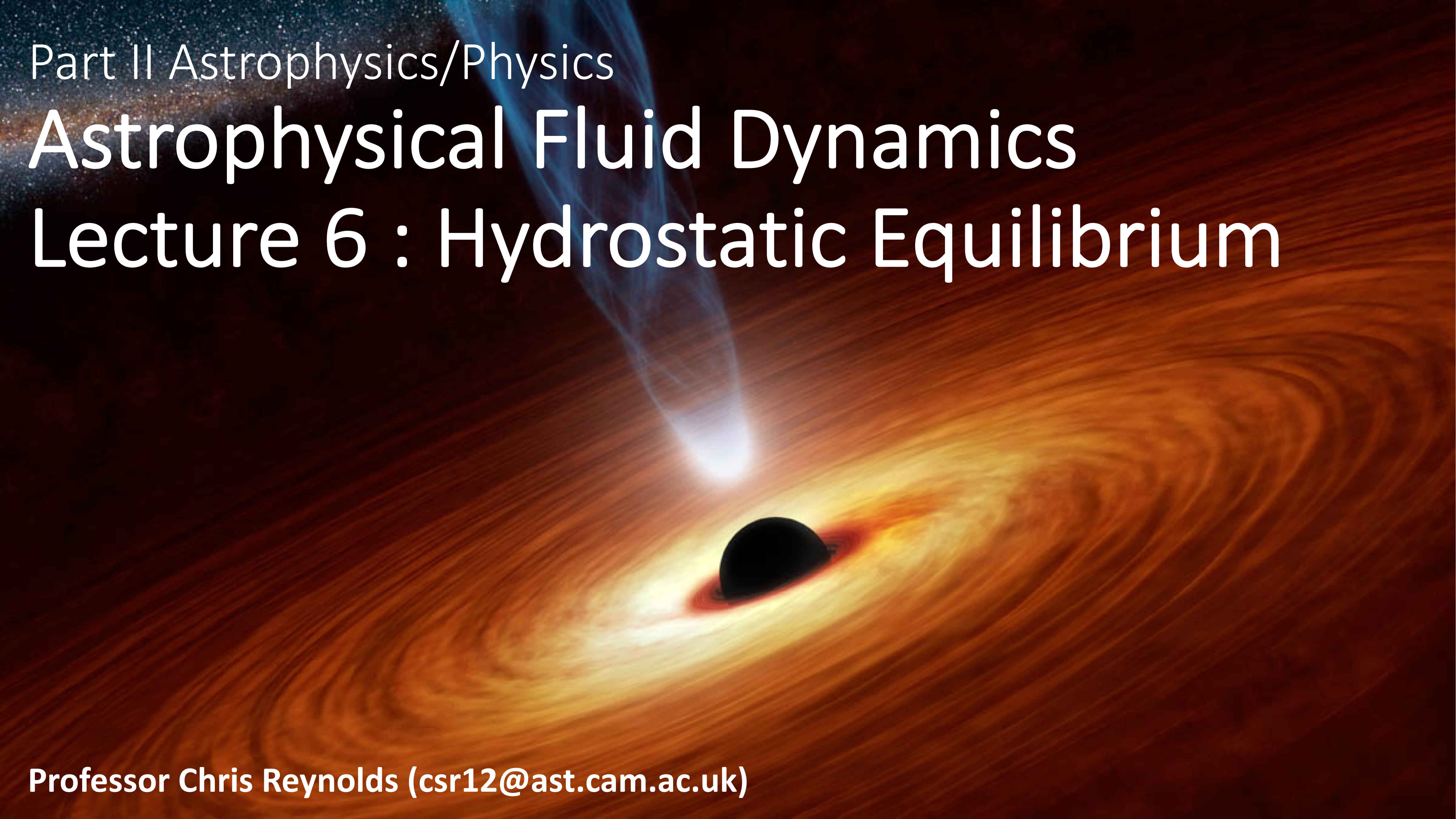

Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 6 : Hydrostatic Equilibrium

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Recap and context

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Continuity Equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g}$$

Momentum Equation

$$\nabla^2 \Psi = 4\pi G \rho$$

Poisson's Equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \rho \frac{\partial \Psi}{\partial t} - \rho \dot{Q}_{\rm cool}$$

Energy Equation

$$E = \rho \left(\frac{1}{2} u^2 + \Psi + \mathcal{E} \right)$$

Defn of total energy

$$p = \frac{k_B}{\mu m_p} \rho T$$

EoS for ideal gas

$$\mathcal{E} = \frac{3p}{2\rho}$$

Internal energy (monoatomic)

This Lecture

- Hydrostatic Equilibrium (Chapter E)
- Basic concepts and examples of hydrostatic equilibrium (E.1)
    - Externally imposed gravitational field
    - Self-gravity
- Stars as self-gravitating polytropes (E.2)
    - Lane-Emden equation
- Isothermal spheres (E.3)
    - Bonnor Ebert spheres

- *After this lecture, you should be able to complete Example Sheet 1*

E.1 : Basics of Hydrostatic Equilibrium

We consider static, equilibrium configurations:

$$\mathbf{u} = 0, \qquad \frac{\partial}{\partial t} = 0$$

Continuity equation is trivially satisfied

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Momentum equation gives:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi = 0$$

$$\boxed{\frac{1}{\rho} \boldsymbol{\nabla} p = -\boldsymbol{\nabla} \Psi} \qquad \text{Equation of Hydrostatic Eqm.}$$

HSE in an external gravitational field

Simplest case to consider is when Ψ is some just specified function of position (so generated by some external agent rather than the fluid mass itself).

If $P = P(\rho)$, this allows us to solve the equation of hydrostatic equilibrium to derive the density structure.

Example : Isothermal atmosphere in a constant gravitational field $\mathbf{g} = -g\hat{\mathbf{z}}$

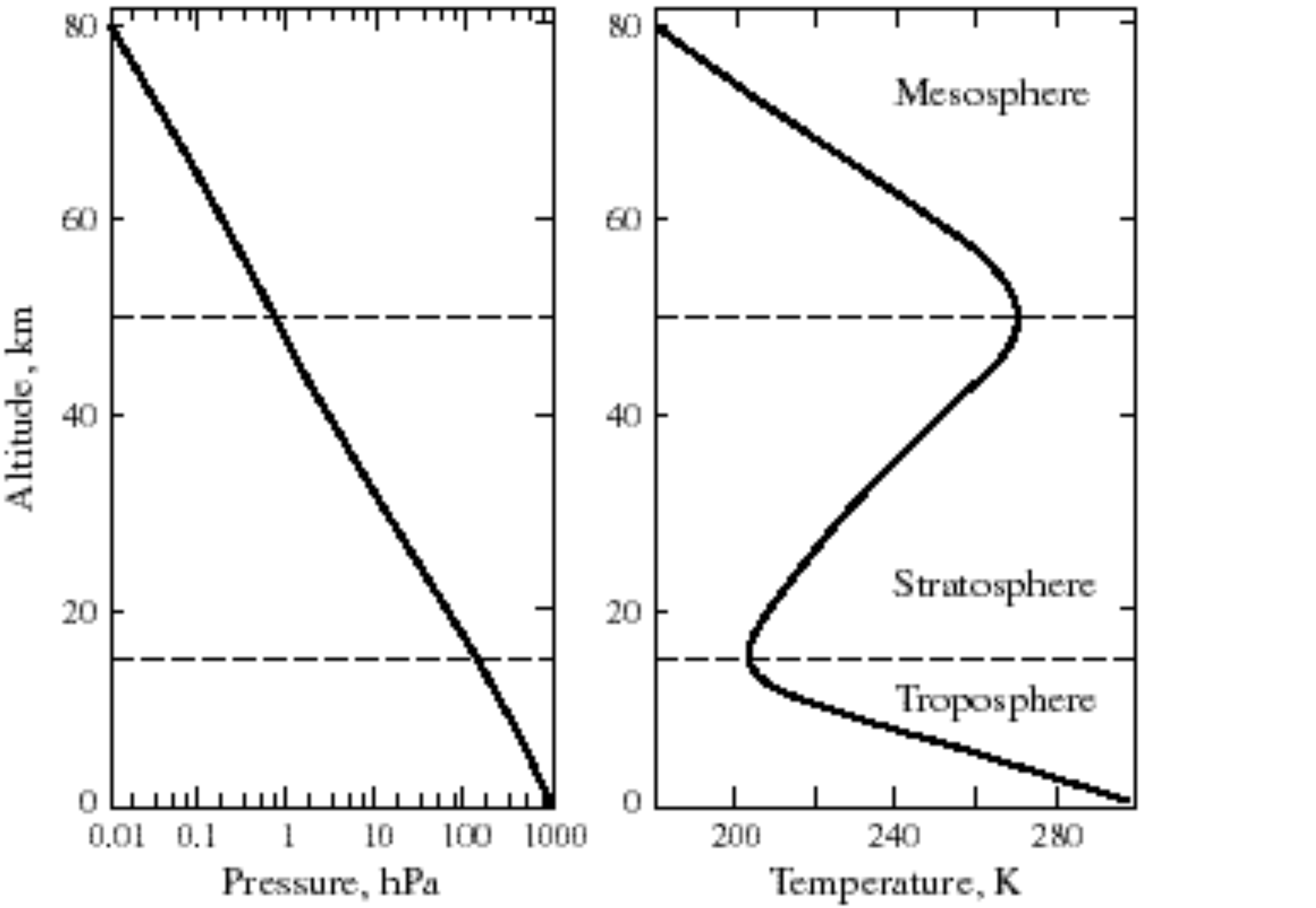
Isothermal : $$p = \frac{\mathcal{R}_*}{\mu}\rho T \qquad \Rightarrow \qquad p = A\rho, \quad A \text{ const.}$$

$$A \cdot \frac{1}{\rho}\nabla\rho = -\nabla\Psi = -g\hat{\mathbf{z}}$$

$$\Rightarrow \qquad \ln\rho = -\frac{gz}{A} + \text{const.}$$

$$\Rightarrow \qquad \rho = \rho_0 \exp\left(-\frac{\mu g}{\mathcal{R}_* T} z\right)$$

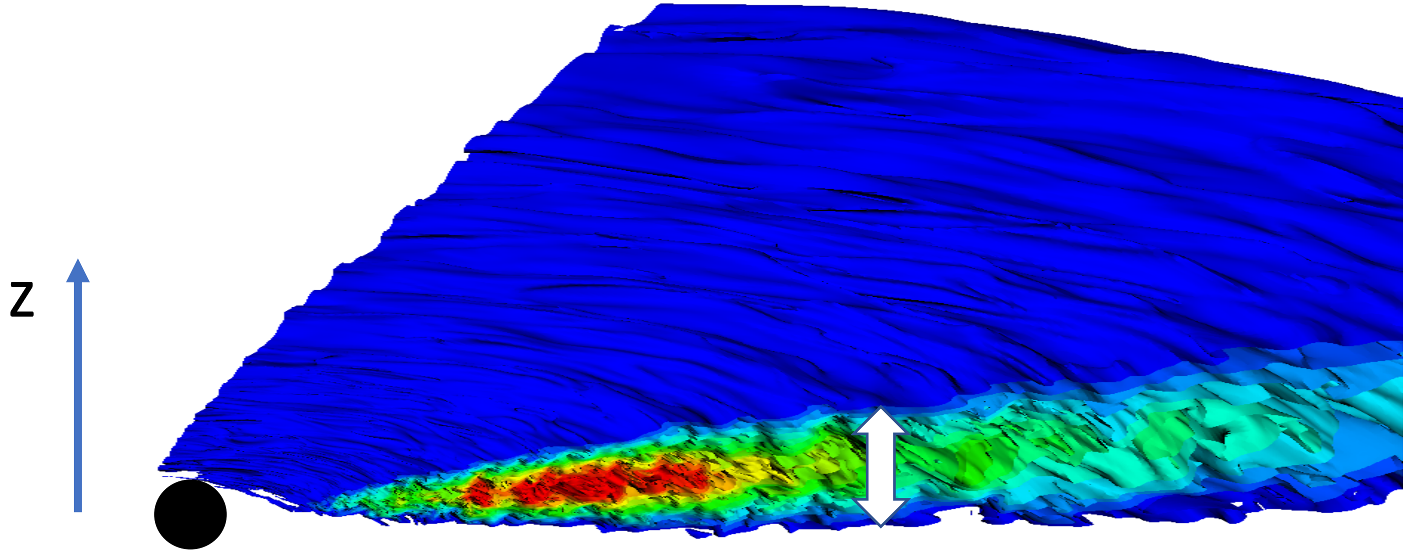
Earth's atmosphere…
T=300K, μ=28, g=9.8 m s^{-2}
e-folding height 9km



http://acmg.seas.harvard.edu/people/faculty/djj/book/bookchap2.html

Mauna Kea Observatories, Hawaii 4200m

Example : Vertical density structure of an isothermal, rotationally-supported, geometrically-thin gas disk orbiting a central mass.



At a given patch of the disk, transform into a locally co-moving and co-rotating frame. In z-direction, pressure forces balance z-cpt of gravity,

$$g_z = -\frac{GM}{r^2}\frac{z}{r} \approx -\frac{GMz}{R^3}$$

So, hydrostatic equilibrium gives

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = g_z$$

$$\Rightarrow \qquad A\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{GM}{R^3}z$$

$$\Rightarrow \qquad \ln\rho = -\frac{GM}{2R^3}z^2 + \text{const.}$$

$$\Rightarrow \qquad \rho = \rho_0 \exp\left(-\frac{GMz^2}{2R^3A}\right)$$

$$\Rightarrow \qquad \rho = \rho_0 \exp\left(-\frac{\Omega^2}{2A}z^2\right) \qquad \text{where} \qquad \Omega^2 = \frac{GM}{R^3}$$

HSE of self-gravitating systems

Now consider static equilibrium of a fluid generating its own gravitational field.
Need to simultaneously solve:

$$\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi \qquad\qquad \nabla^2\Psi = 4\pi G\rho$$

Example : Isothermal self-gravitating slab.

Isothermal : $$p = \frac{\mathcal{R}_*}{\mu}\rho T \qquad \Rightarrow \qquad p = A\rho, \quad A \text{ const.}$$

Hydrostatic Eqm : $$A\frac{1}{\rho}\boldsymbol{\nabla}\rho = -\boldsymbol{\nabla}\Psi$$

$$\Rightarrow \qquad A\frac{\mathrm{d}}{\mathrm{d}z}(\ln\rho) = -\frac{\mathrm{d}\Psi}{\mathrm{d}z}$$

$$\Rightarrow \qquad \Psi = -A\ln(\rho/\rho_0) + \Psi_0 \qquad (\rho_0 = \rho(z=0))$$

$$\Rightarrow \qquad \rho = \rho_0 e^{-(\Psi-\Psi_0)/A}.$$

Poisson Eqn :

$$\frac{d^2\Psi}{dz^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A}.$$

- change variables to $Z = z\sqrt{2\pi G\rho_0/A}$ and $\chi = -(\Psi - \Psi_0)/A$

$$\frac{d^2\chi}{dZ^2} = -2e^\chi \qquad \chi = \frac{d\chi}{dZ} = 0 \text{ at } Z = 0$$

$$\Rightarrow \quad \frac{d\chi}{dZ}\frac{d^2\chi}{dZ^2} = -2\frac{d\chi}{dZ}e^\chi$$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dZ}\left[\left(\frac{d\chi}{dZ}\right)^2\right] = -2\frac{d}{dZ}(e^\chi)$$

$$\Rightarrow \quad \left(\frac{d\chi}{dZ}\right)^2 = C_1 - 4e^\chi. \qquad d\chi/dZ = 0 \text{ when } \chi = 0 \Rightarrow C_1 = 4.$$

- So

$$\therefore \quad \frac{d\chi}{dZ} = 2\sqrt{1-e^\chi} \quad \Rightarrow \quad \int \frac{d\chi}{\sqrt{1-e^\chi}} = 2\int dZ$$

Evaluate χ-integral:

$$\int \frac{d\chi}{\sqrt{1-e^\chi}} = \int \frac{2\cos\theta\, d\theta}{\sin\theta\sqrt{1-\sin^2\theta}} \qquad e^\chi = \sin^2\theta$$

$$= \int \frac{2\, d\theta}{\sin\theta}$$

$$= \int 2 \cdot \frac{1}{2}\frac{1+t^2}{t}\, d\theta \qquad t = \tan\frac{\theta}{2}$$

$$= 2\int \frac{dt}{t}$$

$$= 2\ln t + C_2$$

So, Poisson Equation is

$$2\ln t = 2Z + C_2 \qquad \chi = 0 \text{ at } Z = 0 \Rightarrow \theta = \pi/2, t = 1 \Rightarrow C_2 = 0,$$

$$\Rightarrow \quad t = e^Z \quad \Rightarrow \quad \sin\theta = e^{\chi/2} = \frac{2e^Z}{1+e^{2Z}} = \frac{1}{\cosh Z}$$

Final result:

$$\Psi - \Psi_0 = 2A \ln \cosh\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)$$

$$\rho = \frac{\rho_0}{\cosh^2\left(\sqrt{\frac{2\pi G\rho_0}{A}}z\right)}$$

Stars as self-gravitating polytropes

Consider a spherically-symmetric self-gravitating system in hydrostatic eqm. This is a good approximation to a star. We have

$$\nabla p = -\rho \nabla \Psi$$

$$\Rightarrow \qquad \frac{\mathrm{d}p}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Psi}{\mathrm{d}r} \qquad \text{(spherical polar)}$$

Since ρ>0, this implies that p is a monotonic function of Ψ.

But we also have that

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}p}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Psi}{\mathrm{d}r} \qquad \Rightarrow \qquad \rho = -\frac{\mathrm{d}p}{\mathrm{d}\Psi}$$

So ρ is also a monotonic function of Ψ.

$$\therefore \qquad p = p(\Psi), \rho = \rho(\Psi) \qquad \Rightarrow \qquad p = p(\rho)$$

STARS ARE BAROTROPES

So stars are **barotropes**.

Let's write $p = K\rho^{1+1/n}$ In general, n(ρ).

When n=constant, the structure is called a **polytrope**.

Many real stars can be approximated as polytopes.

Important to note that, in general

$$1 + \frac{1}{n} \neq \gamma.$$

This equality only holds if the star is isentropic (uniform entropy). This is true if, for example, the star is convective throughout.

Analysis of the structure of a polytope:

The equation of hydrostatic equilibrium is

$$-\nabla\Psi = \frac{1}{\rho}\nabla\left(K\rho^{1+1/n}\right) = (n+1)\nabla\left(K\rho^{1/n}\right)$$

$$\Rightarrow \qquad \rho = \left(\frac{\Psi_T - \Psi}{[n+1]K}\right)^n \qquad \Psi_T \equiv \Psi \text{ where } \rho = 0, \text{ the surface.}$$

Central density is

$$\rho_c = \left(\frac{\Psi_T - \Psi_c}{[n+1]K}\right)^n$$

So we write

$$\rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n$$

Feeding this into Poisson's equation:

$$\nabla^2\Psi = 4\pi G\rho_c \left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n$$

$$\Rightarrow \quad \nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n$$

where we remap the potential coordinate $\theta = \dfrac{\Psi_T - \Psi}{\Psi_T - \Psi_c}$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n$$

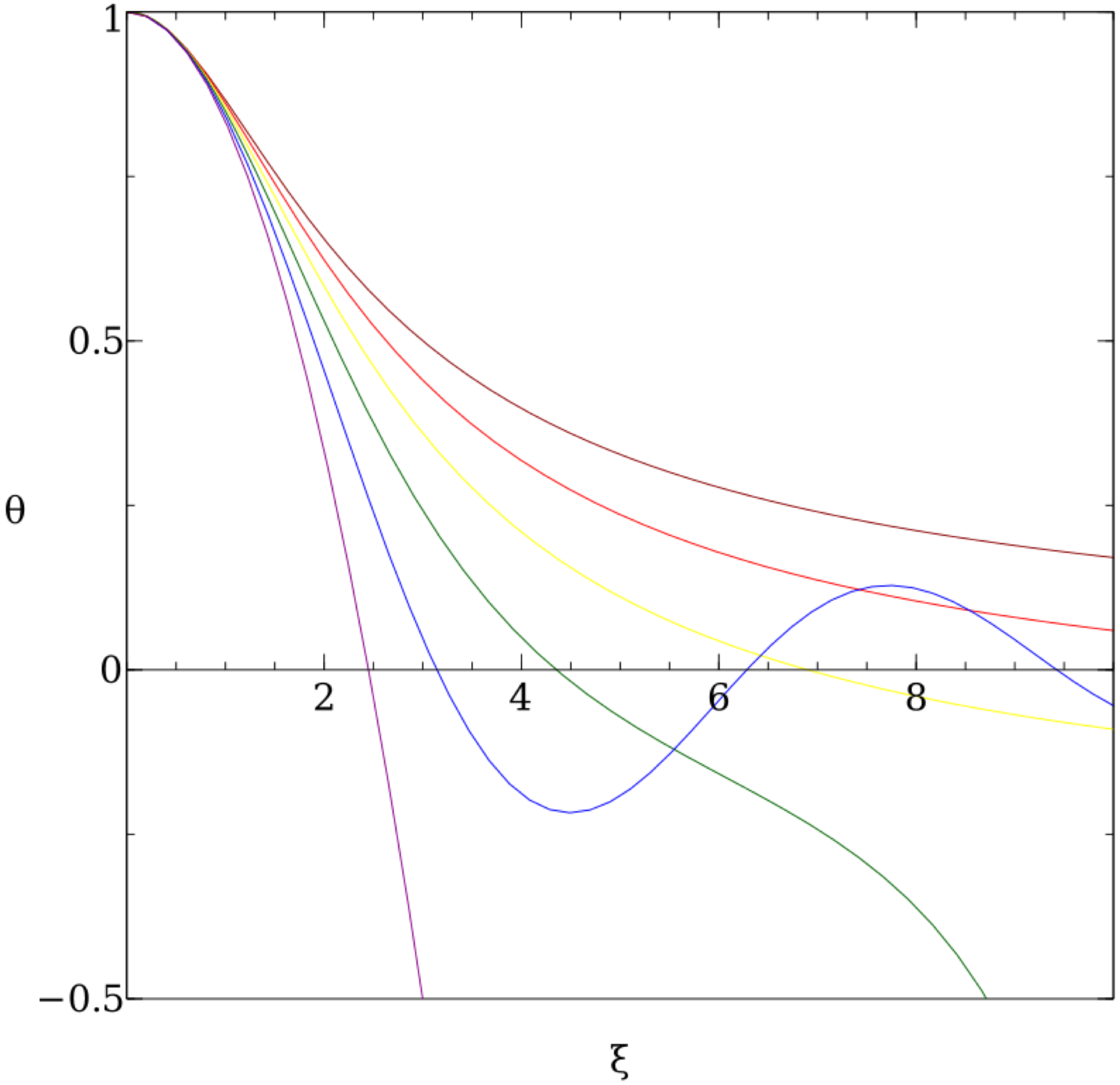
Imposing spherical symmetry and writing in spherical polar coordinates.

Rescale the radial coordinate $\xi = r\sqrt{\dfrac{4\pi G\rho_c}{\Psi_T - \Psi_c}}$

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n}$$

LANE-EMDEN EQN. OF INDEX n

$\theta = 1,\ \mathrm{d}\theta/\mathrm{d}\xi = 0$ at $\xi = 0$.



1
0.5
0
−0.5
2
4
6
8
θ
ξ

Analytic solutions exist for n=0,1,5:

Example, solution for n=0 (corresponding to uniform density)

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n = -1$$

$$p = K\rho^{1+1/n}$$

$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\xi^2$$

$$\Rightarrow \qquad \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{1}{3}\xi^3 - C$$

$$\Rightarrow \qquad \theta = -\frac{\xi^2}{6} + \frac{C}{\xi} + D$$

Apply boundary conditions, C=0, D=1

$$\therefore \qquad \theta = 1 - \frac{\xi^2}{6}$$

Isothermal spheres

Isothermal case corresponds to $n \to \infty$. Hydrostatic equilibrium gives

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Psi}{\mathrm{d}r} \qquad \text{and} \qquad p = K\rho$$

$$\Rightarrow \qquad \frac{\mathrm{d}\Psi}{\mathrm{d}r} = -\frac{K}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

$$\Rightarrow \qquad \Psi - \Psi_c = -K \ln(\rho/\rho_c)$$

Poisson eqn:

$$\nabla^2 \Psi = 4\pi G \rho$$

$$\Rightarrow \qquad \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = 4\pi G \rho$$

$$\Rightarrow \qquad \frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G \rho$$

$$\rho = \rho_c e^{-\Psi} \qquad \xi = r\sqrt{\frac{4\pi G \rho_c}{K}} \qquad \Rightarrow \qquad \boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right) = e^{-\Psi}}$$

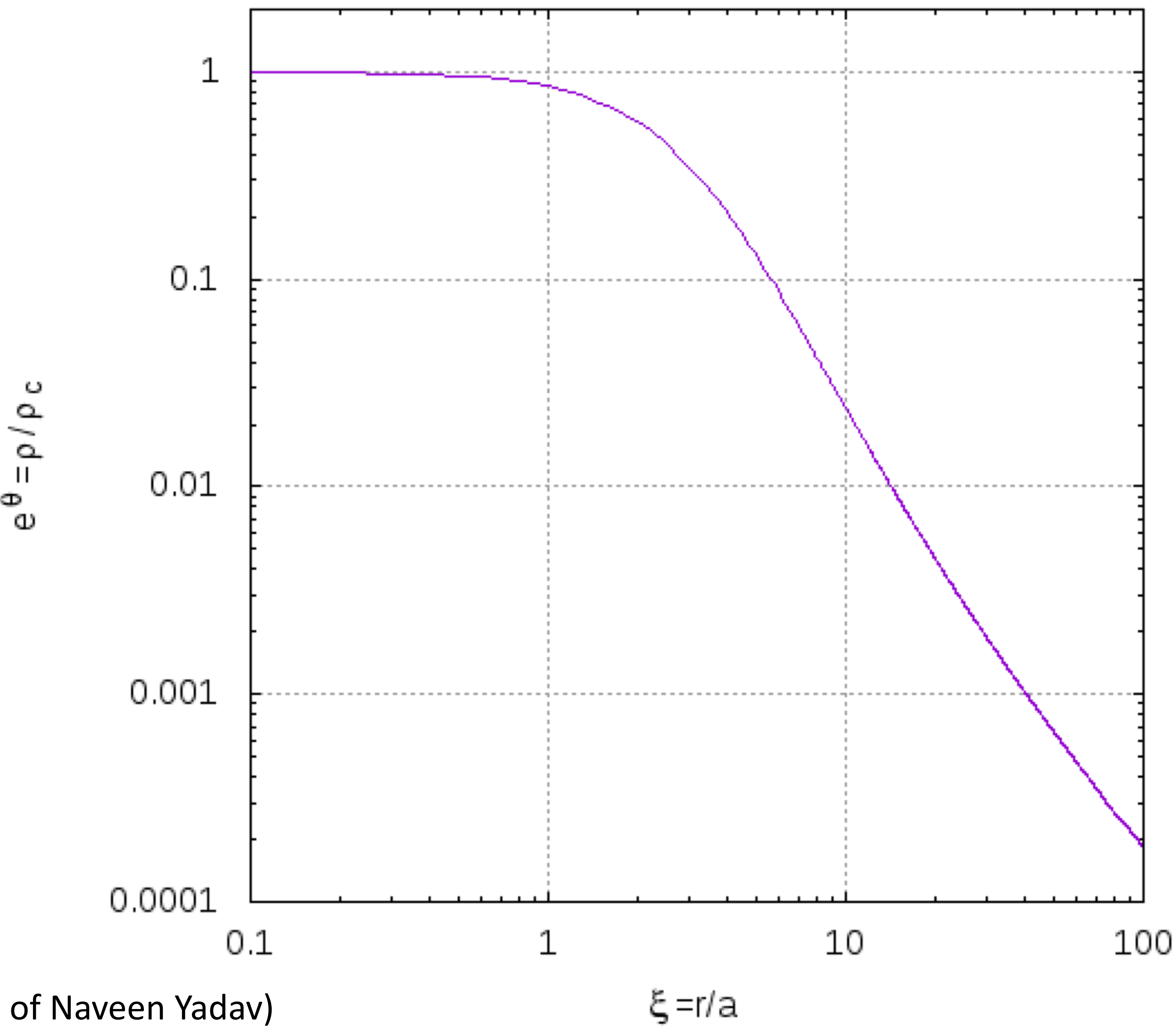
At large radius, this equation implies

$$\rho \propto r^{-2} \qquad \text{as } r \to \infty$$

$$\Rightarrow \qquad M(r) \propto r$$

Physical solutions (finite total mass) require truncation at some finite radius, hence need to be confined by some finite external pressure.

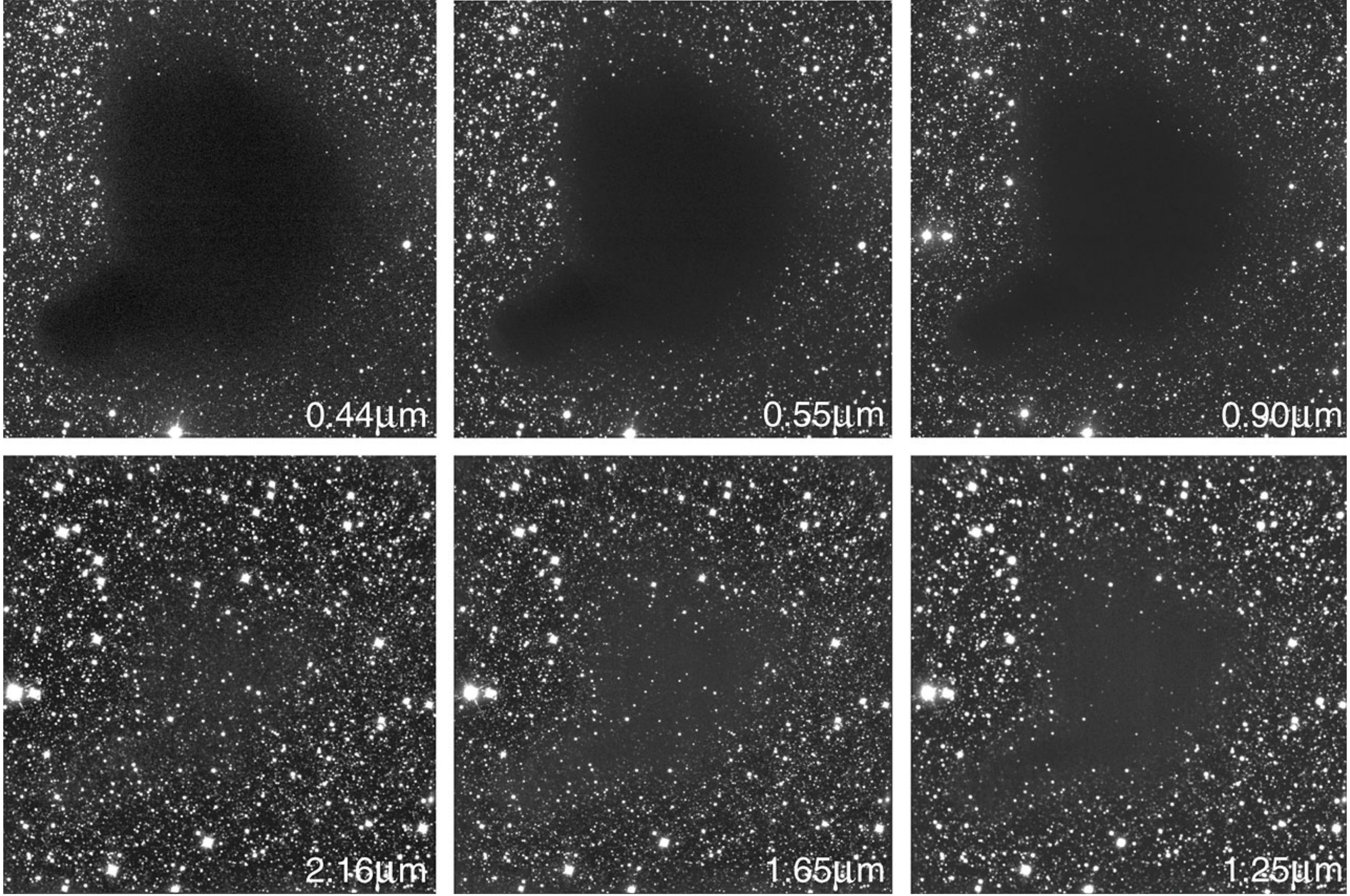
Truncated isothermal spheres are known as **Bonnor-Ebert spheres.**



(From blog of Naveen Yadav)

Stability needs:

$$\frac{\rho_c}{\rho_{\rm ext}} < 14$$



ESO (Dark Clouds in B68)