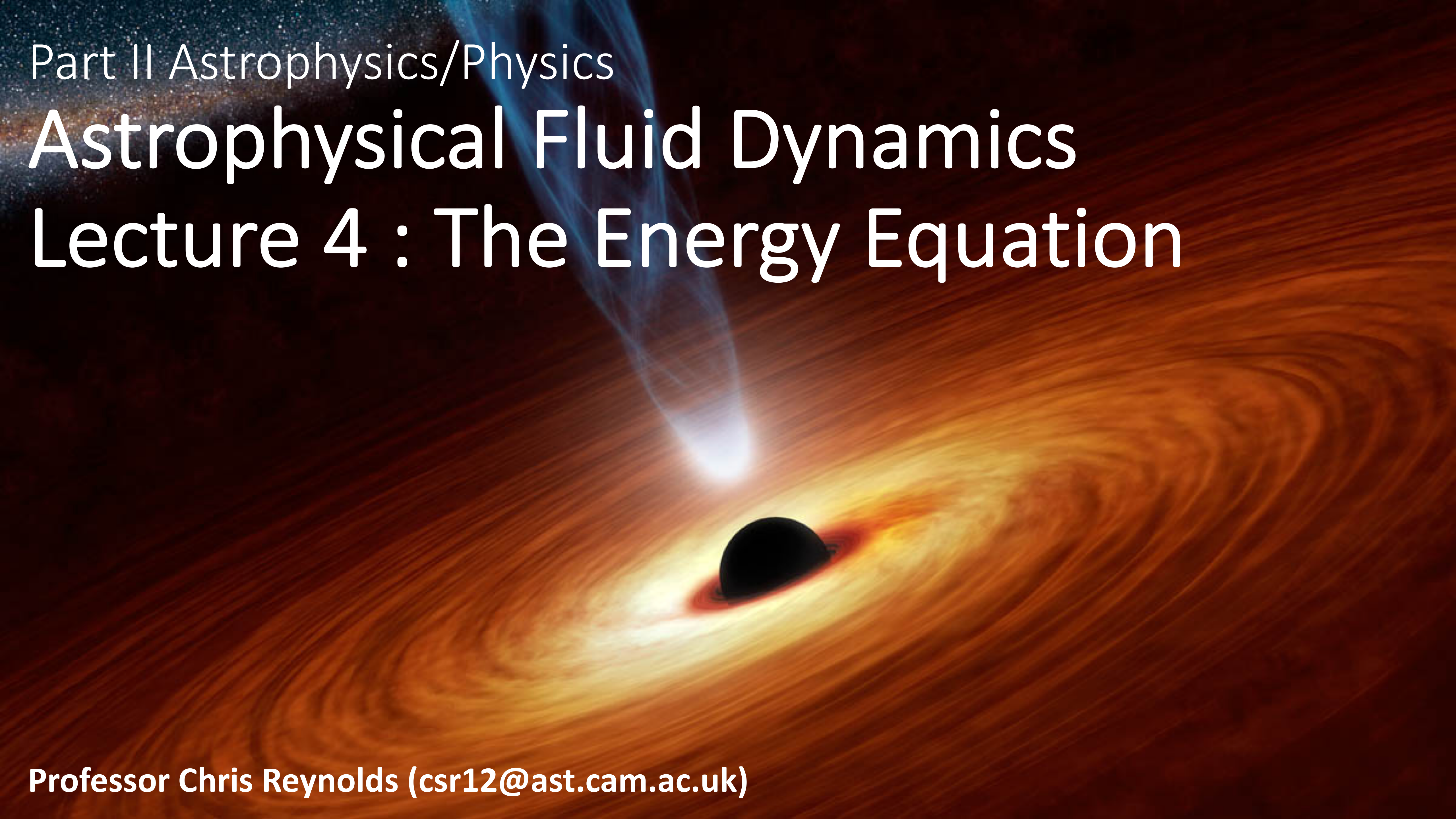
Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 4 : The Energy Equation



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Story so far…

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

CONTINUITY EQUATION

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi$$

MOMENTUM EQUATION

$$\nabla^2 \Psi = 4\pi G \rho$$

POISSON’S EQUATION

This lecture

- The Energy Equation (Chapter D)
- Equation of state (D.1)
- Barotropic fluids (D.1)
    - Examples - isothermal and adiabatic cases
- The energy equation (D.2)

D.1 : Equation of State

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \Psi$$

$$\nabla^2 \Psi = 4\pi G \rho$$

unknowns:

$\rho(\mathbf{r}, t), \mathbf{u}(\mathbf{r}, t), p(\mathbf{r}, t), \Psi(\mathbf{r}, t)$

3 scalar + 1 vector fields

equations:

2 scalar + 1 vector

Must have additional constraints – hence need for an **equation of state,** *p* = …

Equation of state depends upon microphysics of the fluid.

Most important case is that of an ideal gas:

$$p = p(\rho, T) = nk_B T = \frac{k_B}{\mu m_p}\rho T$$

Still doesn't close our system of equations…

- Introduce new unknown scalar field, $T(\mathbf{r},t)$
- Must close equations with a new PDE encoding conservation of energy
- Will discuss such an equation a little later in this lecturE

Barotropic fluids

A **barotropic fluid** is one in which pressure is purely a function of density, $p = p(\rho)$

In this case, the fluid equations form a closed system even without the addition of an energy equation.

<u>**Example**</u> **: Electron degeneracy pressure**

Important in systems with free electrons that are (relatively) cold and dense.

$$p = \frac{\pi^2 \hbar^2}{5 m_e m_{\rm ion}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \rho^{5/3}$$

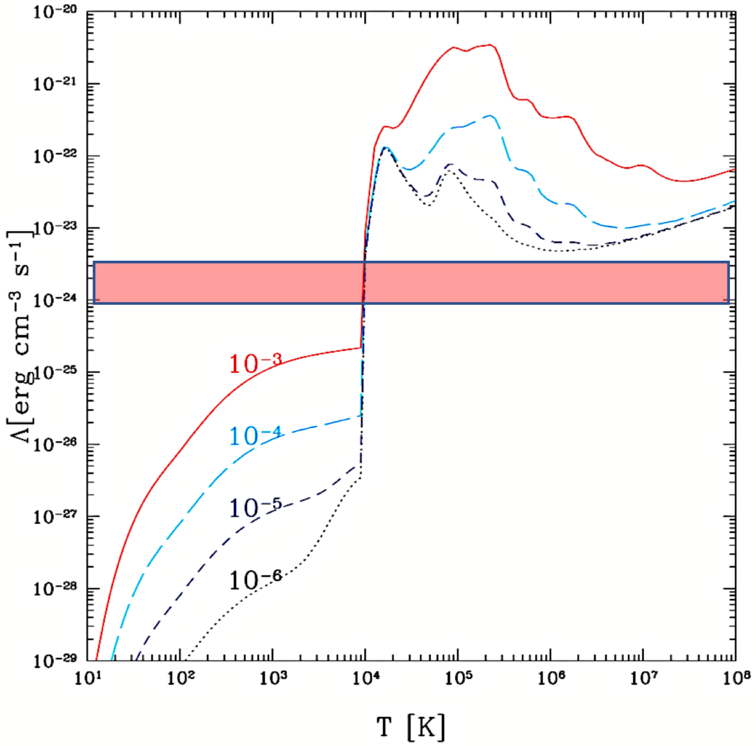
(non-relativistic)

e.g., interiors of white dwarfs, iron core in massive stars, deep interior of Jupiter.

Example : **Isothermal ideal gas**

$$p = A\rho \qquad\qquad A = \frac{k_B T}{\mu m_p} = \text{constant}$$

Most commonly realized when there are strong heating and cooling processes that balance at some well defined temperature.



Example : Adiabatic ideal gas (reversible thermodynamic changes)

$$p = K\rho^{\gamma}$$

where K and γ are constants.

Proof : start with first law of thermodynamics:

$$\underbrace{đQ}_{\substack{\text{heat absorbed by} \\ \text{unit mass of fluid} \\ \text{from surronding}}} = \underbrace{\mathrm{d}\mathcal{E}}_{\substack{\text{change in internal} \\ \text{energy of unit} \\ \text{mass of fluid}}} + \underbrace{p\,\mathrm{d}V}_{\substack{\text{work done by} \\ \text{unit mass of fluid}}}$$

$$p = \frac{\mathcal{R}_*}{\mu}\rho T, \qquad \mathcal{E} = \mathcal{E}(T)$$

Includes microphysical degrees of freedom

$$đQ = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\mathrm{d}T + p\,\mathrm{d}V$$
$$= C_V\,\mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V}\mathrm{d}V$$

For reversible change, $đQ = 0$, so

$$C_V \, \mathrm{d}T + \frac{\mathcal{R}_* T}{\mu V} \, \mathrm{d}V = 0$$

$$\Rightarrow \qquad C_V \, \mathrm{d}(\ln T) + \frac{\mathcal{R}_*}{\mu} \, \mathrm{d}(\ln V) = 0$$

$$\Rightarrow \qquad V \propto T^{-C_V \mu / \mathcal{R}_*}$$

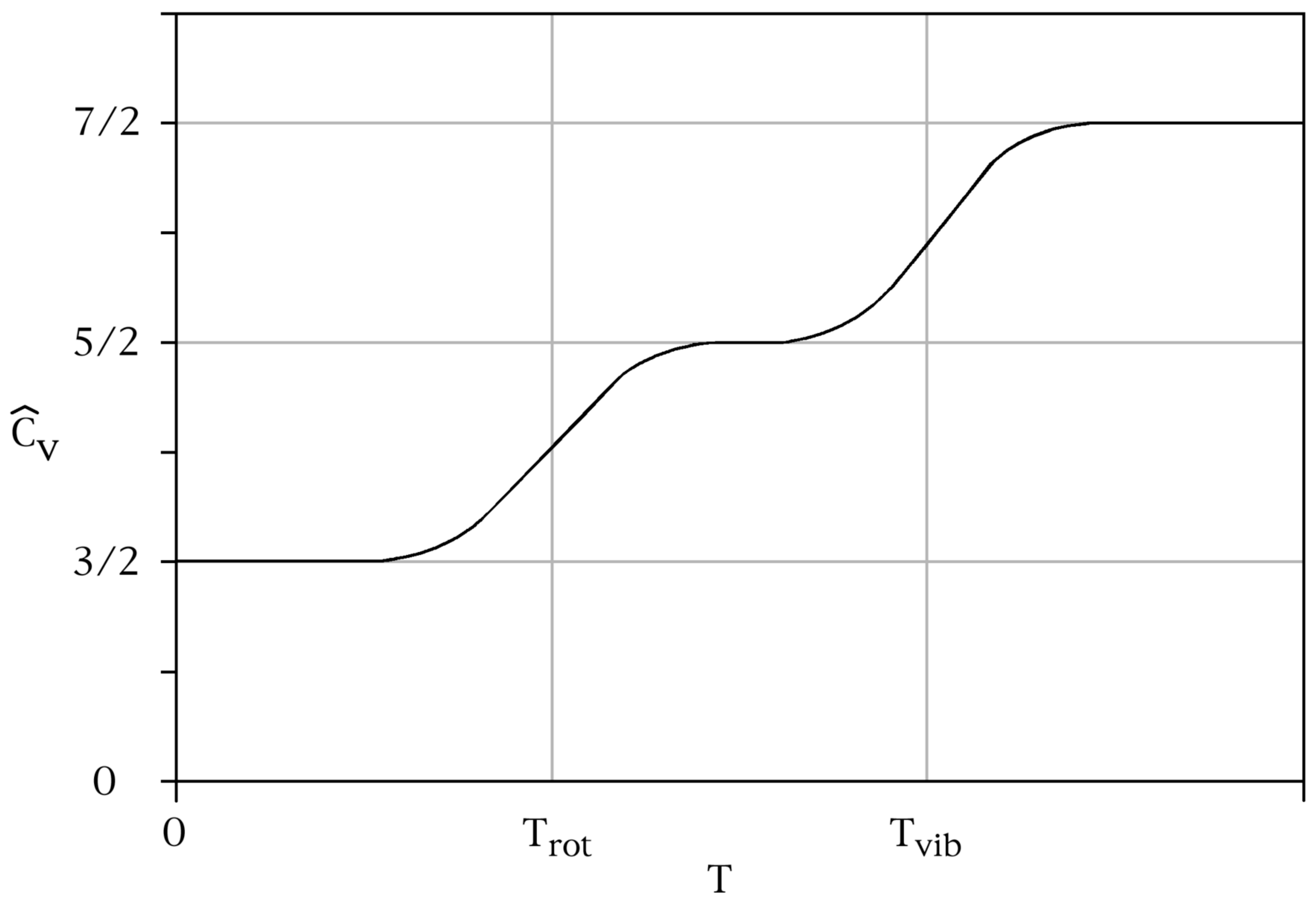
$$\Rightarrow \qquad p \propto T^{1 + C_V \mu / \mathcal{R}_*}$$

If there are *f* degrees of freedom in which the gas can storage internal energy, then

$$C_V = f \frac{\mathcal{R}_*}{2\mu} \qquad \left(\frac{1}{2} k_B T \text{ per particle per degree of freedom} \right)$$

monoatomic; f = 3

diatomic ; f = 5 (if two rotational modes excited)



7/2
5/2
3/2
0
$\widehat{C}_V$
0
T_{rot}
T_{vib}
T

Return to ideal gas law

$$p = \frac{\mathcal{R}_*}{\mu}\rho T \qquad \text{with } \rho = 1/V \text{ for a unit mass of fluid}$$

$$\Rightarrow \qquad pV = \frac{\mathcal{R}_* T}{\mu}$$

$$\Rightarrow \qquad p\,\mathrm{d}V + V\,\mathrm{d}p = \frac{\mathcal{R}_*}{\mu}\,\mathrm{d}T$$

So:

$$\text{đ}Q = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T + p\,\mathrm{d}V$$

$$= \underbrace{\left(\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T} + \frac{\mathcal{R}_*}{\mu}\right)}_{\substack{\text{specific heat capacity} \\ \text{at constant pressure, } C_p}} \quad \mathrm{d}T - V\,\mathrm{d}p$$

We have shown that

$$C_p - C_V = \frac{\mathcal{R}_*}{\mu}$$

Define ratio of specific heat capacities,

$$\gamma \equiv \frac{C_p}{C_V} \qquad = \frac{f+2}{f}$$

So we can see

$$\left.\begin{aligned} p &\propto T^{1+C_V\mu/\mathcal{R}_*} \quad &\Rightarrow \quad p &\propto T^{\gamma/(\gamma-1)} \\ V &\propto T^{-C_V\mu/\mathcal{R}_*} \quad &\Rightarrow \quad V &\propto T^{-1/(\gamma-1)} \end{aligned}\right\} \quad p \propto \rho^\gamma$$

Fluid element behaves adiabatically if $p = K\rho^\gamma$ with *K* = constant

The fluid is isentropic if all fluid elements behave adiabatically with same *K*

(ln *K* proportional to the entropy per unit mass)

D.2 : The Energy Equation

In general case, we will need to account for the heating and cooling of the fluid as it flows. We need to derive a PDE that describes energy conservation in the flow.

Again, start with the first law of thermodynamics but now do not assume dQ=0…

$$đQ = \mathrm{d}\mathcal{E} + \underbrace{p\,\mathrm{d}V}_{\mathrm{d}W=-p\mathrm{d}V}$$

Apply this to a given fluid element (Lagrangian framework)

$$\frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{\mathrm{D}W}{\mathrm{D}t} + \frac{đQ}{\mathrm{d}t} \quad \Rightarrow \quad \frac{\mathrm{D}\mathcal{E}}{\mathrm{D}t} = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}$$

$$\frac{\mathrm{D}W}{\mathrm{D}t} = -p\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

$$\frac{đQ}{\mathrm{d}t} \equiv -\dot{Q}_{cool} \qquad \text{rate of cooling per unit mass}$$

Define total energy per unit volume: $$E = \rho(\underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\Psi}_{\text{potential}} + \underbrace{\mathcal{E}}_{\text{internal}})$$

Evaluate Lagrangian changes of *E*

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\mathrm{D}\rho}{\mathrm{D}t}\frac{E}{\rho} + \rho\left(\mathbf{u}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \frac{\mathrm{D}\Psi}{\mathrm{D}t} + \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t} - \dot{Q}_{\mathrm{cool}}\right) \qquad (**)$$

with

$$\frac{\mathrm{D}E}{\mathrm{D}t} \equiv \frac{\partial E}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}E \qquad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\Psi$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \frac{\mathrm{D}\Psi}{\mathrm{D}t} \equiv \frac{\partial\Psi}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\Psi$$

Substituting into (**):

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{E}{\rho}\rho\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla p - \rho\mathbf{u}\cdot\nabla\Psi + \rho\frac{\partial\Psi}{\partial t}$$

$$+\rho\mathbf{u}\cdot\nabla\Psi - \frac{p}{\rho}\rho\nabla\cdot\mathbf{u} - \rho\dot{Q}_{\mathrm{cool}}$$

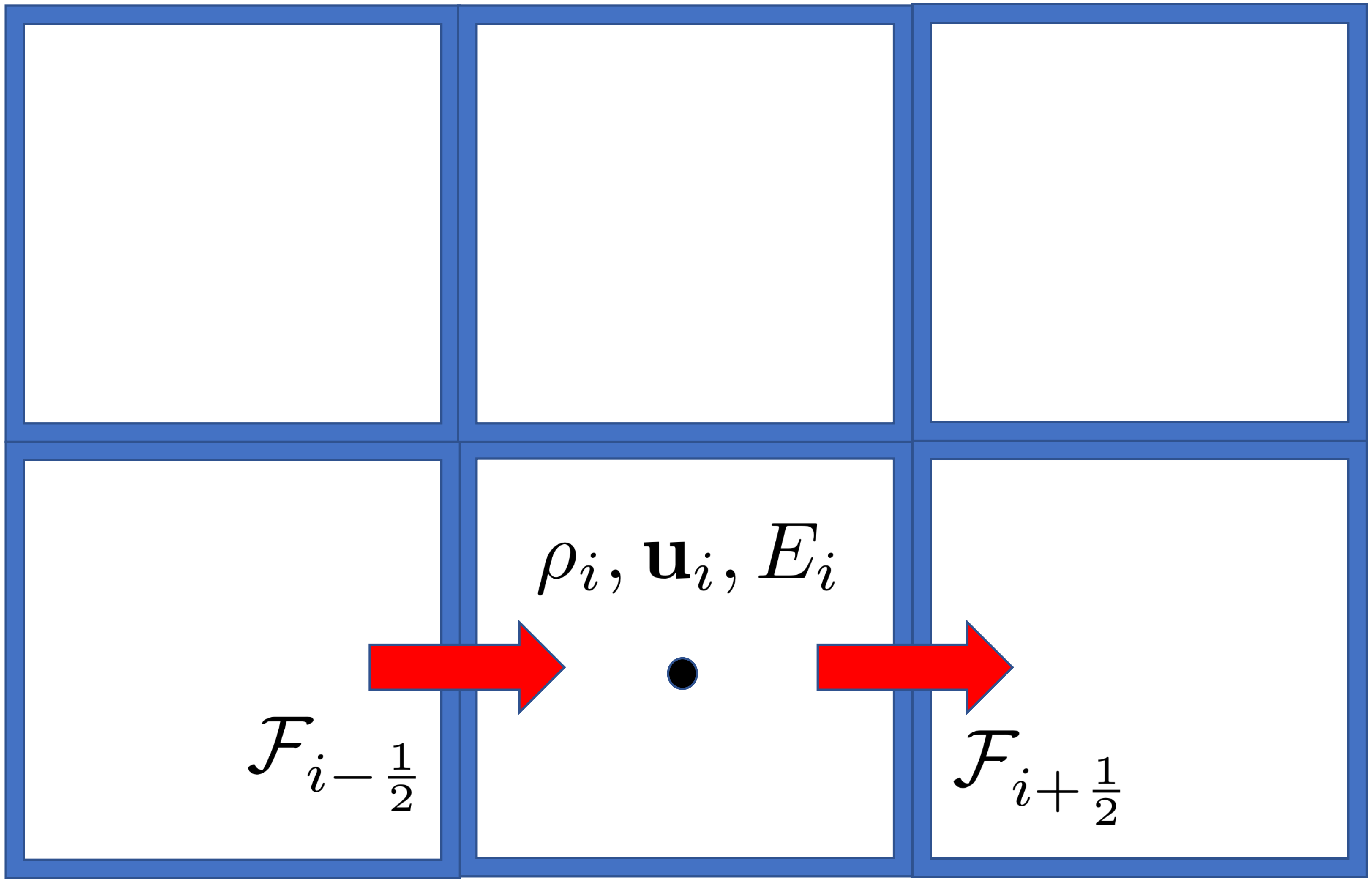
$$\Rightarrow \qquad \frac{\partial E}{\partial t} + \mathbf{u}\cdot\nabla E = -(E+p)\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla p + \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{\mathrm{cool}}$$

$$\boxed{\frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\mathbf{u}] = \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{\mathrm{cool}}}$$

ENERGY EQUATION

LHS has the standard "conservative form" – changes of total energy density due to divergence in the flux of enthalpy (*E*+*p*).

Gravitational field is only a source of total energy if it is changing over time.



$$\mathcal{F} = (E + p)u_x$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot ((E+p)\mathbf{u}) \quad \Rightarrow \quad \frac{E_i(t+\delta t) - E_i(t)}{\delta t}\Delta V = -(\mathcal{F}_{i+\frac{1}{2}}(t) - \mathcal{F}_{i-\frac{1}{2}}(t))\Delta A + ...$$