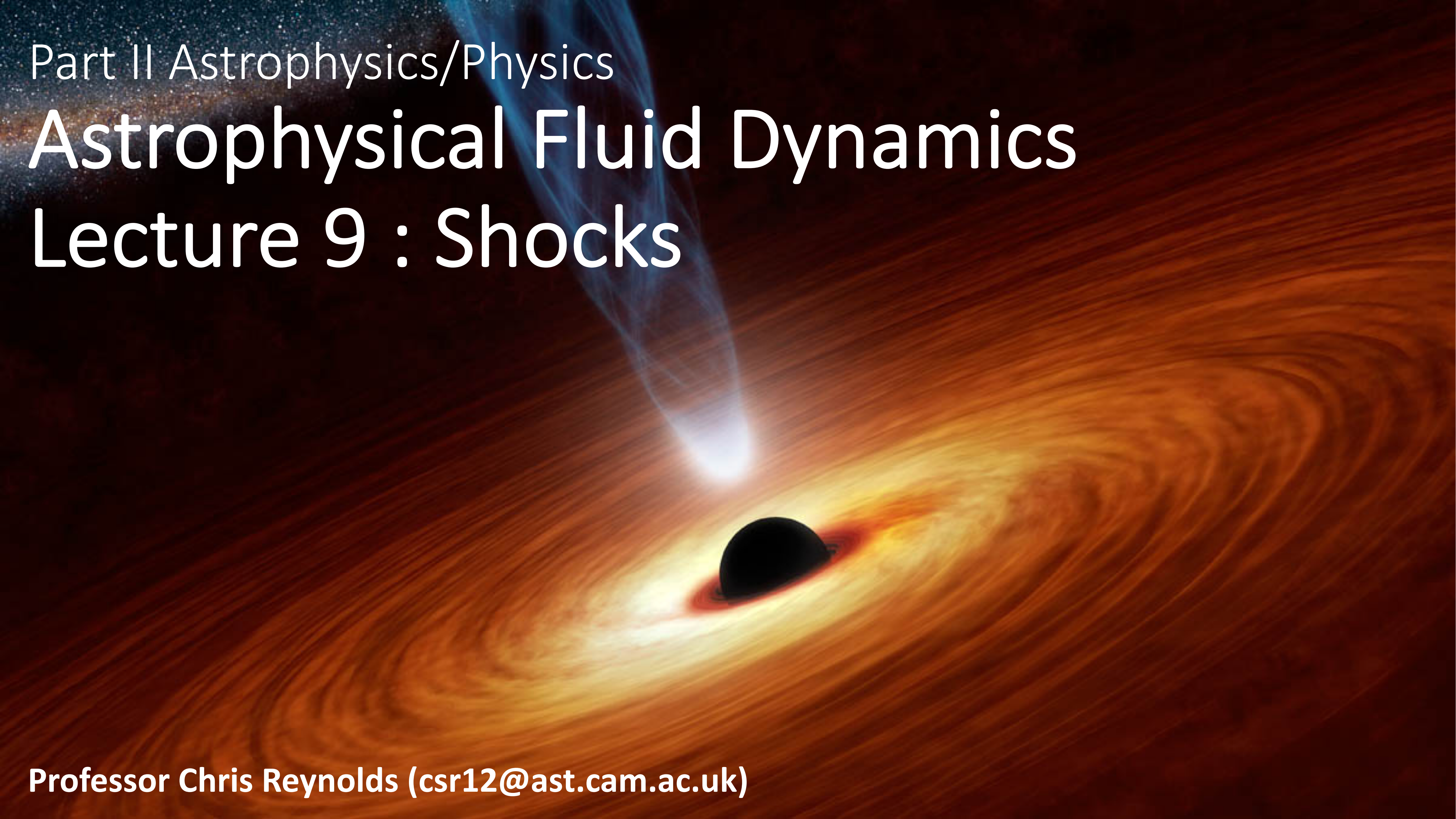
Part II Astrophysics/Physics

Astrophysical Fluid Dynamics

Lecture 9 : Shocks



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Recap

- Developed basic equations of fluid dynamics (for ideal gas)
- Hydrostatic Equilibrium and Stellar Structure
- Sound waves – first look at dynamics
  - Linear perturbation theory… dynamics decomposed into independent modes
  - Waves in uniform medium… dispersion free propagation
  - Waves in stratified systems… dispersion and acoustic cutoffs

This Lecture

- Sound speed as a signal speed and Mach cones
- Motivating the idea of a shock
- The Rankine-Hugoniot jump conditions
  - Adiabatic shocks
  - Isothermal shocks

After today's lecture, you can attempt…

- All of Example Sheets 1 and 2
- Q1 of Example Sheet 3

Mach Cones

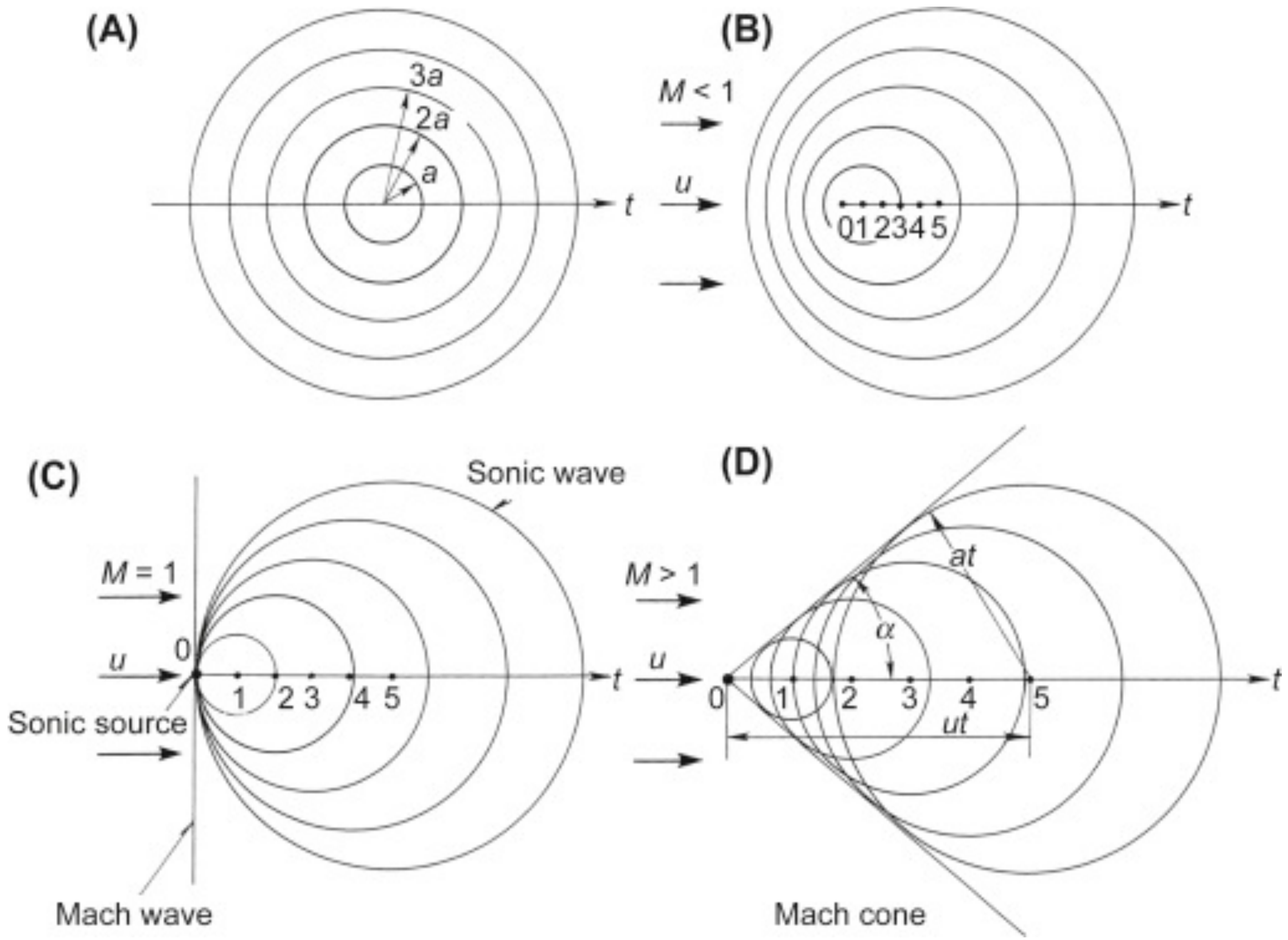
Consider some source of disturbance… signals propagate outwards at speed of sound.

Sound waves are characteristics of the set of hyperbolic equations.

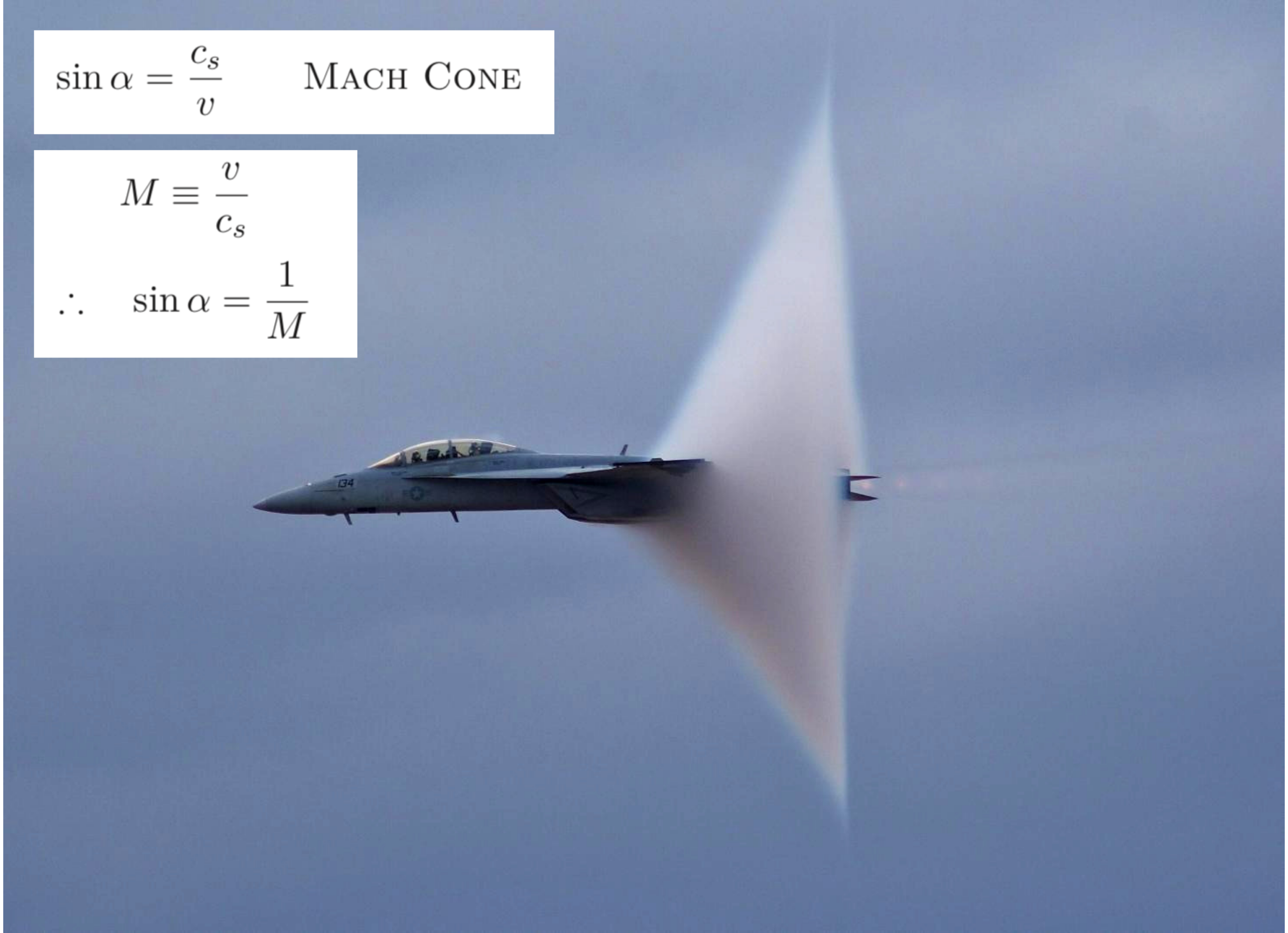
Wavefronts are circular since speed of sound isotropic.

If source is moving, then center of each subsequent wavefront is displaced.

What happens if the source moves at of faster than the sound speed?



From Science Direct



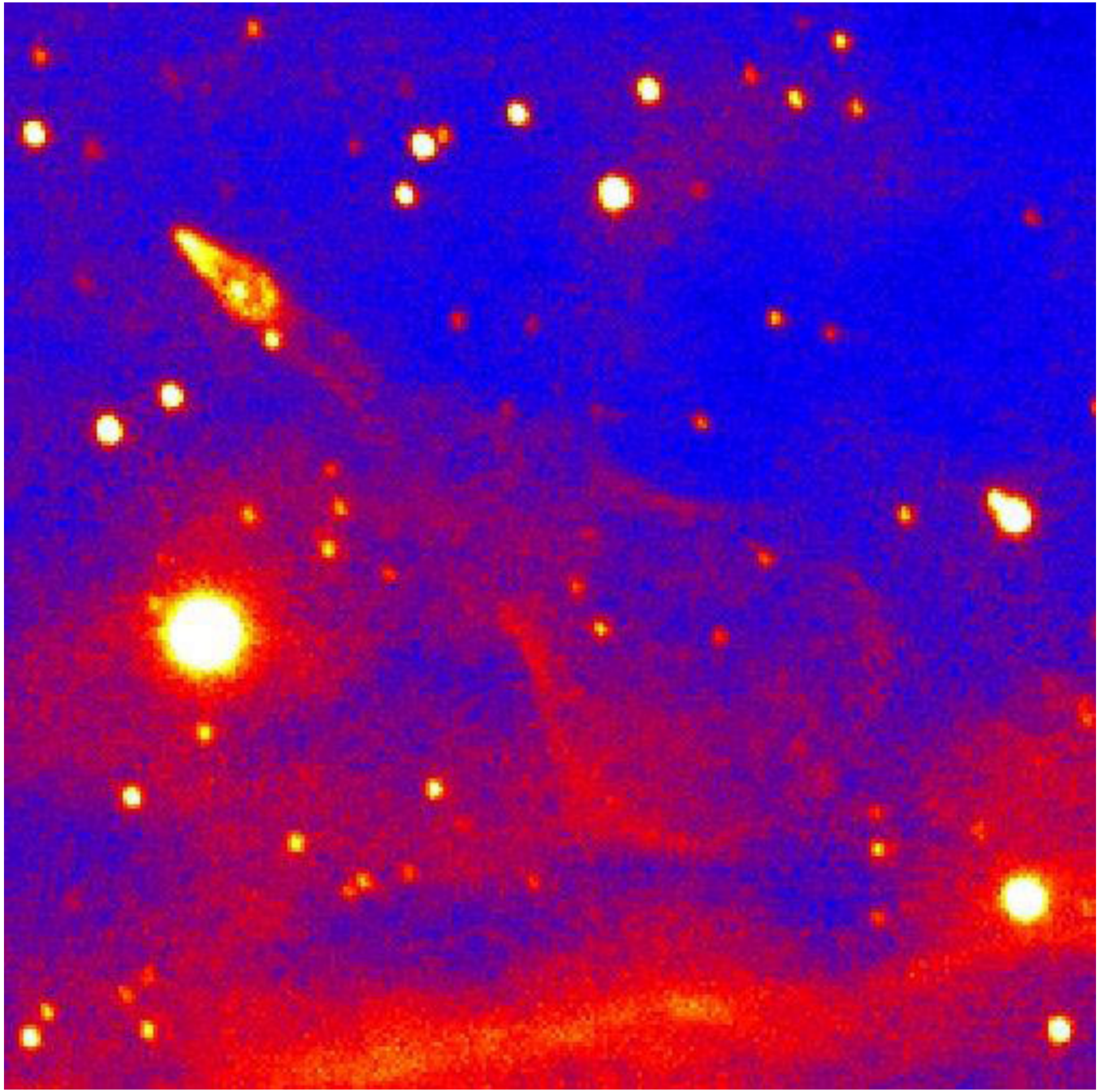
$$\sin \alpha = \frac{c_s}{v}$$ MACH CONE

$$M \equiv \frac{v}{c_s}$$

$$\therefore \quad \sin \alpha = \frac{1}{M}$$

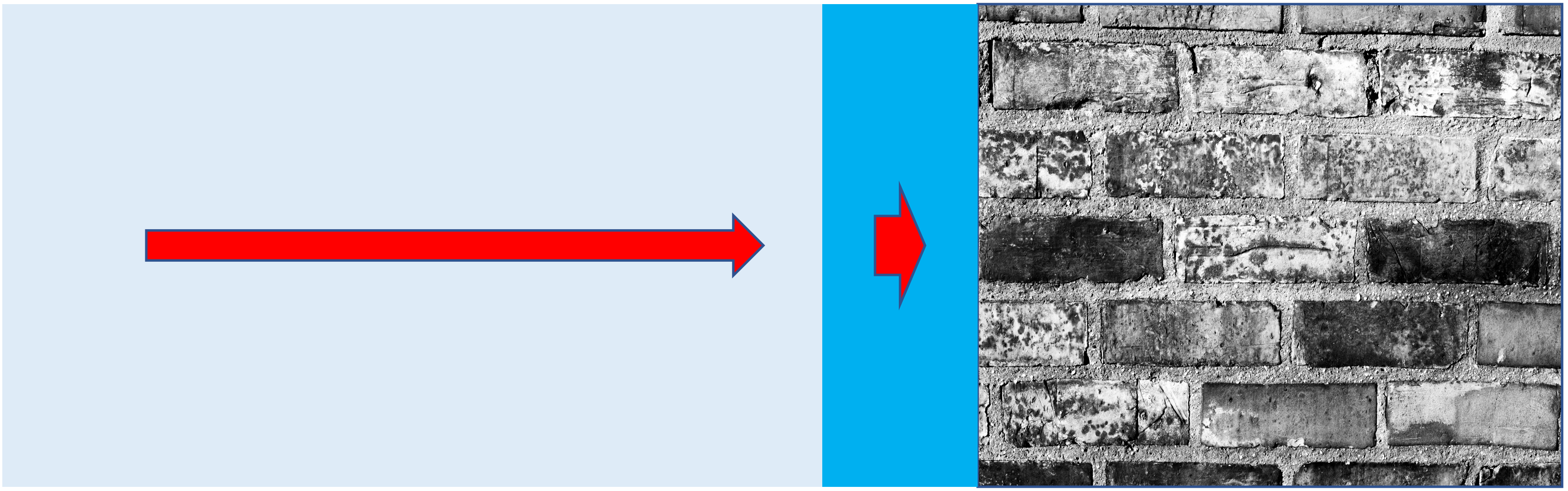
Guitar Nebula

Bow shock created by an ejected pulsar traveling supersonically through surrounding cool medium.



Shocks

A supersonic fluid can “run into things” before it gets any prior signal.



The result are strong discontinuities in the properties of the flow.

These discontinuities are called shock waves and can be an important way of heating gas.

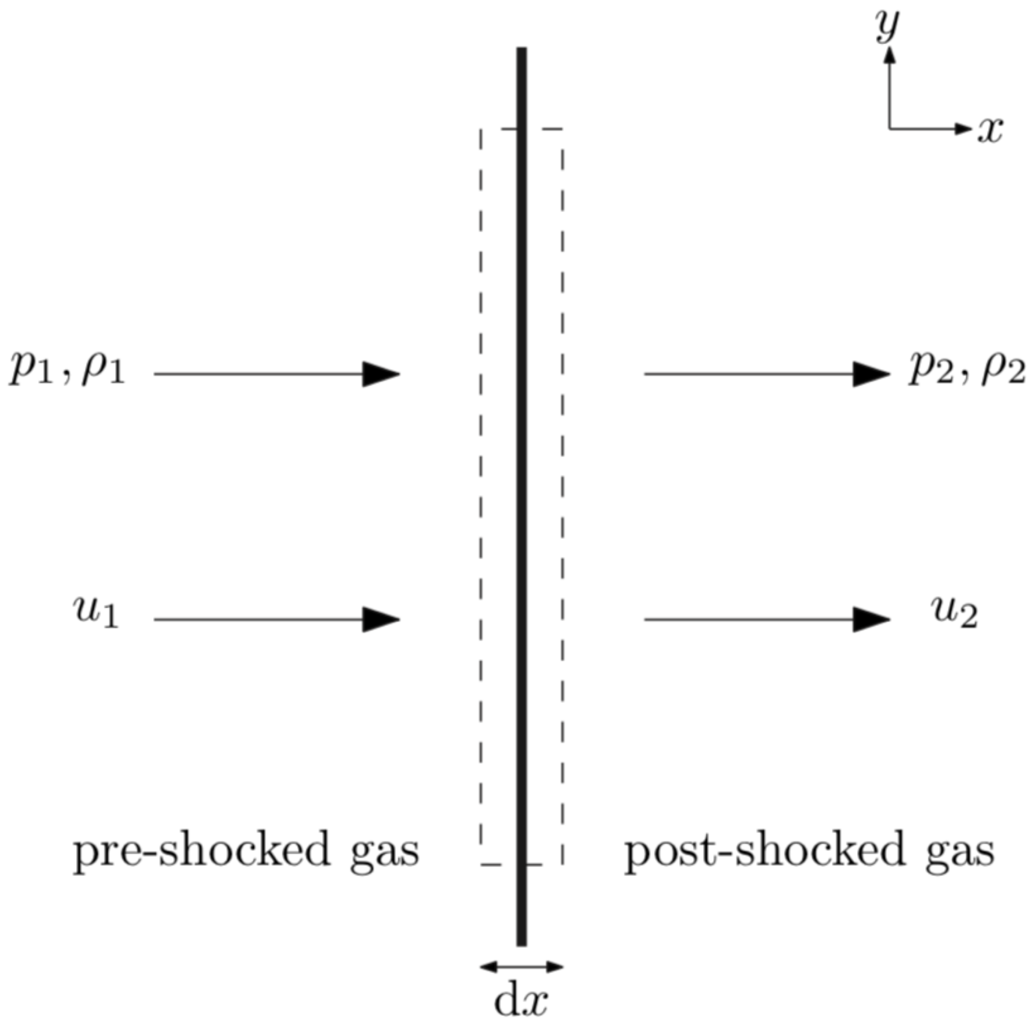
Let’s analyze the properties of shocks.

The Rankine-Hugoniot Jump Conditions

Let's specialize to the case of fluid entering a plane-parallel shock normally.

On each side of the shock, the properties are uniform.

Density, pressure, velocity are discontinuous across shock.



Integrate continuity equation over small volume encompassing the shock:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\left(\int_{-\mathrm{d}x/2}^{\mathrm{d}x/2} \rho \,\mathrm{d}x\right) + \rho u_x\Big|_{x=\mathrm{d}x/2} - \rho u_x\Big|_{x=-\mathrm{d}x/2} = 0$$

In steady-state, mass cannot be piling up at shock front, so

$$\frac{\partial}{\partial t}\left(\int \rho \,\mathrm{d}x\right) = 0$$

$$\Rightarrow \qquad \boxed{\rho_1 u_1 = \rho_2 u_2} \qquad \text{1st Rankine-Hugoniot Relation}$$

Similarly for momentum equation

$$\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x u_x + p) - \rho\frac{\partial \Psi}{\partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\int \rho u_x \,\mathrm{d}x\right) = -\left.(\rho u_x u_x + p)\right|_{\mathrm{d}x/2} + \left.(\rho u_x u_x + p)\right|_{-\mathrm{d}x/2}$$

$$\Rightarrow \quad \boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2} \qquad \text{2ND R-H RELATION}$$

For now, assume **adiabatic shock (no cooling)** and time-steady gravitational field, then apply same analysis to energy equation:

$$\frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\mathbf{u}] = \cancelto{0}{-\rho\dot{Q}_{\mathrm{cool}}} + \cancelto{0}{\rho\frac{\partial \Psi}{\partial t}}$$

$$\Rightarrow \quad \frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\mathbf{u}] = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\int E\,\mathrm{d}x\right) + \left.(E+p)u_x\right|_{\mathrm{d}x/2} - \left.(E+p)u_x\right|_{-\mathrm{d}x/2} = 0$$

$$\Rightarrow \quad (E_1 + p_1)u_1 = (E_2 + p_2)u_2$$

But $E = \rho\left(\frac{1}{2}u^2 + \mathcal{E} + \Psi\right)$

So, the last equation gives

$$\Psi_1 = \Psi_2$$
$$\rho_1 u_1 = \rho_2 u_2$$

$$\frac{1}{2}\rho_1 u_1^3 + \rho_1 \mathcal{E}_1 u_1 + \rho_1 \Psi_1 u_1 + p_1 u_1 = \frac{1}{2}\rho_2 u_2^3 + \rho_2 \mathcal{E}_2 u_2 + \rho_2 \Psi_2 u_2 + p_2 u_2$$

$$\boxed{\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}}$$

3RD R-H RELATION

$$\left.\begin{aligned} \mathcal{E} &= C_V T \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \Rightarrow \quad \mathcal{E} = \frac{C_V \mu}{\mathcal{R}_*}\frac{p}{\rho}$$

$$\left.\begin{aligned} \gamma &= \frac{C_p}{C_V} \\ C_p - C_V &= \frac{\mathcal{R}_*}{\mu} \end{aligned}\right\} \Rightarrow \quad C_V(\gamma - 1) = \frac{\mathcal{R}_*}{\mu}$$

$$\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho} \quad \Rightarrow \quad \boxed{\frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma - 1}}$$

$$c_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_S = \frac{\gamma p}{\rho}$$

Summary of the three R-H jump conditions:

$$\left.\begin{aligned} &\rho_1 u_1 = \rho_2 u_2 \\ &\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\ &\frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma - 1} \end{aligned}\right\}$$

Conservation of mass, momentum and energy through the shock front.

- Define Mach number of incoming flow, $M_1 \equiv u_1/c_{s,1}$. Then we can show

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{((\gamma-1)M_1^2 + 2)(2\gamma M_1^2 - (\gamma - 1))}{(\gamma+1)^2 M_1^2}$$

- We find.

$$\frac{p_1}{\rho_1^\gamma} \neq \frac{p_2}{\rho_2^\gamma} \quad \text{i.e. } K_1 \neq K_2$$

→ Direction of the jump set up 2nd law of thermodynamics

- Always find that flow decelerates from supersonic to subsonic; bulk kinetic energy converted into disorganized motions (heat).

- Strong shocks have M_1>>1. Then

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma+1}{\gamma-1}$$

So there is a maximum density jump for adiabatic strong shocks.

For γ=5/3, strong shocks have $\rho_2/\rho_1 = 4$

- For strong shocks,

$$T_2 = \frac{2\gamma(\gamma-1)M_1^2}{(\gamma+1)^2} T_1$$

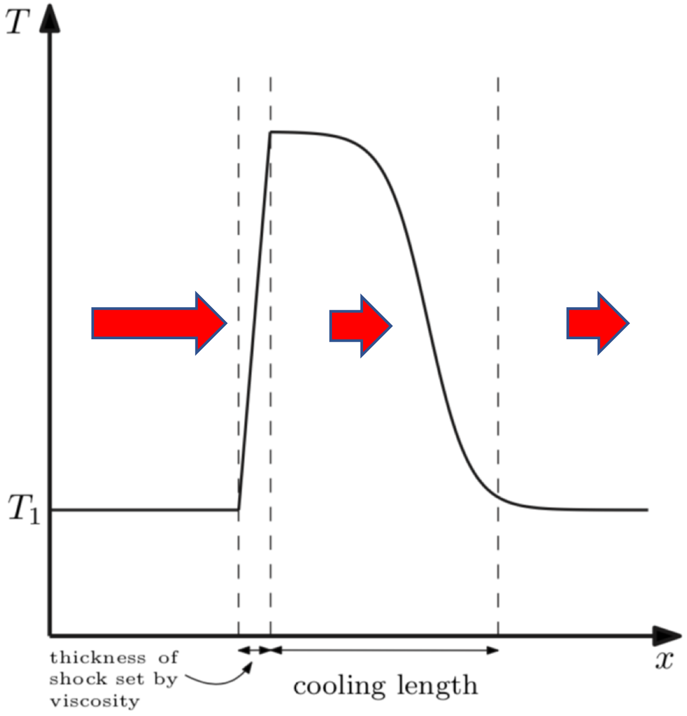
$$\Rightarrow \qquad k_B T_2 = \frac{4(\gamma-1)}{(\gamma+1)^2}\left(\frac{1}{2}\mu m_p u_1^2\right)$$

using $c_{s,1}^2 = \frac{\gamma k_B T_1}{\mu m_p}$

$$\Rightarrow \qquad k_B T_2 = \frac{3}{8}\left(\frac{1}{2}\mu m_p u_1^2\right) \qquad \text{for } \gamma = 5/3$$

Makes explicit the notion that the kinetic energy of the pre-shock fluid is being converted into random motion of post-shock flow.

Another important case to examine are isothermal shocks… relevant when there is strong cooling that cools the post-shock gas back to the pre-shock temperature.



First two Rankine-Hugoniot jump conditions are unchanged:

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

Third R-H relation is replaced with

$$T_1 = T_2 \qquad \Rightarrow \qquad c_{s,1} = c_{s,2}$$

So,

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

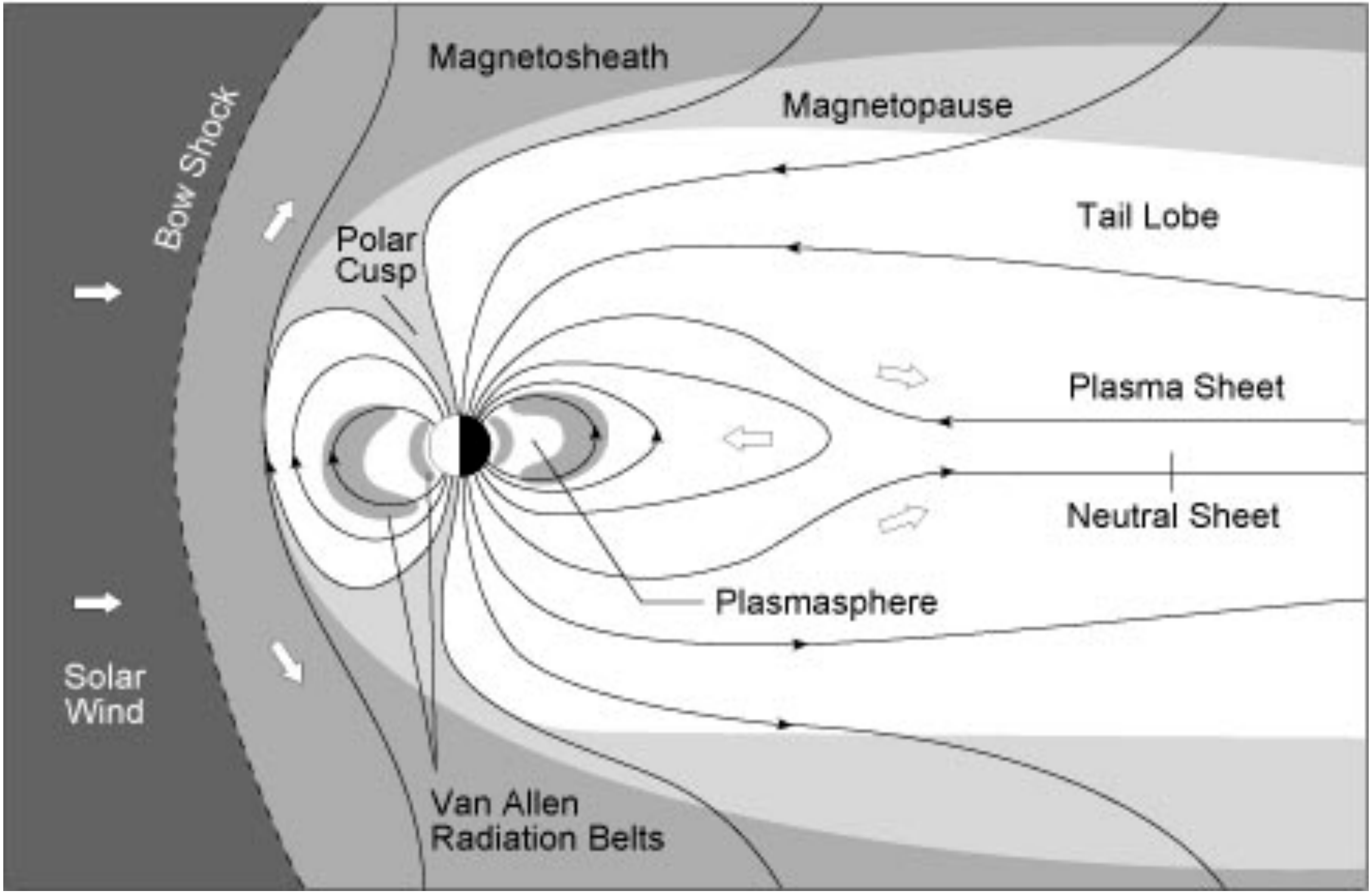
$$\rho_1(u_1^2 + c_s^2) = \rho_2(u_2^2 + c_s^2)$$ since $p_1 = c_{s,1}^2 \rho_1$ etc.

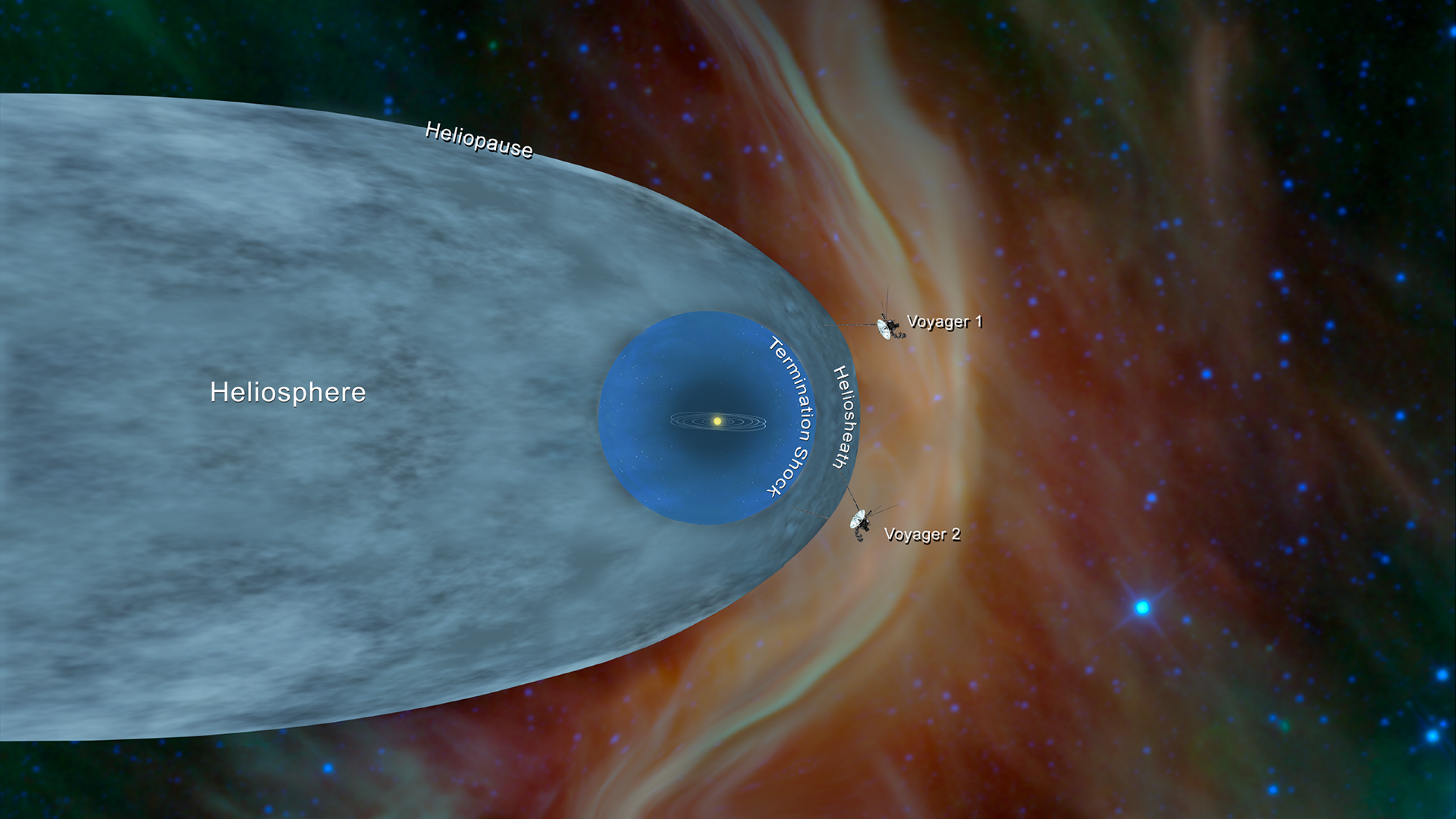
$$\Rightarrow \qquad u_1 + \frac{c_s^2}{u_1} = u_2 + \frac{c_s^2}{u_2} \quad (\text{since } \rho_1 u_1 = \rho_2 u_2)$$

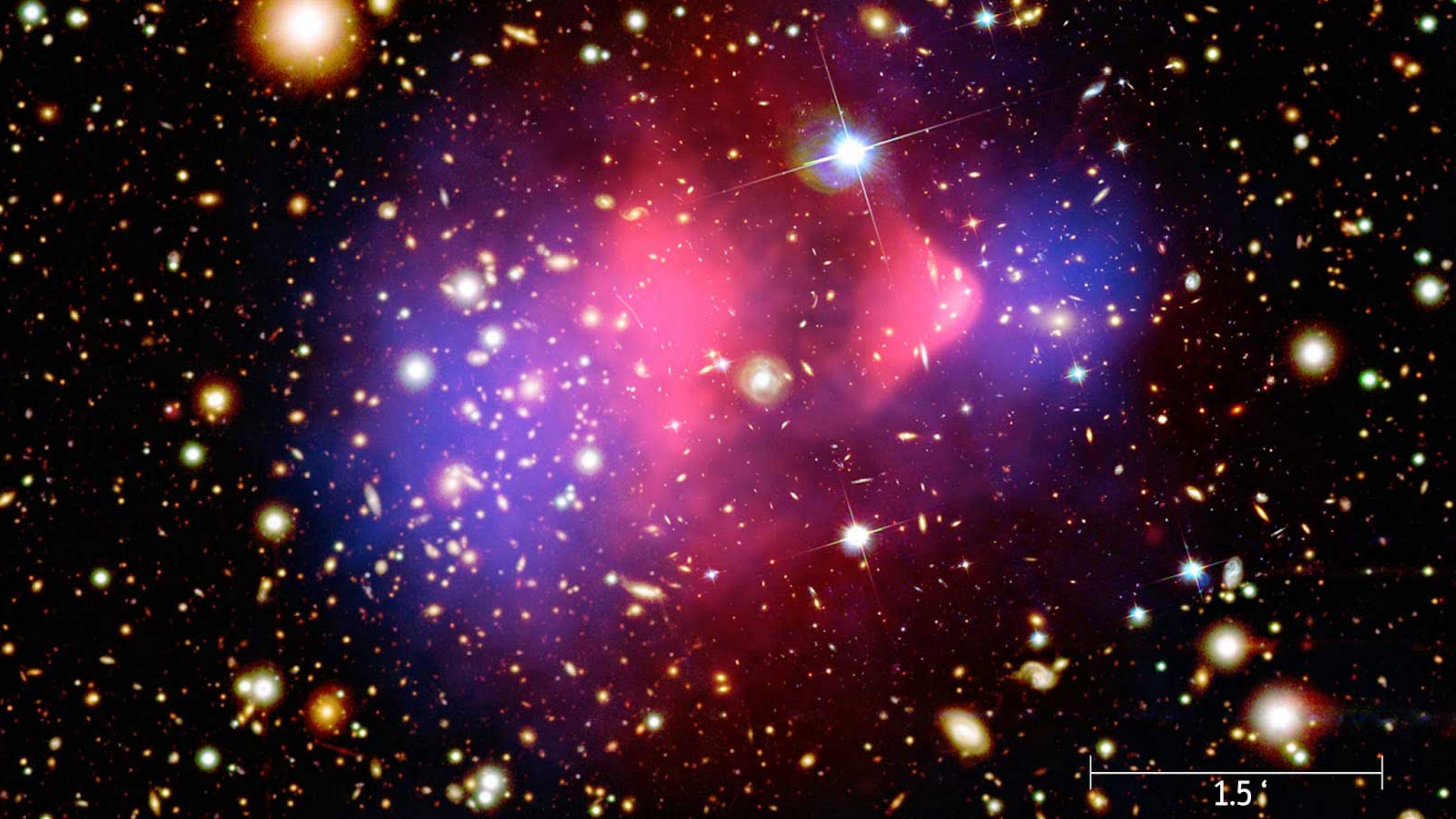
$$\Rightarrow \qquad c_s^2 \left(\frac{1}{u_1} - \frac{1}{u_2} \right) = u_2 - u_1$$

$$\Rightarrow \qquad c_s^2 \frac{u_2 - u_1}{u_1 u_2} = u_2 - u_1$$

$$\Rightarrow \qquad \boxed{c_s^2 = u_1 u_2}$$


Magnetosheath
Magnetopause
Bow Shock
Tail Lobe
Polar
Cusp
Plasma Sheet
Neutral Sheet
Plasmasphere
Solar
Wind
Van Allen
Radiation Belts


Heliopause
Heliosphere
Termination Shock
Heliosheath
Voyager 1
Voyager 2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