Structure and Evolution of Stars

Lecture 7: Equation of State Inside Stars

- Thermodynamic equilibrium – macro- and micro-scopic processes important for understanding stars
- Equation of State for a gas and the effect of composition
  - Mass Fractions
  - Mean Molecular Mass
- Radiation Pressure
- Total Pressure for Classical Gas
- Star(s) of the Week #1: Measuring the Mass of the Galaxy

The Equation of State

- In Lecture 6, used familiar conservation principles (e.g. mass and then momentum for hydrostatic equilibrium) to produce relations governing the global properties of stars – e.g. M(r), T(r),…
- However, even when thinking about the question of what contributes to the pressure in the star, clear that the microscopic properties of the material are also important - we assumed a perfect gas and pure hydrogen composition but found inconsistency with estimate of the temperature from the virial theorem, cf T_{eff}=5800K
- Both macroscopic and microscopic factors are important for understanding the structure and evolution of stars
- Will consider the **equation of state** $P=P(\rho, T$, composition) for the material inside the star

Thermodynamic Equilibrium

- Straightforward estimates of the central pressure and the mean temperature inside the Sun from macroscopic considerations (hydrostatic equilibrium and the virial theorem) indicate high P and T and the mean density is also significant
- Interactions between particles and radiation in such circumstances are such that the "mean free path" (mfp) is small and mfp$<<R$, satisfying the condition for thermal equilibrium locally, i.e. radiation described by a blackbody, depends on T only
- "mean free time" (mft) is very small – radiation has velocity c and matter has large v (high T) – and mft<<timescale_for_macroscopic_change, i.e. system in thermodynamic equilibrium, specified by $T(r)$, $\rho(r)$ and composition(r)

Specifying the Composition

Large T from Virial Theorem suggests that material is ionised, consisting of atomic nuclei and free electrons

Mass fraction, X_i, of nuclear species i:

$$X_i = \frac{\rho_i}{\rho}$$

Number density, n_i, of species i:

$$n_i = \frac{\rho_i}{A_i m_H}$$

Where A_i is the atomic mass, e.g. A_{Carbon}=12

$$n_i = \frac{\rho}{m_H}\frac{X_i}{A_i}; \qquad X_i = \frac{n_i A_i m_H}{\rho}$$

Will see in later lecture that thermonuclear reactions result in change of X_i as nuclei are transformed from one species into another

Equation of State

In general, $P=P(\rho, T, \text{composition})$

For an ideal gas:

$$P = nkT$$

If $m_p=\rho/n$= average mass of a particle:

$$P = \frac{\rho kT}{m_P}$$

So, m_p, is important and can define quantity **mean molecular mass**, μ:

$$\mu = \frac{m_P}{m_H}$$

μ is the mean mass of gas particles in units of the mass of the hydrogen atom. Note - aside from cool outer atmospheres of late-type stars, no molecules!

$$P = \frac{\rho}{\mu m_H} kT$$

μ depends on the composition and ionisation

Calculating, μ, for partially ionised material involves solving Saha's equation to derive the fraction of an atomic species in each state of partial ionisation.

Fully ionised case provides very good approximation for stellar interiors

Calculation for neutral or fully ionised gas is straightforward.

For neutral gas, mean particle mass, m_n:

$$m_n = \frac{\sum_j N_j m_j}{\sum_j N_j} = \frac{\text{total mass}}{\text{total \# particles}}$$

where summation over "*j*" is for different atomic species

Divide each side by m_H to give:

where $A_j = m_j / m_H$

$$\mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j}$$

For fully ionised gas, in an exactly similar way, obtain:

where w_j is the number of electrons liberated by complete ionisation of an atom of type j

$$\mu_{ion} = \frac{\sum_j N_j A_j}{\sum_j N_j (1 + w_j)}$$

Easy to see that for pure hydrogen gas, $\mu_n / \mu_{ion} = 2$

Mass Fractions

The mean molecular mass is normally expressed in terms of mass fractions, with 3 mass fractions defined:

X for hydrogen
Y for helium
Z for metals

where "metals" refers to all atomic species combined *except* hydrogen and helium

$$X = \frac{\text{total mass of H}}{\text{total gas mass}}; \; Y = \frac{\text{total mass of He}}{\text{total gas mass}}$$

$$\mathrm{Z} = \frac{\text{total mass of metals}}{\text{total gas mass}}$$

Mass Fractions

Clearly $X+Y+Z=1$

Relate μ to X, Y, Z for the case of neutral gas:

$$\frac{1}{m_p} = \frac{1}{\mu_n m_H} = \frac{\sum_j N_j}{\sum_j N_j m_j}$$

$$= \frac{\text{total \# particles}}{\text{total mass of gas}}$$

$$= \sum_j \frac{\text{\# of j particles}}{\text{total mass j}} \times \frac{\text{total mass j}}{\text{total mass of gas}}$$

Introducing F_j to designate the mass fraction of species j

$$= \sum_j \frac{N_j}{N_j A_j m_{\mathrm{H}}} \times F_j$$

$$= \sum_j \frac{1}{A_j m_H} F_{\mathrm{j}}$$

Mass Fractions & Mean Molecular Mass

$$\frac{1}{\mu_n m_H} = \sum_j \frac{1}{A_j m_H} F_j$$

Cancel $1/m_H$

$$\Rightarrow \frac{1}{\mu_n} = \sum_j \frac{1}{A_j} F_j$$

$$= X + \frac{1}{4} Y + \left\langle \frac{1}{A} \right\rangle Z$$

where $\left\langle \frac{1}{A} \right\rangle$ is a weighted average for metals

For the Sun, $\left\langle \frac{1}{A} \right\rangle \approx \frac{1}{16}$

Mass Fractions & Mean Molecular Mass

For fully ionised gas (the more relevant case):

$$\frac{1}{\mu_{ion}} = \sum_j \frac{(1+w_j)}{A_j} F_j$$

$$= 2X + \frac{3}{4}Y + \left\langle \frac{1+w}{A} \right\rangle Z$$

For elements significantly greater mass than He then $1+w_j \approx w_j$

Nuclei typically have the same number of protons as neutrons and thus #protons≈#neutrons≈#electrons and $A_j \approx 2w_j$

$$\therefore \left\langle \frac{1+w}{A} \right\rangle \approx \frac{1}{2}$$

Mass Fractions & Mean Molecular Mass

Now have:

$$\frac{1}{\mu_{ion}} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

Z is in fact small,
typically $Z \approx 0.03$, thus:

$$\frac{1}{\mu_{ion}} \approx 2X + \frac{3}{4}Y$$

Y is then $\approx 1-X$

$$\Rightarrow \frac{1}{\mu_{ion}} \approx 2X + \frac{3}{4}(1-X) = \frac{5X+3}{4}$$

For typical value of
X=0.7 gives μ_{ion}=0.62

The Total Pressure

- For material behaving as a perfect gas, with $P=nkT$, have seen that P depends on the composition of the material and the ionisation state
- Both the ions and electrons contribute to the gas pressure and the assumption of complete ionisation for the interior of stars is very good
- Radiation also contributes to the pressure and $P=P_{gas}+P_{rad}$
- Radiation pressure is given by the familiar relation:

$$P_{rad} = \frac{1}{3} a T^4 \quad \text{where} \quad a = \frac{8\pi^5 k^4}{15 h^3 c^3} = 7.6 \times 10^{-16}\ \text{J K}^{-4}\text{m}^{-3}$$

where a is the "radiation constant"

The Total Pressure

Can derive the relations for the pressure from a perfect gas and radiation using the pressure integral:

Calculation is based on considering the rate at which momentum is transferred to an imaginary surface

In the case of a perfect gas, take velocity distribution of particles to be Maxwellian

For radiation, take frequency distribution of photons to be given by the blackbody relation

$$P = \int_0^\infty v p n(p) dp$$

$v =$ velocity

$p =$ momentum

$n(p)dp = \#$ particles per unit volume with momenta in the interval $p \rightarrow p + dp$

The Total Pressure

For a classical system, no quantum-mechanical considerations important, have expression for total pressure:

$$P = nkT + \frac{1}{3}aT^4$$

Valid in many cases when describing the behaviour of material within stars, e.g. the Sun

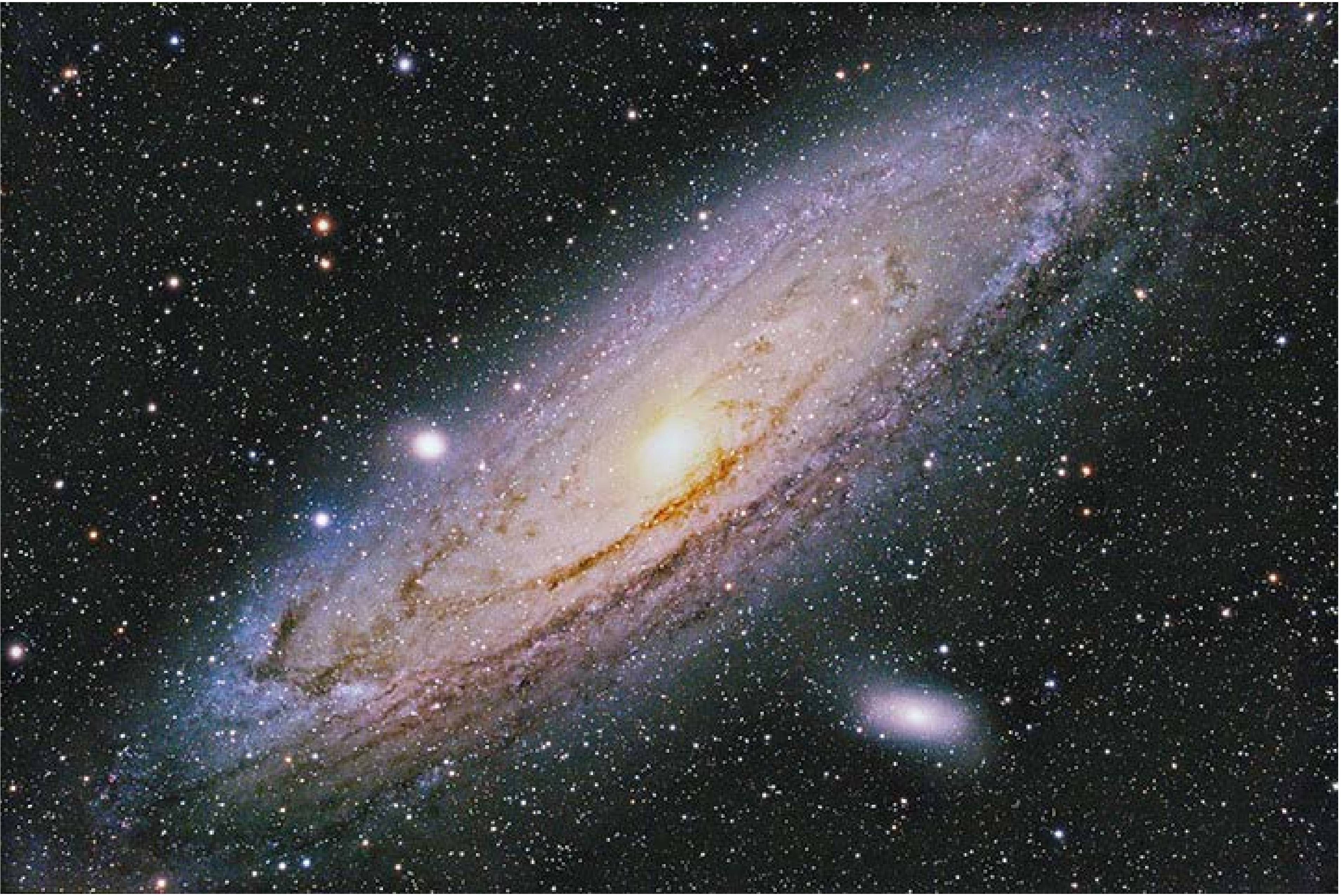
However, quantum mechanical effects do become important in certain circumstances and are key to understanding existence of systems such as white dwarfs and neutron stars – see Lecture 8

Star(s) of the Week: #1

- SDSSJ110639.10-004720.06; *g*=20.2
  SDSSJ122726.77+004641.03; *g*=21.0
- Aim to determine mass of the Galaxy, including dark halo, using stars with photometric distances – from Clewley+ 2005 MNRAS 362 349; now a standard technique – many papers since, e.g. Deason+ 2015 MNRAS 448 77
- Select relatively blue, B-V~0.2, faint, m_V~20.5, stellar objects away from the Galactic plane
- Colour and magnitude of selected objects include horizontal branch stars at distances out to distances of 100kpc. Horizontal branch has ~constant L – "standard candle"

$$d = 10^{(m-M+5)/5}\ \mathrm{pc}$$

with m~20.5, M=+0.5, d=100kpc



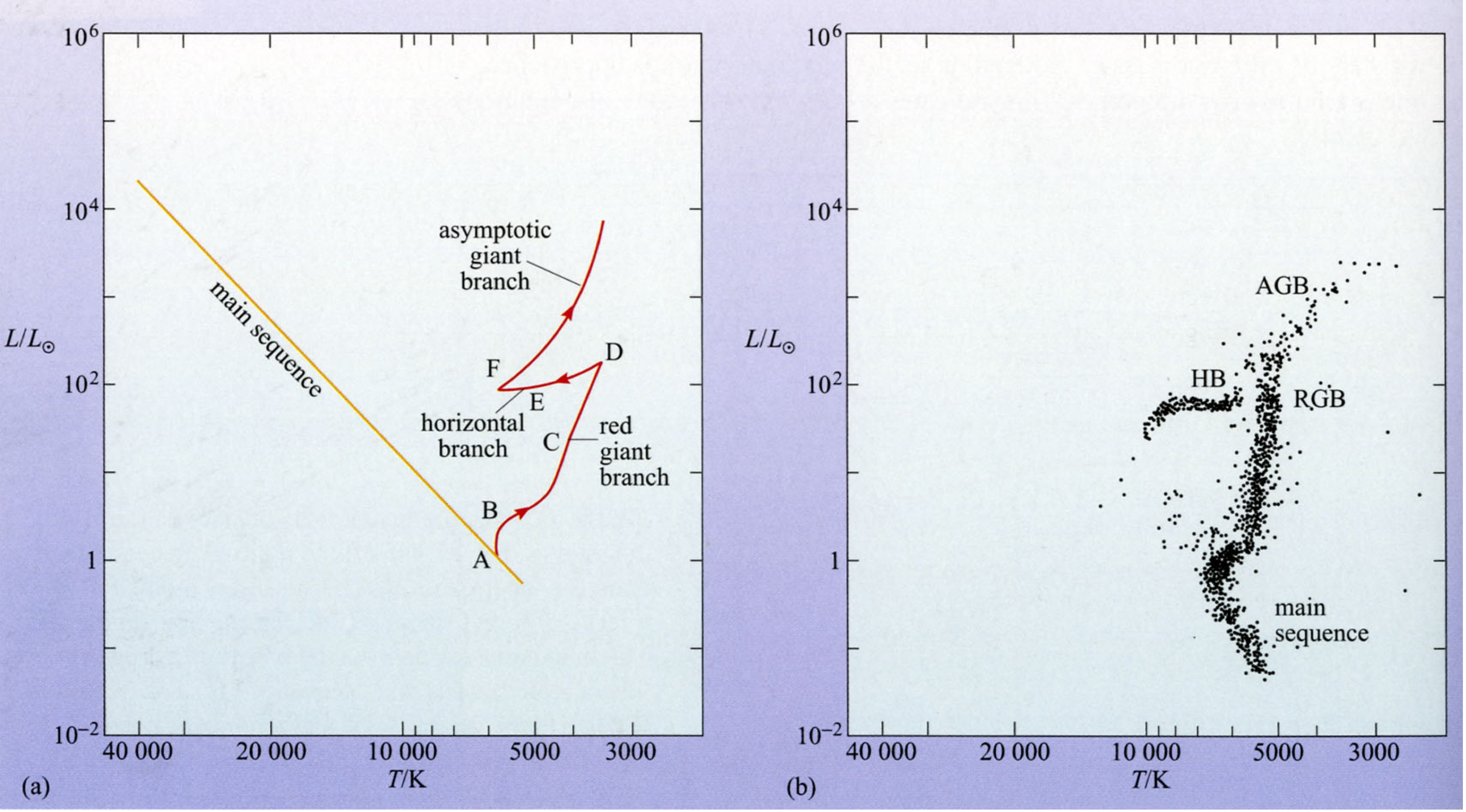
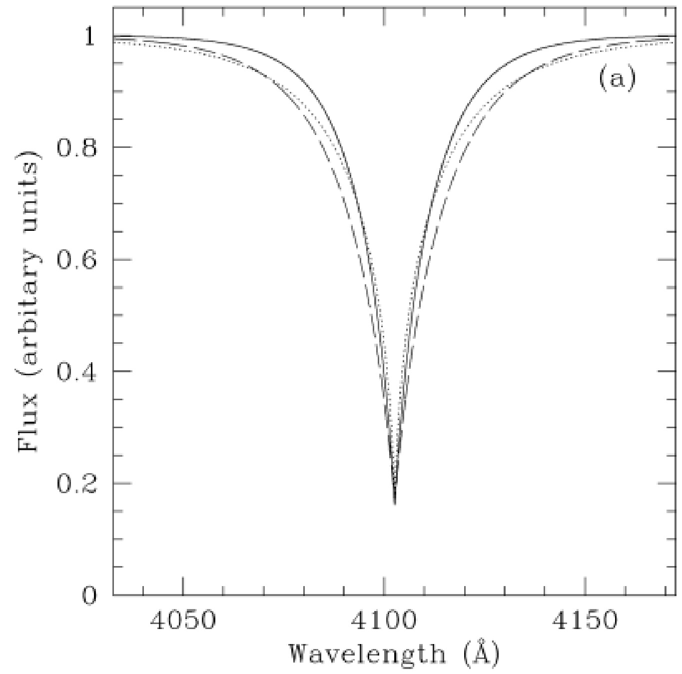


Figure 7.3 (a) The predicted path of a $1 M_\odot$ star, plotted on the same scale with the same labels as Figure 7.2, (A) hydrogen core fusion; (B) onset of hydrogen shell fusion; (C) hydrogen shell fusion continues; (D) helium core fusion starts; (E) helium core fusion continues; (F) helium shell fusion starts. (b) The H–R diagram of a globular cluster which illustrates how stars tend to concentrate in these regions.

Star of the Week: #1

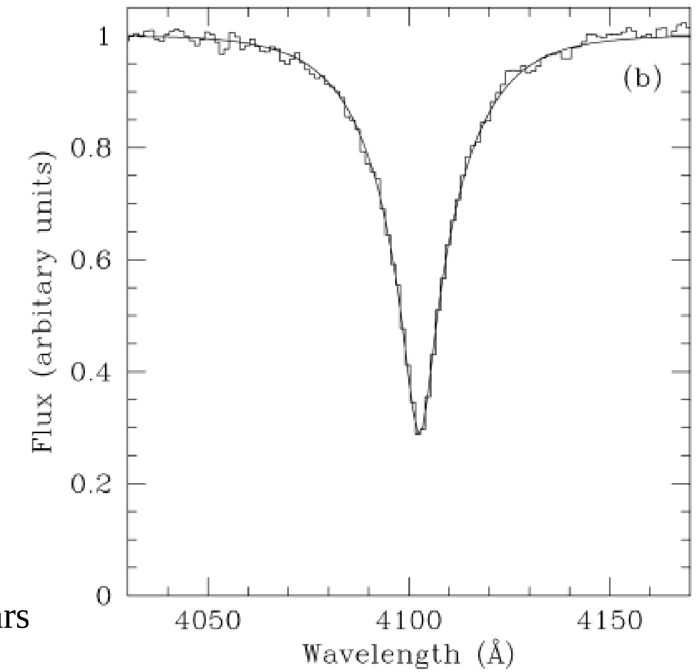
- Obtain spectra of targets with ESO VLT. Measure radial velocity of objects via Doppler shift
- Now have sample of stars with measured radial velocities and good distance estimates
- Use virial theorem – $0.5v^2 = GM_{Galaxy}/R$ – to estimate mass of Galaxy within R<100kpc
- What are the uncertainties?
  - Velocities – have radial but not tangential values
  - Distances
  - Halo properties
- Which objects are on the horizontal branch?
- SDSSJ110639.10-004720.06 v_r=260±13kms^{-1} d=36±5kpc
  SDSSJ122726.77+004641.03 v_r=-21±5kms^{-1} d=103±5kpc

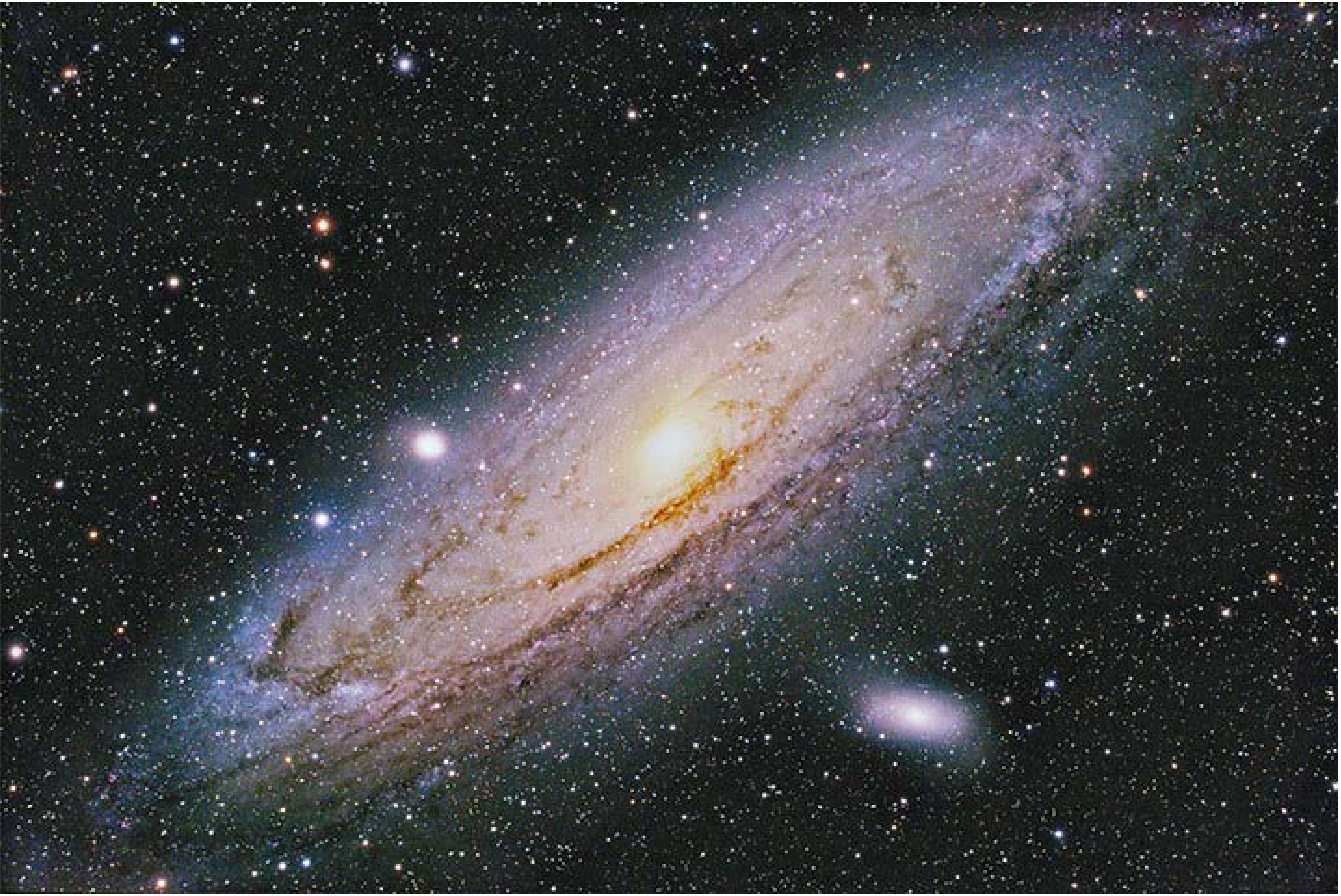
Illustration of the difference between the absorption line profile of the Hδ line in horizontal branch stars and blue stragglers (solid – BHB; dashed – BS; dotted – BS with different temperature)

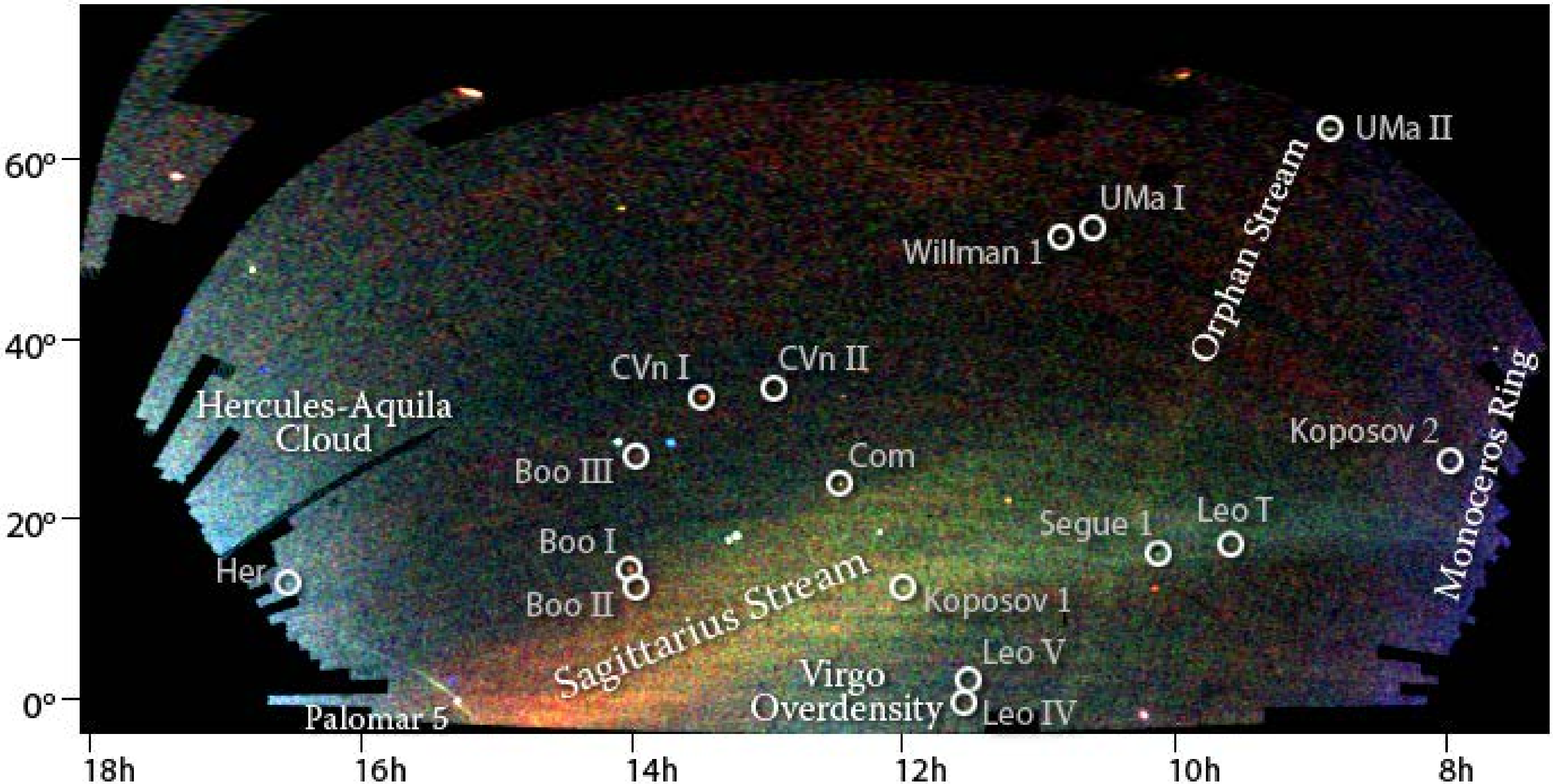


Actual fit to absorption line in a BHB star. The classification scheme involves absorption line parameters from fits to a number of different Balmer absorption lines. The new line-width determination scheme is the key to the project.

From Clewley et al., 2002, *MNRAS*, 337, 87






60°
40°
20°
0°
18h
16h
14h
12h
10h
8h
UMa II
UMa I
Willman 1
Orphan Stream
CVn I
CVn II
Hercules-Aquila
Cloud
Koposov 2
Monoceros Ring
Boo III
Com
Leo T
Segue 1
Boo I
Her
Boo II
Sagittarius Stream
Koposov 1
Leo V
Virgo
Overdensity
Leo IV
Palomar 5

Lecture 7: Summary

- Important interplay between both macro- and microscopic processes for determining properties of stars
- Should understand the meaning of mass fractions and mean molecular mass and how latter depends on composition and ionisation
- Total Pressure for a classical gas involves contributions from particles and radiation
- Idea of Degeneracy Pressure as consequence of quantum mechanics (more detail in Lecture 8)
- Star(s) of the Week #1: Measuring the Mass of the Galaxy
  - Use of photometric distances
  - Practical importance of determining luminosity class via absorption line widths