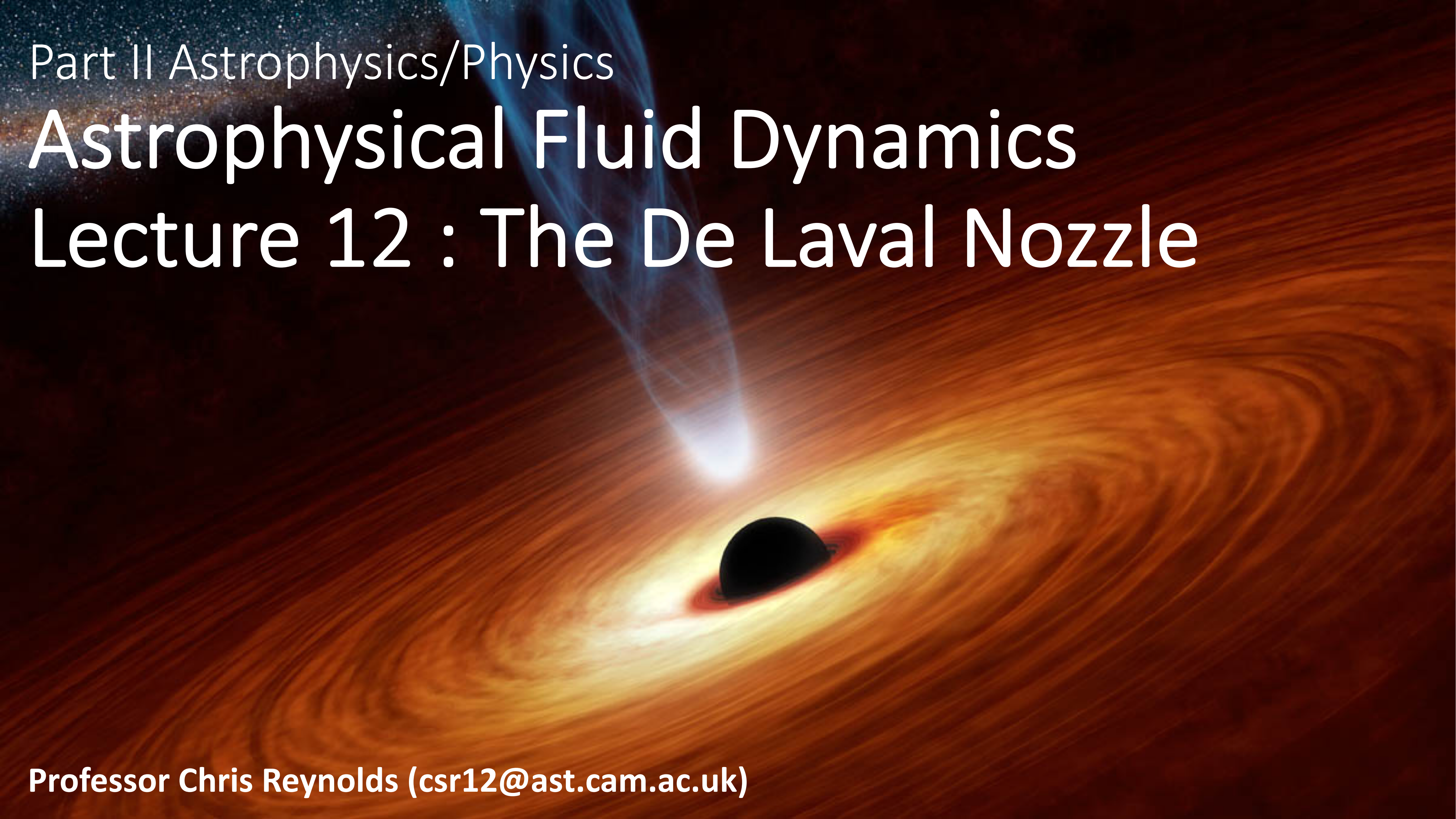
Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 12 : The De Laval Nozzle



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Recap – last lecture

- Concept of vorticity ⇒ rotational and irrotational flows
- Barotropic (but otherwise general) flow obeys Helmholtz eqn

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})$$

(lines of vorticity carried around in flow)

- If also steady-state, then

$$\mathbf{u} \cdot \boldsymbol{\nabla} \underbrace{\left[\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi\right]}_{H} = 0$$

(*H* = constant along a streamline)

- If also irrotational, then

$$\boldsymbol{\nabla} H = 0$$

(*H* = constant throughout fluid)

This Lecture

- Important applications
- De Laval nozzle (G.3)
  - Basic formulism
  - Critical points and sonic transitions
  - Physical interpretation
  - Applications
- (set us up for discussion of spherical accretion next lecture)

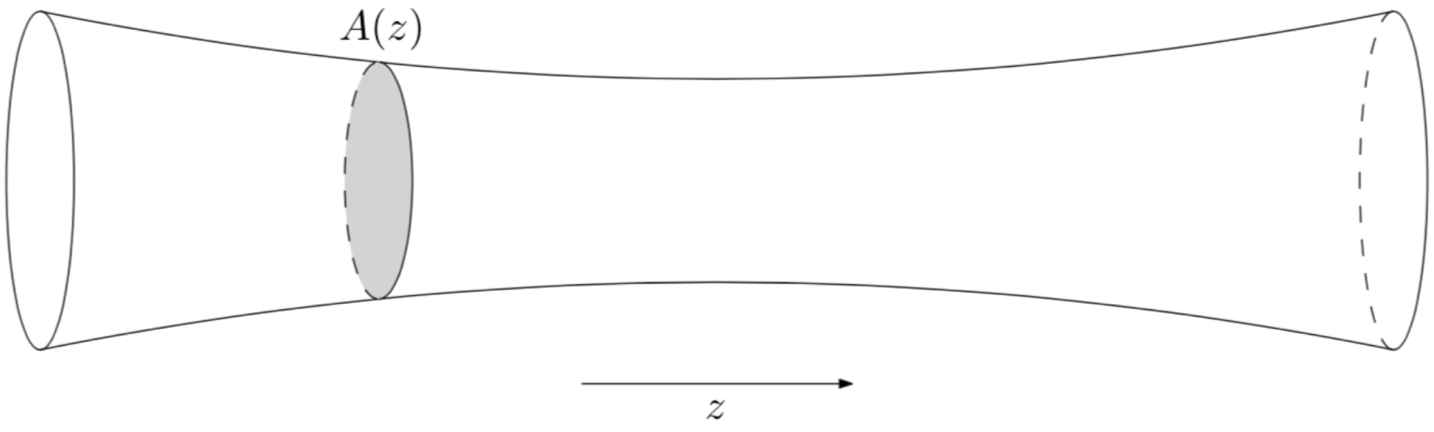
- By end of this lecture, you can do…
  - All of Examples Sheet 1 and 2
  - Example Sheet 3, Q1, Q4, Q9

G.3 : The De Laval Nozzle

Consider steady-state, barotropic flow through a restricted nozzle (no gravity):

Momentum eqn:

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\rho}\nabla\rho = -\frac{1}{\rho}c_s^2\nabla\rho$$



$$\rho u A = \text{constant } \dot{M} \qquad \text{(mass flow per second)}$$

$$\Rightarrow \quad \ln\rho + \ln u + \ln A = \ln\dot{M}$$

$$\Rightarrow \quad \frac{1}{\rho}\boldsymbol{\nabla}\rho + \boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A = 0$$

$$\Rightarrow \quad \frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\ln u - \boldsymbol{\nabla}\ln A$$

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \left[\boldsymbol{\nabla}\ln u + \boldsymbol{\nabla}\ln A\right]c_s^2$$

Carrying over:

$$\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \left[\boldsymbol{\nabla} \ln u + \boldsymbol{\nabla} \ln A\right] c_s^2$$

Now we assume that the flow is irrotational, so we can write

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) = \frac{1}{2}u^2 \boldsymbol{\nabla}\left(\ln u^2\right) = u^2 \boldsymbol{\nabla} \ln u$$

So,

$$u^2 \boldsymbol{\nabla} \ln u = \left[\boldsymbol{\nabla} \ln u + \boldsymbol{\nabla} \ln A\right] c_s^2$$

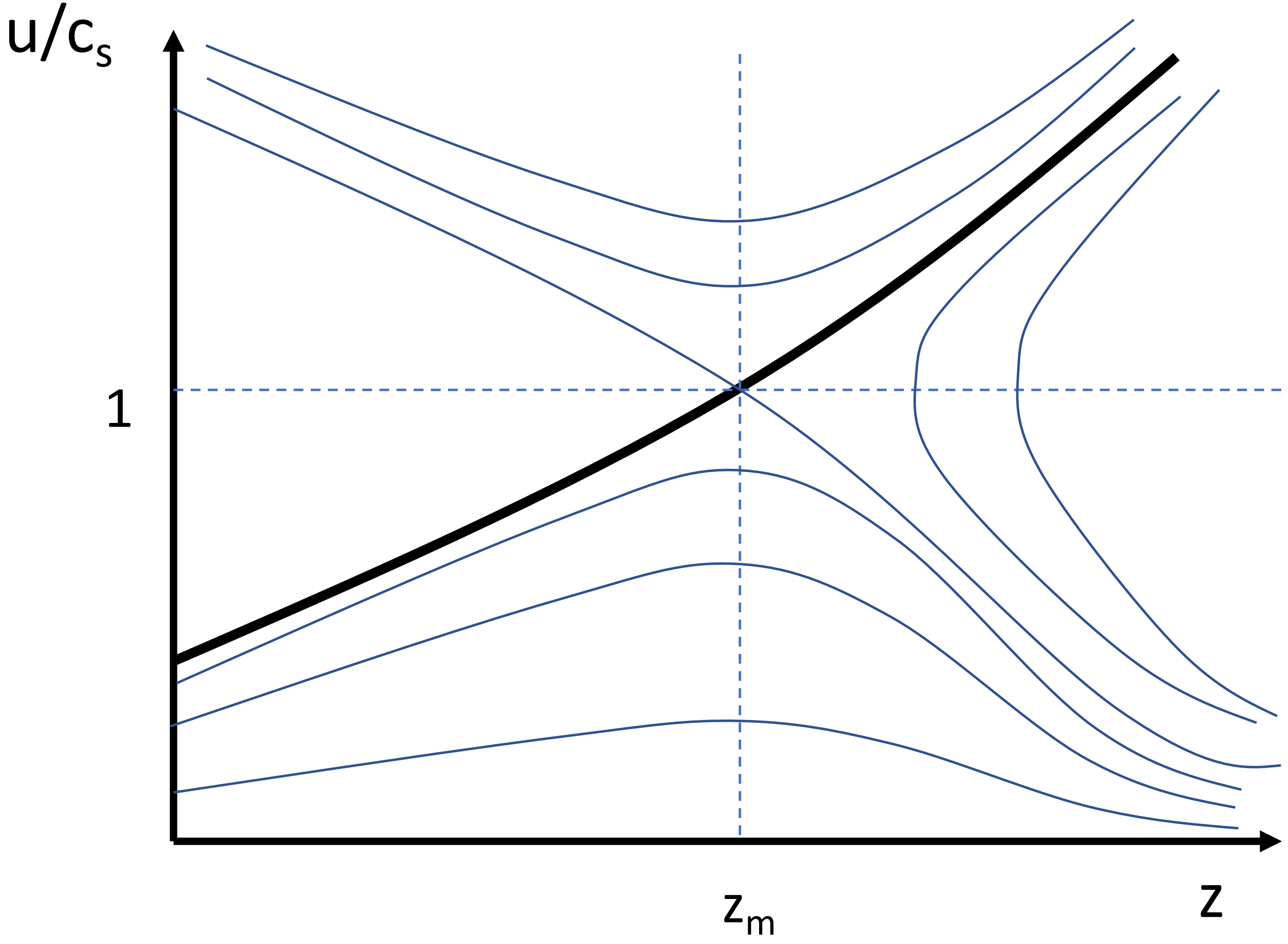
$$\Rightarrow \qquad (u^2 - c_s^2)\boldsymbol{\nabla} \ln u = c_s^2 \boldsymbol{\nabla} \ln A$$

Important finding: when A(z) is an extremum, we must have either

- **minimum or maximum of u, or**
- **$u = c_s$**

We have the potential for subsonic→supersonic transition at extrema of A(z)!

$$(u^2 - c_s^2)\nabla \ln u = c_s^2 \nabla \ln A$$



To analyze this system further, we use use Bernoulli's equation and choose a barotropic equation of state to evaluate the pressure integral

$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = H, \ \text{constant} \qquad \text{[no gravity, steady, irrotational]}$$

Example 1 : Isothermal equation of state

$$p = \frac{\mathcal{R}_* \rho T}{\mu}, \qquad T = \text{const.}$$

$$\Rightarrow \quad \int \frac{\mathrm{d}p}{\rho} = \int \frac{\mathcal{R}_* T}{\mu} \frac{\mathrm{d}\rho}{\rho} = \frac{\mathcal{R}_* T}{\mu} \ln \rho = c_s^2 \ln \rho$$

Suppose that A(z) has a minimum (when A=Am) that allows flow to undergo a sonic transition:

$$\frac{1}{2}u^2 + c_s^2 \ln \rho = \frac{1}{2}c_s^2 + c_s^2 \ln \rho \Big|_{A=A_m}$$

$$\Rightarrow \quad u^2 = c_s^2 \left[1 + 2 \ln \left(\frac{\rho|_{A=A_m}}{\rho} \right)\right] = c_s^2 \left[1 + 2 \ln \left(\frac{uA}{c_s A_m} \right)\right]$$

using

$$\rho u A = \text{constant}$$

For given A(z), can solve for structure of flow $\rho(z), u(z)$ as a function of c_s and $\dot{M}$.

Example 2 : Polytropic equation of state

Complication is that c_s varies (as density changes) so cannot be treated as a parameter of the problem…

$$p = K\rho^{1+1/n} \quad \Rightarrow \quad c_s^2 = \frac{n+1}{n}K\rho^{1/n}$$

So…

$$\begin{aligned}\int \frac{\mathrm{d}p}{\rho} &= \int \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\rho} \\ &= \int K\frac{n+1}{n}\rho^{1/n}\frac{\mathrm{d}\rho}{\rho} \\ &= K\frac{n+1}{n}\int \rho^{1/n-1}\,\mathrm{d}\rho \\ &= K\frac{n+1}{n}n\rho^{1/n} \\ &= nc_s^2\end{aligned}$$

Mass conservation:

$$\rho u A = \rho\Big|_{A_m} c_s\Big|_{A_m} A_m = \dot{M}$$

$$\Rightarrow \quad \rho\Big|_{A_m}\left(\frac{n+1}{n}K\right)^{1/2}\rho^{1/2n}\Big|_{A_m} A_m = \dot{M}$$

$$\Rightarrow \quad \rho^{2+1/n}\Big|_{A_m}\left(\frac{n+1}{n}K\right)A_m^2 = \dot{M}^2$$

$$\Rightarrow \quad \rho\Big|_{A_m} = \left[\left(\frac{\dot{M}}{A_m}\right)^2\frac{n}{K(n+1)}\right]^{n/(2n+1)}$$

Then…

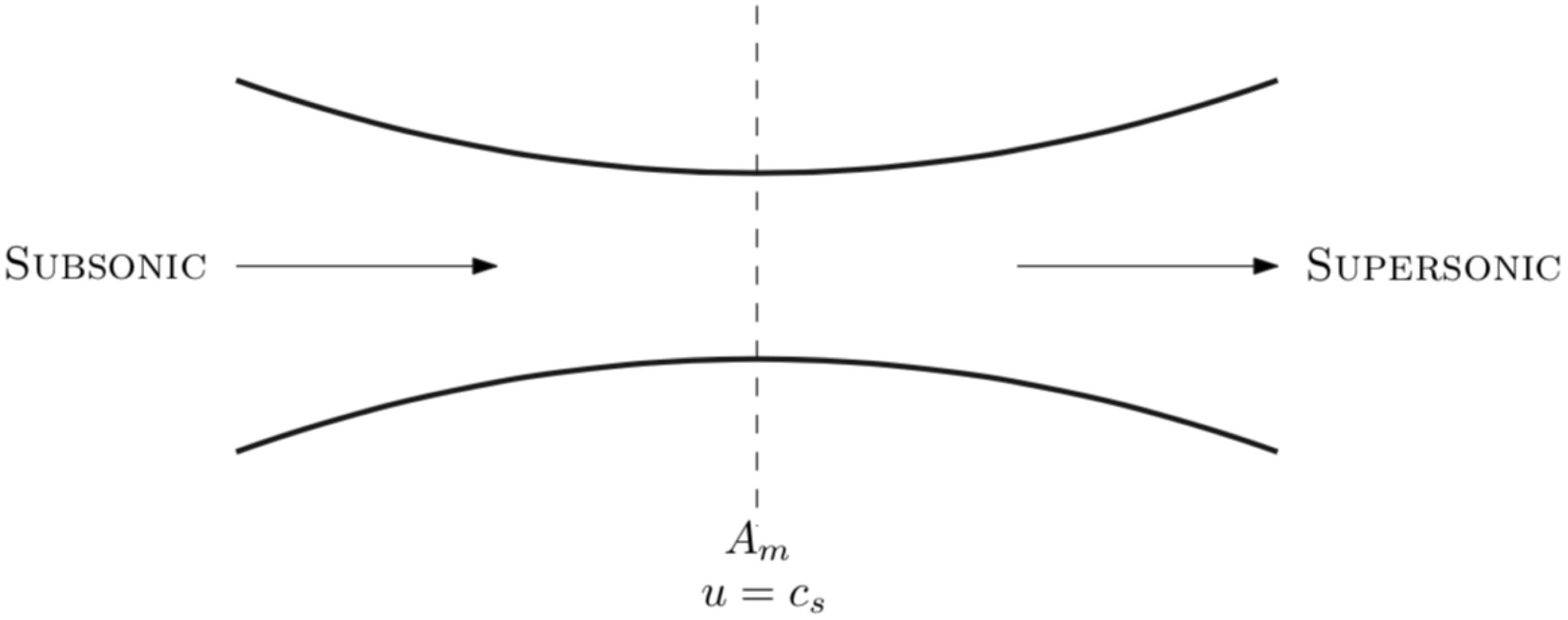
$$\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = \mathrm{const.}$$

$$\begin{aligned}
\Rightarrow \qquad \frac{1}{2}\left(\frac{\dot{M}}{A\rho}\right)^2 + K(n+1)\rho^{1/n} &= \frac{1}{2}\, c_s^2\Big|_{A_m} + K(n+1)\ \rho^{1/n}\Big|_{A_m} \\
&= \frac{1}{2}\left(\frac{n+1}{n}\right) K\ \rho^{1/n}\Big|_{A_m} + K(n+1)\ \rho^{1/n}\Big|_{A_m} \\
&= \left(\frac{1}{2}+n\right)\left(\frac{n+1}{n}\right) K\ \rho^{1/n}\Big|_{A_m}
\end{aligned}$$

For given A(z), this is an implicit equation for $\rho(z)$, so can determine the structure of the flow.

General points of interest:

$$(u^2 - c_s^2)\nabla \ln u = c_s^2 \nabla \ln A$$



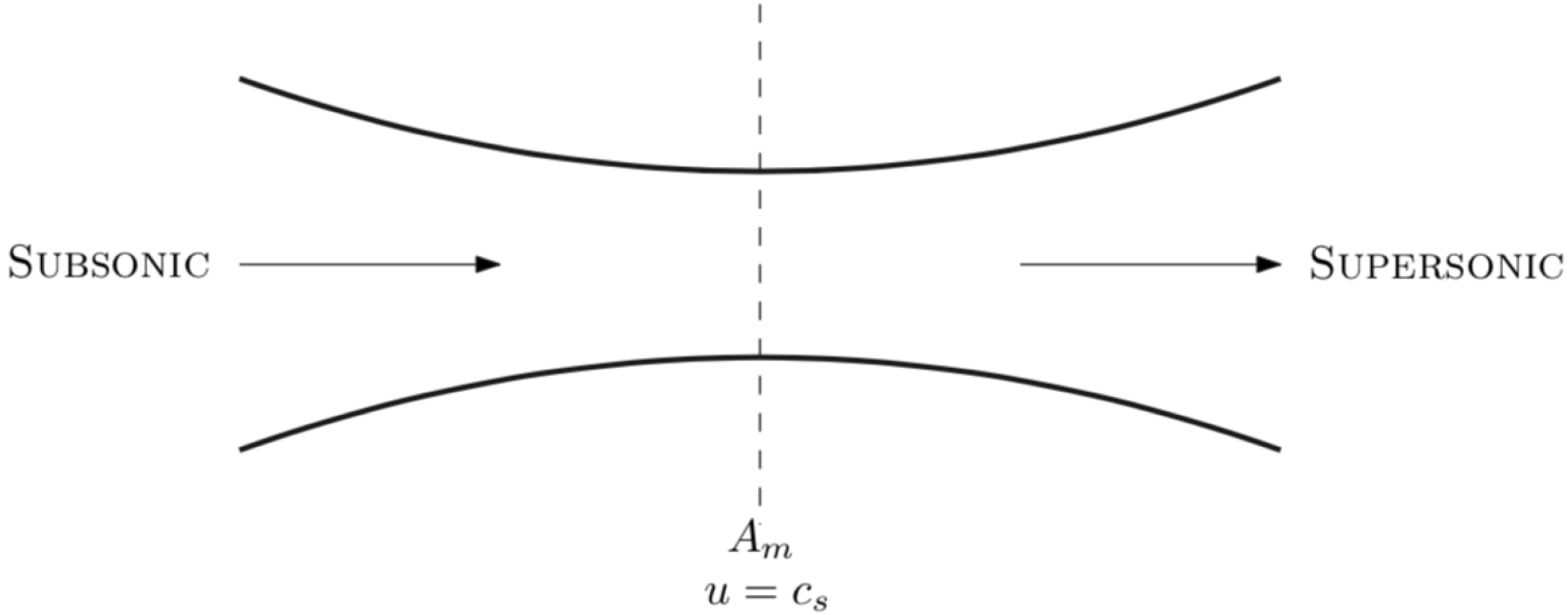
A decrease $\Rightarrow$ $\nabla \ln u$ positive
$\Rightarrow$ u accelerates along streamline

A increases $\Rightarrow$ $\nabla \ln u$ positive
$\Rightarrow$ u accelerates along streamline

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}\rho\frac{\mathrm{d}p}{\mathrm{d}\rho} = -c_s^2\boldsymbol{\nabla}\ln\rho$$

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = u^2\boldsymbol{\nabla}\ln u$$

$$\Rightarrow \qquad u^2\boldsymbol{\nabla}\ln u = -c_s^2\boldsymbol{\nabla}\ln\rho$$



$u \ll c_s, \nabla \ln u \gg \nabla \ln \rho,$

So density almost constant…
i.e. incompressible flow

$u \gg c_s, \nabla \ln u \ll \nabla \ln \rho,$

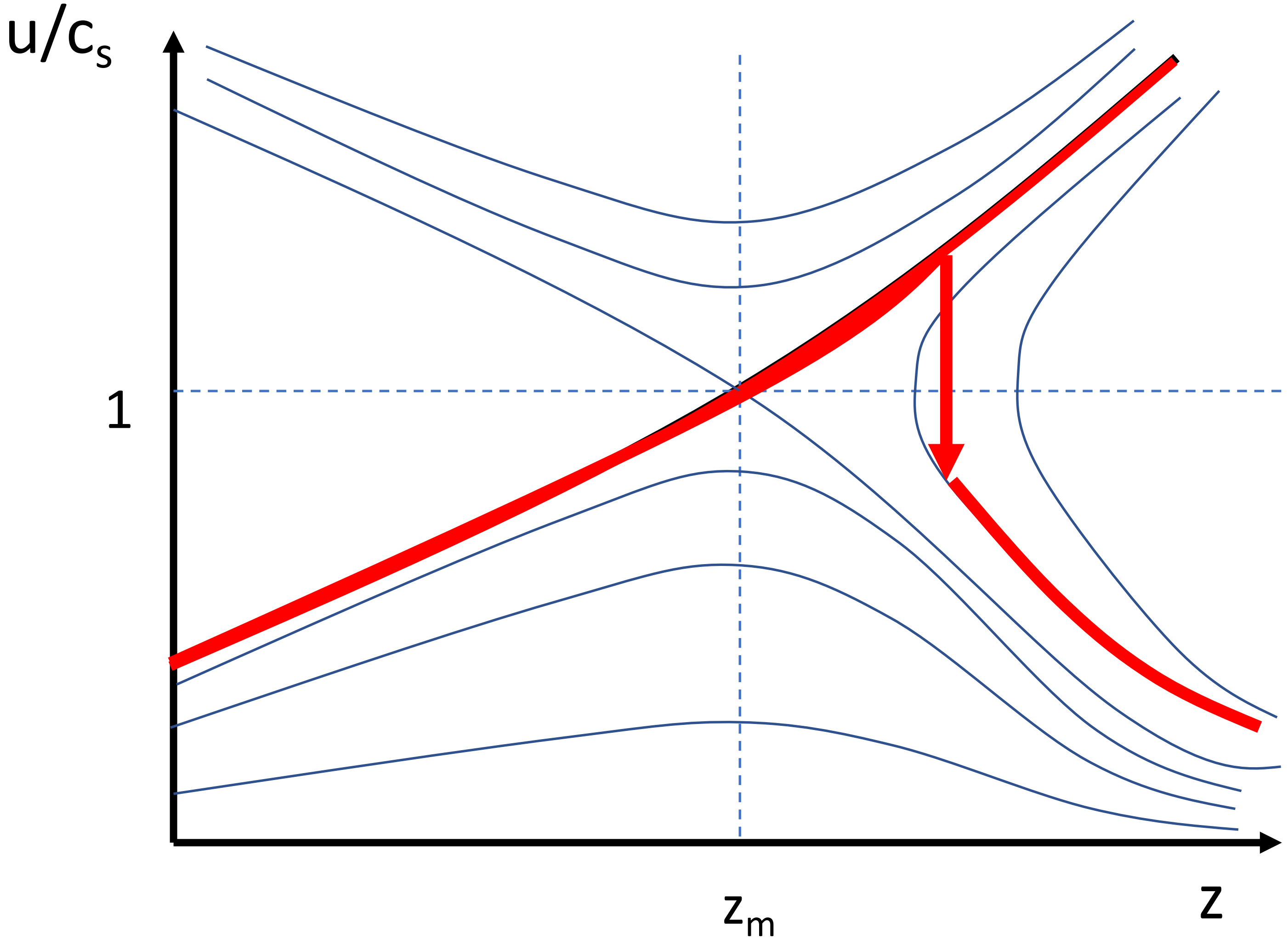
So velocity almost constant…
but density and sound speed drops

Water analogy

NINJA simulation

Space-X test

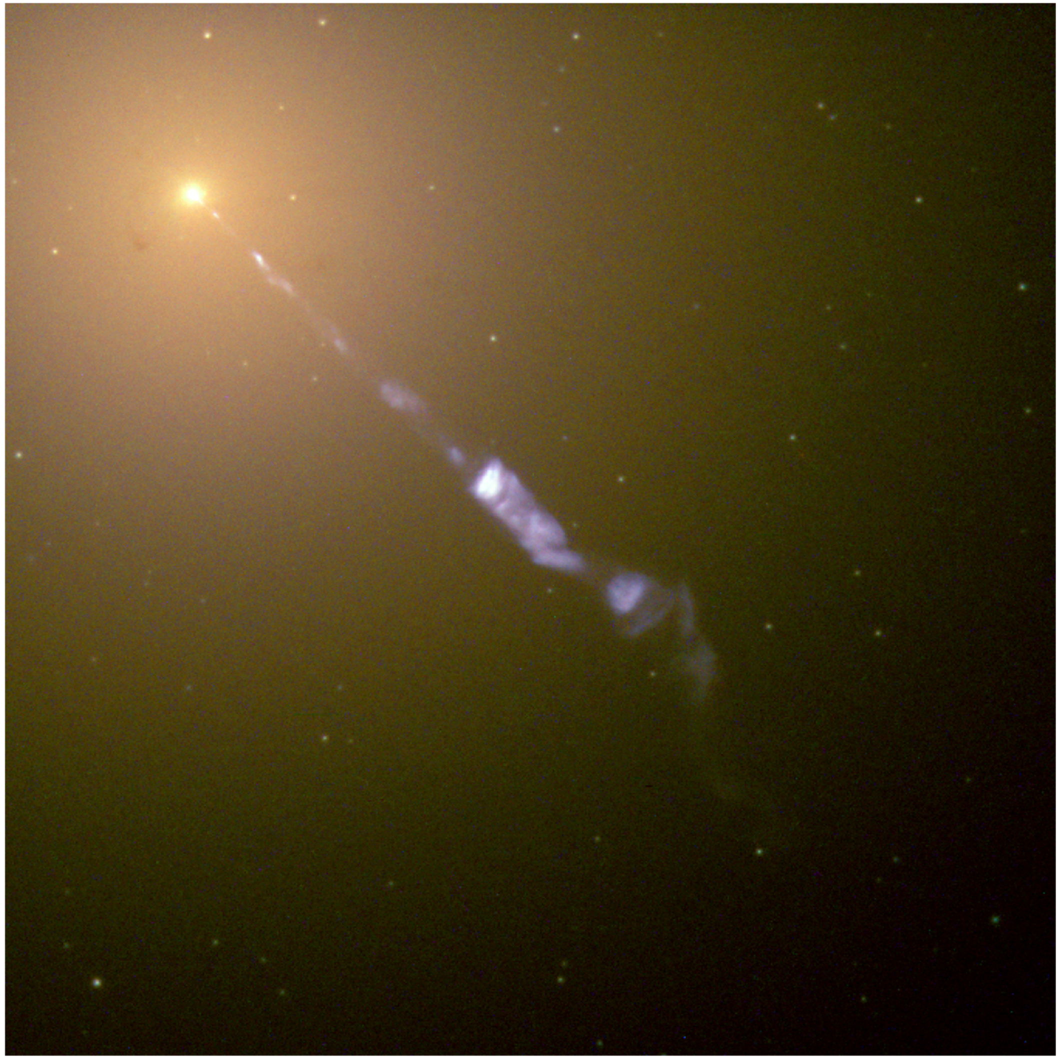
$$(u^2 - c_s^2)\nabla \ln u = c_s^2 \nabla \ln A$$



Nozzles can choke due to shock formation in the exhaust

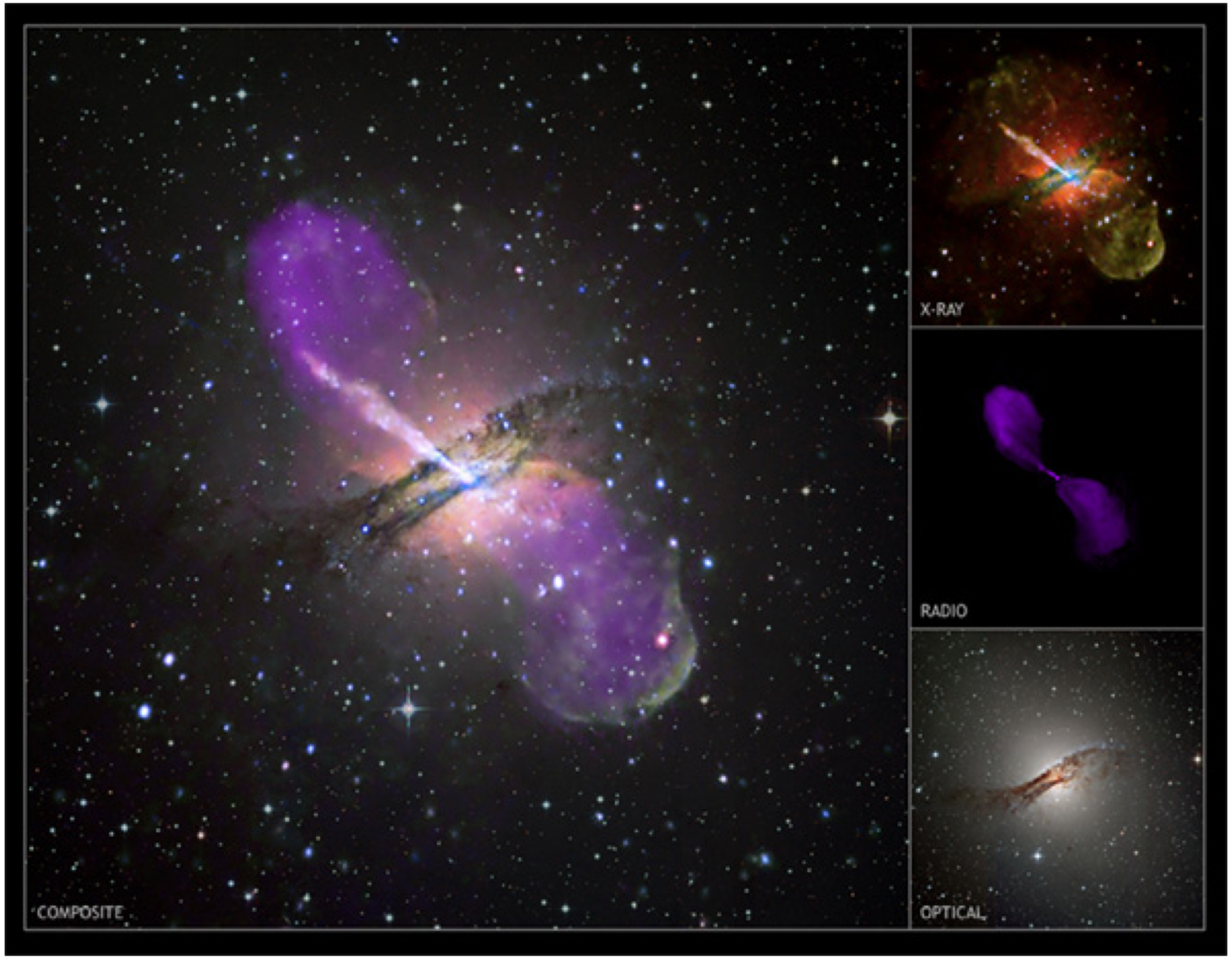
Stability

Galaxy clusters: jet in M87



Credit:HST M87

Galaxy clusters: jet in Cen A



Credit: X-ray - NASA, CXC, R.Kraft (CfA), et al.; Radio - NSF, VLA, M.Hardcastle (U Hertfordshire) et al.; Optical - ESO, M.Rejkub

De Laval nozzle applied to jets… an early but unsuccessful model (Blandford & Rees 1974)