Stellar Dynamics and Structure of Galaxies

Lagrangian and Hamiltonian Dynamics

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Outline I

1. Lagrangian
   - Newton's law = Euler-Lagrange Equation = Hamilton's principle
   - What is the point of Lagrange equations?
   - Hamilton Equations
   - Motion in axisymmetric potentials

According to Newton

Remember Newton's law?

$$\frac{d}{dt}(m\dot{\mathbf{x}}) = \mathbf{F} \tag{8.1}$$

i.e.

$$m\ddot{\mathbf{x}} = \mathbf{F} \tag{8.2}$$

If $\mathbf{F}$ is due to a gravitational potential $\Phi(\mathbf{r})$, then

$$\mathbf{F} = m\mathbf{f} = -m\nabla\Phi \tag{8.3}$$

According to Lagrange

For the **Lagrangian** $\mathcal{L}$, defined as difference between the kinetic and potential energies:

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t) \equiv K - V = \frac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x}, t) \tag{8.4}$$

the following equation is valid:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \tag{8.5}$$

which is clearly just a restatement of the Newton's second law, since

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -\frac{\partial V}{\partial \mathbf{x}}, \quad \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}}, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\ddot{\mathbf{x}} \tag{8.6}$$

Extremal Action

The **Euler-Lagrange equation**

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = 0 \tag{8.7}$$

can be obtained by observing that the motion of the particle will be along the path that is an extremal of the action:

$$S \equiv \int_{t_0}^{t_1} \mathcal{L} dt \tag{8.8}$$

which is known as **Hamilton's principle** or the **principle of least action**.

Changing coordinates

Because the Lagrangian is a **scalar**, extremising the action allows us to show that the form of Euler-Lagrange equations remains the same in **arbitrary** coordinates!

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \tag{8.9}$$

Looking at this equation, it is prudent to define the so-called **generalized momenta**:

$$\mathbf{p} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \tag{8.10}$$

Then another useful construction is the **Hamiltonian**

$$H(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv \mathbf{p}\dot{\mathbf{q}} - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{8.11}$$

Hamilton Equations

Using the Hamiltonian definition, the Lagrange equations simplify to:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \tag{8.12}$$

For a particle moving on an orbit in a time independent potential, according to the Lagrange equations (using chain rule and setting $\partial \mathcal{L}/\partial t = 0$, the Hamiltonian H is conserved. This is not a surprise since the Hamiltonian can be shown to be equal to the total Energy:

$$H(\mathbf{x}, \mathbf{p}) = \mathbf{p}\dot{\mathbf{x}} - \mathcal{L} = K + V \tag{8.13}$$

Motion in axisymmetric potentials

The Lagrangian:

$$\mathcal{L} = \frac{1}{2}[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2] - \Phi(R, z) \tag{8.14}$$

The momenta:

$$p_R = \dot{R}, \quad p_\phi = R^2\dot{\phi}, \quad p_z = \dot{z} \tag{8.15}$$

Therefore, the Hamiltonian:

$$H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R, z) \tag{8.16}$$

Motion in axisymmetric potentials

Given $H = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R, z)$, the Hamilton equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

become

$$\dot{p}_R = \ddot{R} = \frac{p_\phi^2}{R^3} - \frac{\partial \Phi}{\partial R} \quad (8.17)$$

$$\dot{p}_\phi = \frac{d}{dt}(R^2\dot{\phi}) = 0 \quad (8.18)$$

$$\dot{p}_z = \ddot{z} = -\frac{\partial \Phi}{\partial z} \quad (8.19)$$