Stellar Dynamics and Structure of Galaxies

Collisionless systems

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* based on slides prepared by Vasily Belokurov and lecture notes by Jim Pringle

Outline I

Outline II

Collisionless Systems

Introduction

NB The use of the word "collisionless" is a technical one, specific to stellar dynamics. It does not simply mean there are no physical collisions between stars - it is a stronger statement than that.

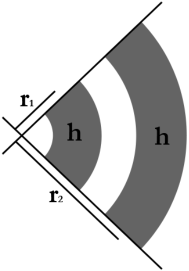
Aiming to describe the strucure of a self-gravitating collection of stars, such as a star cluster or a galaxy.

e.g. globular cluster $N \sim 10^6$ stars, $r_t \sim 10$ pc $\sim 3 \times 10^{17}$ m.

Collisionless Systems

Introduction

1. Gravity is a long range force. For example, if the star density is uniform, then a star at the apex of a cone sees the same force from a region of a given thickness independent of its distance.



$$m_1 \propto r_1^2 h$$
$$m_2 \propto r_2^2 h$$

and

$$f_1 \propto -\frac{Gm_1}{r_1^2} \propto h$$
$$f_2 \propto -\frac{Gm_2}{r_2^2} \propto h$$

$\Rightarrow$ the force acting on a star is determined by distant stars and large-scale structure of the galaxy. The force is zero if uniform density everywhere, but $\neq 0$ if the density falls off in one direction, for example. This is unlike molecules of gas where forces are strong only during close collisions

Collisionless Systems
Introduction

2 Stars almost never collide physically.

Distance to nearest star in a globular cluster is $d \sim \frac{10}{(10^6)^{\frac{1}{3}}} \sim 0.1$ pc $\sim 3 \times 10^{15}$ m $>> r_* \sim 10^9$ m.

$$r_* \ll d \ll r_t$$

This means that we can mentally smooth out the stars into a mean density $\bar{\rho}$ and use that to calculate a mean gravitational potential $\bar{\Phi}$ and use that to calculate the orbits of the individual stars. The forces on a given star do not vary rapidly.

If this is a good approximation then the system is said to be "collisionless"..

Relaxation time

Between 2 and ∞

$N = \infty$ If the system consisted of an infinite number of stars which are themselves point masses then the collisionless approximation would be perfect.

$N = 2$ If instead we have a binary system then the approximation is dire - it does not work at all.

So somewhere between $N = 2$ and $N = \infty$ it becomes OK. What is the criterion for this?

Consider a system of N stars each of mass m, and look at the motion of one star as it crosses the system. Now look at

1. the path under the assumption that the mass of the stars is smoothed out
2. the real path using individual stars

What we want to do is estimate the difference between the two - or, in particular, the difference in the resultant transverse (relative to the initial motion) velocity of the star we have chosen to follow.

Relaxation time

Weak encounters

For the real path we will use an impulse approximation to start with. On the real path the star undergoes encounters with other stars which perturb the straight path. One encounter with a star of mass m at $(0, b)$, i.e. impact parameter b as shown:

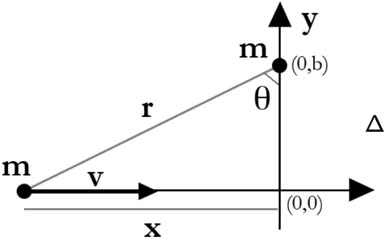
on the path $x = vt$

$$F_y = \frac{Gm^2}{r^2}\cos\theta = \frac{Gm^2 b}{(x^2+b^2)^{\frac{3}{2}}}$$

$$F_y = \frac{Gm^2}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-\frac{3}{2}} = m\dot{v}_y$$

$$\Delta v_y = \frac{Gm}{b^2}\int_{-\infty}^{\infty}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-\frac{3}{2}} dt$$

$$= \frac{Gm}{bv}\int_{-\infty}^{\infty}(1+s^2)^{-\frac{3}{2}}\, ds$$



Setting $s = \tan\theta$ allows us to integrate this

$$\Delta v_y = \frac{2Gm}{bv}$$

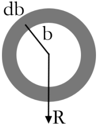
Relaxation time

Weak encounters

We could have obtained this sort of approximation more quickly by noting that $|\Delta v_\perp| \approx$ Force at closest approach $\times$ time spent near perturber $= \frac{Gm}{b^2} \times \frac{2b}{v}$.

How many encounters at distance b are there? The surface density of stars is $\sim \frac{N}{\pi R^2}$,

so the number of stars with b in the range$(b, b+db)$ is



$$\delta n = \frac{N}{\pi R^2} 2\pi b \; db$$

Each encounter produces an effect $\Delta v_\perp$, but the vectors are randomly oriented. Therefore the mean value of the effect is zero, but the sum of the $\delta v_\perp^2$ is non-zero.

So $v_\perp^2$ changes by an amount

$$\left(\frac{2Gm}{bv}\right)^2 \frac{2N}{R^2} b \; db$$

We need to integrate this over all b, so

$$\Delta v_\perp^2 = \int_0^R 8N \left(\frac{Gm}{Rv}\right)^2 \frac{db}{b}$$

Relaxation time

Weak encounters

$$\Delta v_\perp^2 = \int_0^R 8N \left(\frac{Gm}{Rv}\right)^2 \frac{db}{b}$$

There is a problem here, and that is the lower limit 0 for the integral. The approximation we have used breaks down then, so replace 0 by b_{min}, the expected closest approach - i.e. such that

$$\frac{N}{\pi R^2}(b_{\mathrm{min}}^2 \pi) = 1$$

so

$$b_{\mathrm{min}} \sim R/N^{\frac{1}{2}}$$

Then

$$\Delta v_\perp^2 \approx 8N \left(\frac{Gm}{Rv}\right)^2 \ln \Lambda$$

where $\Lambda = R/b_{\mathrm{min}}$.

Relaxation time

Weak encounters

Let us check for consistency that approximation we have used is OK.

When $b = b_{\rm min}$. We have the requirement that $\delta v_\perp / v << 1$, so require $2Gm/bv^2 << 1$, or $b >> 2Gm/v^2$.

But from the Virial theorem $v^2 \sim GM/R \sim GNm/R$, so need $b >> 2GmR/GNm = 2R/N$, i.e. $b/R >> 2/N$.

For $b_{\rm min}$ have $b_{\rm min}/R \sim 1/N^{\frac{1}{2}} >> 1/N$, so the approximation is OK.

Relaxation time

The time to erase the memory of the past motion

So we conclude that $v_\perp^2$ changes by an amount $\Delta v_\perp^2 \approx 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda$ at each crossing.

The collisionless approximation will fail after $n_{\rm relax}$ crossings, where

$$n_{\rm relax}\Delta v_\perp^2 \sim v^2 \quad \text{i.e.} \quad n_{\rm relax} 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda \sim v^2$$

and using $v^2 \simeq \frac{GNm}{R}$ this becomes

$$n_{\rm relax} 8N\left(\frac{v^2}{Nv}\right)^2 \ln\Lambda \sim v^2 \quad \text{i.e.} \quad n_{\rm relax} \sim \frac{N}{8\ln\Lambda}$$

The **relaxation time** is

$$t_{\rm relax} = n_{\rm relax} \times t_{\rm cross} \approx n_{\rm relax}\frac{R}{v}$$

and the crossing time

$$t_{\rm cross} \sim \sqrt{\frac{R^3}{GNm}}$$

Relaxation time

Notes:

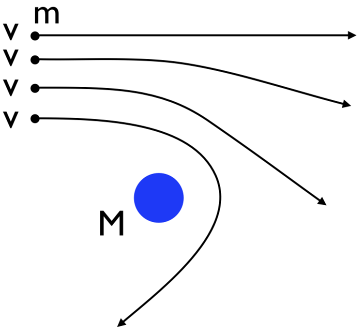
1. $\ln \Lambda \sim \ln N$, so $n_{\rm relax} \sim \frac{N}{8 \ln N}$.
2. relaxation time is the timescale on which stars share energy with each other.
3. can model a system as collisionless only if $t << t_{\rm relax}$.

Estimates of timescales:

- Galaxies: $N \sim 10^{11}$, $t_{\rm cross} \sim 10^8$ yr, $n_{\rm relax} \sim 5 \times 10^8$, so $t_{\rm relax} \sim 5 \times 10^8 t_{\rm cross} \sim 5 \times 10^{16}$ yr. This is much greater than a Hubble time, so galaxies are not relaxed.
- Globular clusters: $N \sim 10^6$, $t_{\rm cross} \sim 10{\rm pc}/20{\rm km\ s}^{-1} \sim 5 \times 10^5$ yr, so $t_{\rm relax} \sim 4 \times 10^9$ yr. Their ages are somewhat greater than this, so globular clusters are relaxed, and hence spherical.

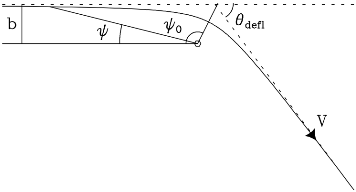
Gravitational Drag / Focusing

Consider a large mass M moving with speed v through a sea of stationary masses m, density ρ. In the frame of the mass M:



.. so not only is $v_\perp$ affected, but there is also a contribution to $v_\parallel$.

Gravitational Drag / Focusing

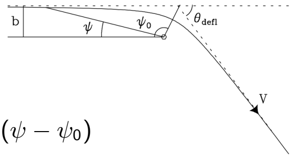


Relative to M have a Keplerian orbit with the angular momentum $h = bv = r^2\dot{\psi}$ The orbit, as you remember:

$$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{GM}{h^2}$$

Gravitational Drag / Focusing

$$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{GM}{h^2}$$



Get C, ψ_0 by differentiating this $\uparrow$

$$\frac{dr}{dt} = Cr^2\dot{\psi}\sin(\psi - \psi_0)$$

As $\psi \to 0$ $\frac{dr}{dt} \to -v$ so

$$-v = Cbv\sin(-\psi_0)$$

Also, since $r \to \infty$ then

$$0 = C\cos\psi_0 + \frac{GM}{b^2v^2}$$

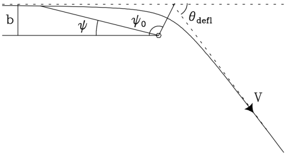
so

$$\tan\psi_0 = -bv^2/GM$$

Gravitational Drag / Focusing

Now $\pi - \theta_{\mathrm{defl}} = 2(\pi - \psi_0)$, so
$\theta_{\mathrm{defl}} = 2\psi_0 - \pi$. $\Rightarrow$

$$\tan\left(\frac{\theta_{\mathrm{defl}}}{2}\right) = -\frac{1}{\tan\psi_0}$$



and so

$$\tan\left(\frac{\theta_{\mathrm{defl}}}{2}\right) = \frac{GM}{bv^2}$$

Then $\theta_{\mathrm{defl}} = \frac{\pi}{2}$ if $\psi_0 = \frac{3\pi}{4}$, or $\tan\psi_0 = -1 \Rightarrow$

$$b_\perp \sim \frac{GM}{v^2}$$

Gravitational Drag / Focusing

To estimate the drag force, we assume that all particles with $b < b_\perp$ lose all their momentum to M (i.e. $\delta v \approx v$ at $b_\perp$)
So the force on $M =$ rate of change of momentum $= \pi b_\perp^2 \rho v^2$
(consider cylinder $vdt \times \pi b^2$ within which each star contributes v)
So

$$M\frac{dv}{dt} = -\pi\rho v^2 \left(\frac{GM}{v^2}\right)^2$$

or

$$\frac{dv}{dt} \simeq -\pi\rho\frac{G^2 M}{v^2}$$

This is known as dynamical friction.

Gravitational Drag / Focusing

Note:

1. We have assumed that the mass is moving at velocity v with respect to the background. In general the background will have a velocity dispersion σ. We have effectively assumed in the above that $v >> \sigma$. If $v << \sigma$ then we expect negligible drag since the particle barely "knows" it is moving. The general result (see Binney & Tremaine, p643 onwards) is that drag is caused by particles with velocities $0 < u < v$.
2. Force $F \propto M^2$, and the wake mass is $\propto M$
3. $F \propto \frac{1}{v^2}$.

Gravitational Drag / Focusing
NGC 2207

Gravitational Drag / Focusing

Applications of dynamical friction

- Galactic cannibalism
  A satellite with $\sigma \sim 50$ km/s in a galaxy with $\sigma \sim 200$ km/s will spiral from 30 kpc in 10 Gyr.
- Decay of black-hole orbits
  for $M_{BH} > 10^6 M_\odot$ only few Gyr to go from 10 kpc to 0
- Friction between the Galactic bar and the Dark Matter halo
- Formation and evolution of binary black holes
- The fates of globular clusters

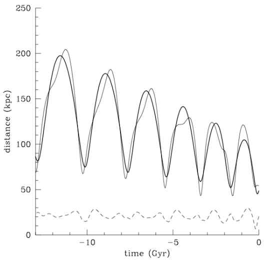
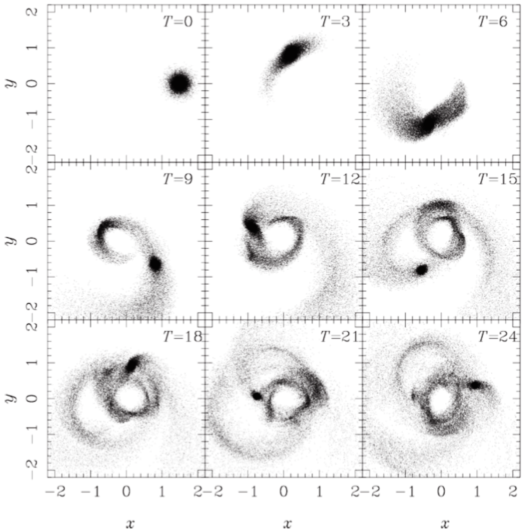


Figure 8.3 The decay of the orbits of the Magellanic Clouds around our Galaxy. The upper curves show the radius of the Clouds from the Galactic center (thick line for the Large Cloud and thin line for the Small Cloud), and the lower, dashed curve shows the distance between the Large and Small Cloud. The Galaxy potential is that of a singular isothermal sphere with circular speed $v_c = 220\,\mathrm{km\,s^{-1}}$, and the drag force is computed using Chandrasekhar's formula (8.7). The initial conditions at $t = 0$ are chosen to reproduce the observed distances and radial velocities of the Clouds and the kinematics of the Magellanic Stream (Gardiner, Sawa, & Fujimoto 1994).

Gravitational Drag / Focusing

Simulation of dwarf satellite accretion



The Collisionless Boltzman Equation

- If the interactions are rare, then the orbit of any star can be calculated as if the system's mass was distributed smoothly.
- But, as we just saw, eventually the **true orbit** deviates from the model orbit.
- Luckily, as long as we consider timescales $< t_{relax}$ we are fine
- In fact, for galaxies, $t_{relax} >> t_{Hubble}$. Perfect!
- However, when modelling a collisionless system such as an elliptical galaxy it is not practical to follow the motions of all constituent stars. Because there are too many of them!

The Collisionless Boltzman Equation

The Distribution Function

Let us assume that the stellar systems consist of a large number N of identical particles with mass m (could be stars, could be dark matter) moving under a smooth gravitational potential $\Phi(\mathbf{x}, t)$.

Most problems are to do with working out the probability of finding a star in particular geographical location about the galaxy, moving at a particular speed.

Or, in other words, the probability of finding the star in the six-dimensional phase-space volume $\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$, which is a small volume $\mathrm{d}^3\mathbf{x}$ centred on $\mathbf{x}$ in the small velocity range $\mathrm{d}^3\mathbf{v}$ centred on $\mathbf{v}$.

At any time t a full description of the state of this system is given by specifying the number of stars $f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$, where $f(\mathbf{x}, \mathbf{v}, t)$ is called the "distribution function" (or "phase space density") of the system.

Obviously, $f \geq 0$ everywhere, since we do not allow negative star densities.

The Collisionless Boltzman Equation

The Distribution Function

Naturally, integrating over all phase space:

$$\int f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v} = N \tag{5.1}$$

Alternatively, we can normalize it to have:

$$\int f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v} = 1 \tag{5.2}$$

Then $f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$ is the probability that at time t a randomly chosen star has phase-space coordinates in the given range.

The Collisionless Boltzman Equation
Phase space flow

If we know the initial coordinates and velocities of every star, then we can use Newton's laws to evaluate their positions and velocities at any other time i.e. given $f(\mathbf{x}, \mathbf{v}, t_0)$ then we should be able to determine $f(\mathbf{x}, \mathbf{v}, t)$ for any t. With this aim, we consider the flow of points in phase space, with coordinates $(\mathbf{x}, \mathbf{v})$, that arises as stars move along in their orbits. We can set the phase space coordinates

$$(\mathbf{x}, \mathbf{v}) \equiv \mathbf{w} \equiv (w_1, w_2, w_3, w_4, w_5, w_6)$$

so the velocity of the flow (which is the time derivative of the coordinates) may be written as

$$\dot{\mathbf{w}} = (\dot{\mathbf{x}}, \dot{\mathbf{v}}) = (\mathbf{v}, -\nabla\Phi).$$

$\dot{\mathbf{w}}$ is a six-dimensional vector which bears the same relationship to the six-dimensional vector $\mathbf{w}$ as the three-dimensional fluid flow velocity $\mathbf{v} = \dot{\mathbf{x}}$.

The Collisionless Boltzman Equation

Phase space flow

Any given star moves through phase space, so the probability of finding it at any given phase-space location changes with time. In what way?

However, the flow in phase space conserves stars, hence we can derive the equation of conservation of the phase space probability analogous to the fluid continuity equation.

The Collisionless Boltzman Equation

The fluid continuity equation

For an arbitrary closed volume V fixed in space and bounded by surface S, the mass of fluid in the volume is

$$M(t) = \int_V \mathrm{d}^3\mathbf{x}\rho(\mathbf{x}, t) \tag{5.3}$$

The fluid mass changes with time at a rate

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_V \mathrm{d}^3\mathbf{x}\frac{\partial\rho}{\partial t} \tag{5.4}$$

But, the mass flowing out through the surface area element d^2S per unit time $\rho\mathbf{v}\cdot\mathrm{d}^2\mathbf{S}$. Thus:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\oint_S \mathrm{d}^2\mathbf{S}\cdot(\rho\mathbf{v}) \tag{5.5}$$

Or

$$\int_V \mathrm{d}^3\mathbf{x}\frac{\partial\rho}{\partial t} + \oint_S \mathrm{d}^2\mathbf{S}\cdot(\rho\mathbf{v}) = 0 \tag{5.6}$$

$$\int_V \mathrm{d}^3\mathbf{x}\frac{\partial\rho}{\partial t} + \oint_S \mathrm{d}^2\mathbf{S}\cdot(\rho\mathbf{v}) = 0$$

can be re-written with the use of the divergence theorem:

$$\int_V \mathrm{d}^3\mathbf{x}\left[\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right] = 0 \tag{5.7}$$

Since the result holds for any volume:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0 \tag{5.8}$$

Which in Cartesian coordinates looks like this:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0 \tag{5.9}$$

using the summation convention

$$\mathbf{A}\cdot\mathbf{B} = \sum_{i=1}^{3} A_i B_i = A_i B_i$$

The Collisionless Boltzman Equation

The continuity of flow in phase space

Since $\dot{\mathbf{x}} = v$, for fluids:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (f\dot{\mathbf{x}}) = 0$$

The analogous equation for the conservation of probability in phase space is:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot (f\dot{\mathbf{w}}) = 0 \qquad (5.10)$$

Note that writing it as a continuity equation carries with it the assumption that the function f is differentiable. This means that close stellar encounters where a star can jump from one point in phase space to another are excluded from this description.

The Collisionless Boltzman Equation

The continuity of flow in phase space

Let us have a closer look at the second term in $\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot (f\dot{\mathbf{w}}) = 0$

$$\frac{\partial (f\dot{w}_i)}{\partial w_i} = \dot{w}_i \frac{\partial f}{\partial w_i} + f \frac{\partial \dot{w}_i}{\partial w_i} \tag{5.11}$$

The flow in six-space is an interesting one, since

$$\sum_{i=1}^{6} \frac{\partial (\dot{w}_i)}{\partial w_i} = \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = \sum_{i=1}^{3} -\frac{\partial}{\partial v_i} \left(\frac{\partial \Phi}{\partial x_i} \right) = 0 \tag{5.12}$$

Here $\frac{\partial v_i}{\partial x_i} = 0$ because in this space v_i and x_i are independent coordinates, and the last step follows because Φ, and hence $\nabla\Phi$ does not depend on the velocities. We can use this equation to simplify the continuity equation, which now becomes

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f}{\partial w_i} = 0 \tag{5.13}$$

The Collisionless Boltzman Equation

The continuity of flow in phase space

or,

$$\frac{\partial f}{\partial t} + \dot{\mathbf{w}} \cdot \nabla_6 f = 0,$$

or (in terms of x_i and v_i, and using summation convention with $i = 1$ to 3.)

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0,$$

or (in vector form)

Collisionless Boltzmann Equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \qquad (5.14)$$

where $\frac{\partial f}{\partial \mathbf{v}}$ is like ∇f, but in the velocity coordinate $\mathbf{v}$ rather than the spatial coordinate $\mathbf{x}$.

The Collisionless Boltzman Equation

Liouville's Theorem

The meaning of the collisionless Boltzmann equation can be seen by extending to six dimensions the concept of the Lagrangian derivative. We define (using the summation convention here and forever more)

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\partial f}{\partial t} + \dot{w}_i \frac{\partial f}{\partial w_i} \tag{5.15}$$

$\frac{\mathrm{d}f}{\mathrm{d}t}$ represents the rate of change of density in phase space as seen by an observer who moves through phase space with a star with phase space velocity $\dot{\mathbf{w}}$. The collisionless Boltzmann equation is then simply

$$\frac{\mathrm{D}f}{\mathrm{D}t} = 0 \tag{5.16}$$

Therefore the flow of stellar phase points through phase space is incompressible – the phase-space density of points around a given star is always the same.

The Collisionless Boltzman Equation

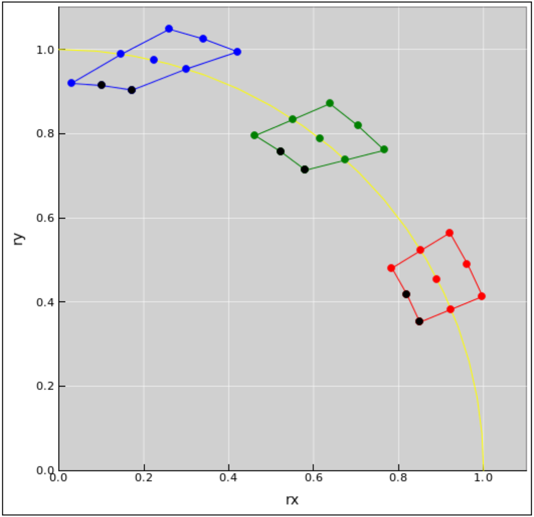
Liouville's Theorem = Preservation of the phase space density

Compare start...

...and finish

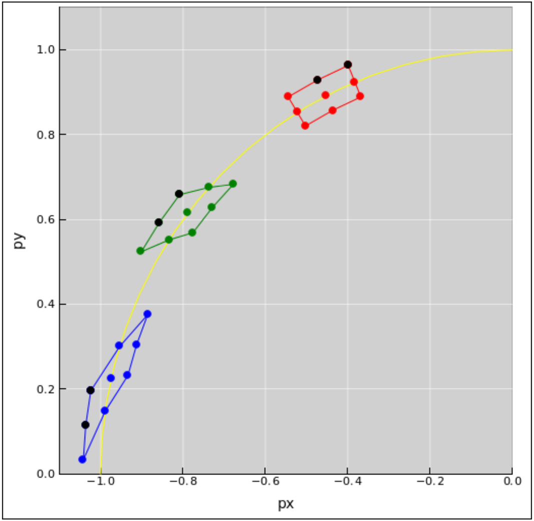
The Collisionless Boltzman Equation

Liouville's Theorem = Preservation of the phase space density



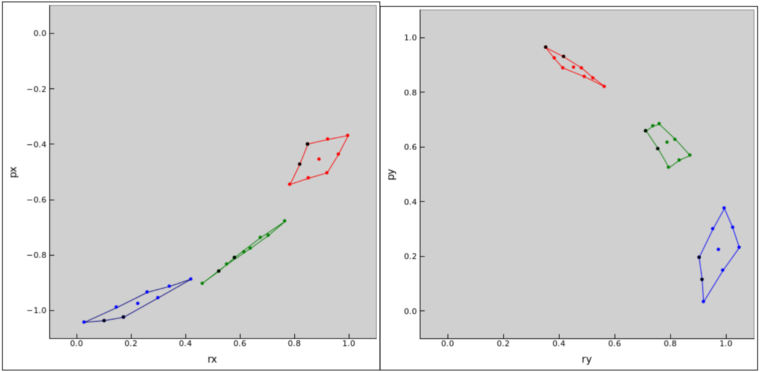
The Collisionless Boltzman Equation

Liouville's Theorem = Preservation of the phase space density



The Collision-less Boltzman Equation

Liouville's Theorem = Preservation of the phase space density



The Collisionless Boltzman Equation

In cylindrical polars

Be careful when writing down the collisionless Boltzmann equation in non-Cartesian coordinates! For example, in cylindrical polars (axial symmetry)

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial \Phi}{\partial R}$$

$$\frac{1}{R}\frac{d}{dt}\left(R^2\dot{\phi}\right) = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi}$$

$$\ddot{z} = -\frac{\partial \Phi}{\partial z}$$

with

$$v_R = \dot{R}$$

$$v_\phi = R\dot{\phi} \quad (\text{ not just } \dot{\phi})$$

$$v_z = \dot{z}$$

Since $\mathrm{d}\mathbf{x} = \mathrm{d}R\mathbf{e}_R + R\mathrm{d}\phi\mathbf{e}_\phi + \mathrm{d}z\mathbf{e}_z$

The Collisionless Boltzman Equation

In cylindrical polars

$\boxed{\ddot{R} - R\dot{\phi}^2 = -\frac{\partial \Phi}{\partial R}}$ $\boxed{\frac{1}{R}\frac{d}{dt}\left(R^2\dot{\phi}\right) = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi}}$ $\boxed{\ddot{z} = -\frac{\partial \Phi}{\partial z}}$ $\boxed{v_R = \dot{R}}$ $\boxed{v_\phi = R\dot{\phi}}$ $\boxed{v_z = \dot{z}}$

Then start with

$$\frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0$$

and this becomes

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z\frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R}$$
$$- \frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \tag{5.17}$$

The Collisionless Boltzman Equation

Limitations and links with the real world

1. Stars are born and die! Hence they are not really conserved. Therefore, more appropriately:

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\frac{\partial f}{\partial \mathbf{v}} = B - D \qquad (5.18)$$

where $B(\mathbf{x}, \mathbf{v}, t)$ and $D(\mathbf{x}, \mathbf{v}, t)$ are the rates per unit phase-space volume at which stars are born and die.

But $\mathbf{v}\partial f/\partial \mathbf{x} \approx vf/R = f/t_{cross}$

Similarly, $\partial\Phi/\partial\mathbf{x} \approx a \approx v/t_{cross}$, hence
$\partial\Phi/\partial\mathbf{x}\ \partial f/\partial\mathbf{v} \approx af/v \approx f/t_{cross}$
Therefore, the important ratio

$$\gamma = \left|\frac{B-D}{f/t_{cross}}\right| \ll 1 \qquad (5.19)$$

i.e. the fractional change in the number of stars per crossing time is small

The Collisionless Boltzman Equation

Limitations and links with the real world

Density of stars at a particular location **x**

$$\nu(x) \equiv \int \mathrm{d}^3\mathbf{v} f(\mathbf{x}, \mathbf{v}) \qquad (5.20)$$

Probability distribution of stellar velocities at **x**

$$P_x(\mathbf{v}) = \frac{f(\mathbf{x}, \mathbf{v})}{\nu(\mathbf{x})} \qquad (5.21)$$

For lines of sight through the galaxy, defined by **s** - a unit vector from observer to the galaxy.
The components of **x** and **v** vectors parallel and perpendicular to the line of sight are:

$$x_{\|} \equiv \mathbf{s} \cdot \mathbf{x}$$

$$v_{\|} \equiv \mathbf{s} \cdot \mathbf{v}$$

$$x_{\perp} \equiv \mathbf{x} - x_{\|}\mathbf{s}$$

$$v_{\perp} \equiv \mathbf{v} - v_{\|}\mathbf{s}$$

The Collisionless Boltzman Equation

Limitations and links with the real world

The distribution of the line-of-sight velocities at $\mathbf{x}_\perp$

$$F(\mathbf{x}_\perp, v_\parallel) = \frac{\int \mathrm{d}x_\parallel \nu(\mathbf{x}) \int \mathrm{d}^2\mathbf{v}_\perp P_x(v_\parallel \mathbf{s} + \mathbf{v}_\perp)}{\int \mathrm{d}x_\parallel \nu(\mathbf{x})}$$

The mean line-of-sight velocity:

$$\bar{v}_\parallel(\mathbf{x}_\perp) \equiv \int \mathrm{d}v_\parallel v_\parallel F(\mathbf{x}_\perp, v_\parallel)$$

The line-of-sight velocity dispersion:

$$\sigma_\parallel^2(\mathbf{x}_\perp) \equiv \int \mathrm{d}v_\parallel (v_\parallel - \bar{v}_\parallel)^2 F(\mathbf{x}_\perp, v_\parallel)$$

The Jeans Equations

- The distribution function f is a function of seven variables, so solving the collisionless Boltzmann equation in general is hard.
- So need either simplifying assumptions (usually symmetry), or try to get insights by taking moments of the equation.
- We cannot observe f, but can determine ρ and line profile (which is the average velocity along a line of sight $\overline{v}_r$ and $\overline{v_r^2}$.

The Jeans Equations
Zeroth moment

Start with the collisionless Boltzmann equation -using the summation convention

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0 \tag{5.22}$$

and take the zeroth moment integrating over $d^3\mathbf{v}$.

$$\frac{\partial}{\partial t} \int\int\int_{\infty}^{\infty} f d^3\mathbf{v} + \int\int\int v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int\int\int \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0 \tag{5.23}$$

where for the first term we can take the differential with respect to time out of the integral since the limits are independent of t, and in the third term Φ is independent of $\mathbf{v}$ so the $\frac{\partial \Phi}{\partial x_i}$ term comes out.

The Jeans Equations

Zeroth moment

$$\boxed{\frac{\partial}{\partial t}\int\int\int_{\infty}^{\infty} f d^3\mathbf{v} + \int\int\int v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int\int\int \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0}$$

Now

$$\nu(\mathbf{x}, t) = \int\int\int_{-\infty}^{\infty} f d^3\mathbf{v}$$

is just the number density of stars at $\mathbf{x}$ (and if all stars have the same mass m then $\rho(\mathbf{x}, t) = m\nu(\mathbf{x}, t)$). So the first term is just

$$\frac{\partial \nu}{\partial t}$$

The Jeans Equations

Zeroth moment

$$\boxed{\frac{\partial}{\partial t}\iiint_{\infty}^{\infty} f d^3\mathbf{v} + \iiint v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\iiint \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0}$$

Also

$$\frac{\partial}{\partial x_i}(v_i f) = \frac{\partial v_i}{\partial x_i} f + v_i \frac{\partial f}{\partial x_i}$$

and

$$\frac{\partial v_i}{\partial x_i} = 0$$

since v_i and x_i are independent coordinates, and so

$$\frac{\partial}{\partial x_i}(v_i f) = v_i \frac{\partial f}{\partial x_i}$$

The Jeans Equations
Zeroth moment

$$\boxed{\frac{\partial}{\partial t}\iiint_{\infty}^{\infty} f d^3\mathbf{v} + \iiint v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\iiint \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0}$$

Hence the second term above becomes

$$\frac{\partial}{\partial x_i}\iiint v_i f d^3\mathbf{v}$$

and if we define an average velocity $\overline{v}_i$ by

$$\overline{v}_i = \frac{1}{\nu}\iiint v_i f d^3\mathbf{v}$$

(so interpret f as a probability density) then the term we are considering becomes

$$\frac{\partial}{\partial x_i}\left(\nu \overline{v}_i\right)$$

The Jeans Equations

Zeroth moment

$$\boxed{\frac{\partial}{\partial t}\iiint_{\infty}^{\infty} f d^3\mathbf{v} + \iiint v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\iiint \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0}$$

The last term involving

$$\iiint \frac{\partial f}{\partial v_i} d^3\mathbf{v} = f|_{-\infty}^{\infty} = 0$$

since we demand that $f \to 0$ as $\mathbf{v} \to \infty$.

And so the zeroth moment equation becomes

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu \overline{v}_i) = 0 \tag{5.24}$$

which looks very like the usual fluid continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

The Jeans Equations

First moment

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

Multiply the collisionless Boltzmann equation ↑ by v_j and then integrate over $d^3\mathbf{v}$.

Then since

$$\frac{\partial v_j}{\partial t} = 0$$

we have

$$\int v_j \frac{\partial f}{\partial t} d^3\mathbf{v} = \frac{\partial}{\partial t}\int f v_j d^3\mathbf{v}$$

So the first moment equation becomes

$$\frac{\partial}{\partial t}\int f v_j d^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0 \qquad (5.25)$$

The Jeans Equations

First moment

$$\boxed{\frac{\partial}{\partial t}\int f v_j d^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0}$$

Looking at each of the terms in equation (5.25):

First term $= \frac{\partial}{\partial t}(\nu \overline{v}_j)$ by definition.

Second term $= \frac{\partial}{\partial x_i}(\nu \overline{v_i v_j})$, where

$$\overline{v_i v_j} = \frac{1}{\nu}\int v_i v_j f d^3\mathbf{v}$$

Third term:

$$\int v_j \frac{\partial f}{\partial v_i} d^3\mathbf{v} = [f v_j]_{-\infty}^{\infty} - \int \frac{\partial v_j}{\partial v_i} f d^3\mathbf{v} = -\delta_{ij}\nu$$

The Jeans Equations

First moment

$$\boxed{\frac{\partial}{\partial t}\int f v_j d^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0}$$

So first moment equation is

$$\frac{\partial}{\partial t}(\nu \overline{v}_j) + \frac{\partial}{\partial x_i}\left(\nu \overline{v_i v_j}\right) + \nu \frac{\partial \Phi}{\partial x_j} = 0 \qquad (5.26)$$

We can manipulate this a bit further - subtracting

$\overline{v}_j \times \boxed{\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}\left(\nu \overline{v}_i\right) = 0}$

gives

$$\nu \frac{\partial \overline{v}_j}{\partial t} - \overline{v}_j \frac{\partial}{\partial x_i}\left(\nu \overline{v_i}\right) + \frac{\partial}{\partial x_i}\left(\nu \overline{v_i v_j}\right) = -\nu \frac{\partial \Phi}{\partial x_j} \qquad (5.27)$$

The Jeans Equations

First moment

$$\boxed{\nu \frac{\partial \overline{v}_j}{\partial t} - \overline{v}_j \frac{\partial}{\partial x_i}\left(\nu \overline{v_i}\right) + \frac{\partial}{\partial x_i}\left(\nu \overline{v_i v_j}\right) = -\nu \frac{\partial \Phi}{\partial x_j}}$$

Now define

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v_i}\ \overline{v_j}$$

(this is a sort of dispersion). Thus $\overline{v_i v_j} = \overline{v_i}\ \overline{v_j} + \sigma_{ij}^2$ where the $\overline{v_i}\ \overline{v_j}$ refers to streaming motion and the σ_{ij}^2 to random motion at the point of interest. Using this we can tidy up (5.27) to obtain

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\nu \sigma_{ij}^2\right) \qquad (5.28)$$

This has a familiar look to it cf the fluid equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla)\mathbf{u} = -\rho \nabla \Phi - \nabla p$$

So the term in σ_{ij}^2 is a "stress tensor" and describes anisotrpoic pressure.

The Jeans Equations

Note that σ_{ij}^2 is symmetric, so it can be diagonalised. Ellipsoid with axes σ_{11}, σ_{22}, σ_{33} where 1, 2, 3 are the diagonalising coordinates is called the velocity ellipsoid.

If the velocity distribution is isotropic then we can write $\sigma_{ij}^2 = \left(\frac{p}{\nu}\right)\delta_{ij}$ for some p, and the get $-\nabla p$ in equation (5.28).

(5.24) and (5.26) are the **Jeans equations**. (5.26) can be replaced by (5.28).

These equations are valuable because they relate observationally accessible quantities.

The Jeans Equations

James Hopwood Jeans

The Jeans Equations

However...

The trouble is we have not solved anything. In a fluid we use thermodynamics to relate p and ρ, but do not have that here. These equations can give some understanding, and can be useful in building models, but not a great deal more.

Importantly, the solutions of the Jeans equation(s) are not guaranteed to be physical as there is no condition $f > 0$ imposed.

Moreover, this is an incomplete set of equations. If Φ and ν are known, there are still <u>nine</u> unknown functions to determine: 3 components of the mean velocity $\bar{\mathbf{v}}$ and 6 components of the velocity dispersion tensor σ^2. Yet we only have 4 equations: one zeroth order and 3 first order moments.

Multiplying CBE further through by $v_i v_k$ and integrating over all velocities will not supply the missing information.

We need to truncate or close the regression to even higher moments of the velocity distribution.

Such closure is possible in special circumstances

Application of Jeans equations

Isotropic velocity dispersion

Take equation (5.28)

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i} \left(\nu \sigma_{ij}^2 \right)$$

and assume at each point:

- steady state $\frac{\partial}{\partial t} = 0$
- isotropic $\sigma_{ij}^2 = \sigma^2 \delta_{ij}$
- non-rotating $\overline{v_i} = 0$

So no mean flow, and velocity dispersion is the same in all directions (but $\sigma^2 = \sigma^2(r)$).

Application of Jeans equations

Isotropic velocity dispersion

Then

$$\nu \frac{\partial \overline{v}_j}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\nu \sigma_{ij}^2\right)$$

becomes

$$-\nu \nabla \Phi = \nabla (\nu \sigma^2)$$

- Cluster with spherical symmetry - if we know $\nu(r)$ or $\rho(r) = m\nu(r)$, then from Poisson's equation $\nabla^2 \Phi = 4\pi G \rho$, the potential $\Phi(r)$ can be determined. Then can solve for $\sigma^2(r)$
- So given a density distribution $\rho(r)$ and the assumption of isotropy we can find $\sigma(r)$, *i.e.* can find a fully self-consistent model for the internal velocity structure of the cluster / galaxy.
- Minor difficulties: no guarantee (1) it is correct (is isotropic everywhere possible?) or (2) it works (what if $\sigma^2 < 0$ in the formal solution?).

Application of Jeans equations

Jeans equations for cylindrically symmetric systems

Start with the collisionless Boltzmann equation and set $\frac{\partial}{\partial \phi} = 0$ [not $v_\phi = 0$!]. So we have, from the cylindrical polar version of the equation (5.17)

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right) \frac{\partial f}{\partial v_R} - \frac{1}{R} \left(v_R v_\phi \right) \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0$$

Then for the zeroth moment equation $\iiint dv_R dv_\phi dv_z$.
Time derivative term:

$$\iiint \frac{\partial f}{\partial t} dv_R dv_\phi dv_z = \frac{\partial}{\partial t} \iiint f dv_R dv_\phi dv_z = \frac{\partial \nu}{\partial t}$$

Application of Jeans equations

Jeans equations for cylindrically symmetric systems

Velocity terms:

$$\iiint \left(v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \frac{v_\phi^2}{R} \frac{\partial f}{\partial v_R} - \frac{1}{R} v_R v_\phi \frac{\partial f}{\partial v_\phi} \right) dv_R dv_\phi dv_z$$

$$= \frac{\partial}{\partial R} \iiint v_R f \, dv_R dv_\phi dv_z + \frac{\partial}{\partial z} \iiint v_z f \, dv_R dv_\phi dv_z$$

$$+ \frac{1}{R} \iiint v_\phi^2 \frac{\partial f}{\partial v_R} dv_R dv_\phi dv_z$$

$$- \iiint \left[\frac{\partial}{\partial v_\phi} \left(\frac{v_R v_\phi f}{R} \right) - f \frac{\partial}{\partial v_\phi} \left(\frac{v_R v_\phi}{R} \right) \right] dv_R dv_\phi dv_z \qquad \uparrow$$

$$\uparrow 0 \text{ (div theorem)} \qquad\qquad 0 \text{ (div theorem)}$$

$$= \frac{\partial}{\partial R} \iiint v_R f \, dv_R dv_\phi dv_z + \frac{1}{R} \int\int\int v_R f \, dv_R dv_\phi dv_z$$

$$+ \frac{\partial}{\partial z} \iiint v_z f \, dv_R dv_\phi dv_z$$

$$= \frac{1}{R} \frac{\partial}{\partial R} (R \nu \, \overline{v_R}) + \frac{\partial}{\partial z} (\nu \, \overline{v_z})$$

where $\overline{v_R} = \frac{1}{\nu} \iiint v_R f \, dv_R dv_\phi dv_z$ and $\overline{v_z} = \frac{1}{\nu} \iiint v_z f \, dv_R dv_\phi dv_z$

Application of Jeans equations

Jeans equations for cylindrically symmetric systems

Terms with the potential Φ:

$$\iiint \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} dv_R dv_\phi dv_z = \frac{\partial \Phi}{\partial z}\iiint \frac{\partial f}{\partial v_z} dv_R dv_\phi dv_z = 0$$

and

$$\iiint \frac{\partial \Phi}{\partial R}\frac{\partial f}{\partial v_R} dv_R dv_\phi dv_z = \frac{\partial \Phi}{\partial R}\iiint \frac{\partial f}{\partial v_R} dv_R dv_\phi dv_z = 0$$

Hence

$$\frac{\partial \nu}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\nu\overline{v}_R) + \frac{\partial}{\partial z}(\nu\overline{v}_z) = 0 \tag{5.29}$$

This is the zeroth order moment equation.

Application of Jeans equations

Jeans equations for cylindrically symmetric systems

There are three first moment equations, corresponding to each of the v components, where we take the collisionless Boltzmann equation $\times v_R$, v_ϕ, v_z and $\iiint dv_R dv_\phi dv_z$.

The results are

$$\frac{\partial(\nu\overline{v}_R)}{\partial t} + \frac{\partial(\nu\overline{v_R^2})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z})}{\partial z} + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R}\right) = 0 \quad (5.30)$$

$$\frac{\partial(\nu\overline{v}_\phi)}{\partial t} + \frac{\partial(\nu\overline{v_R v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_\phi v_z})}{\partial z} + \frac{2\nu}{R}\overline{v_\phi v_R} = 0 \quad (5.31)$$

and

$$\frac{\partial(\nu\overline{v}_z)}{\partial t} + \frac{\partial(\nu\overline{v_R v_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{\nu\overline{v_R v_z}}{R} + \nu\frac{\partial\Phi}{\partial z} = 0. \quad (5.32)$$

Now, this is something **powerful**.

Application of Jeans equations

- Spheroidal components with isotropic velocity dispersion
- Asymmetric drift
- Local mass density
- Local velocity ellipsoid
- Mass distribution in the Galaxy out to large radii

Application of Jeans equations

Asymmetric drift

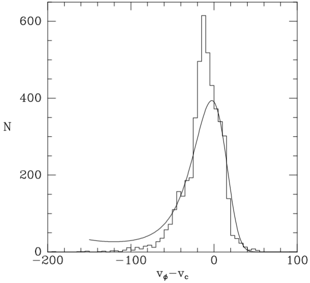


Figure 4.17 The distribution of v_ϕ components of 4787 F and G stars that have space velocities in Nordström et al. (2004). Stars with a high probability of having variable radial velocities are excluded. The smooth curve shows the distribution predicted by the Schwarzschild DF for a population with the same value of $\overline{v_R^2}^{1/2} = 34\,\mathrm{km\,s^{-1}}$.

There is a lag and the lag increases with the age of the stellar tracers and so does the random component of their motion.

Application of Jeans equations

Asymmetric drift

The distribution of azimuthal velocities $\tilde{v}_\phi = v_\phi - v_c$ is very skew. This asymmetry arises from two effects.

- Stars near the Sun with $\tilde{v}_\phi < 0$ have less angular momentum and thus have $R_g < R_0$ compared to stars with $\tilde{v}_\phi > 0$ and $R_g > R_0$. The surface density of stars declines exponentially, hence there are more stars with smaller R_g.
- The velocity dispersion σ_R declines with R, so the fraction of stars with $R_g = R_0 - \delta R$ is larger than the fraction of stars with $R_g = R_0 + \delta R$. Thus there are more stars on eccentric orbits that can reach the Sun with $\tilde{v}_\phi < 0$

Application of Jeans equations

Asymmetric drift

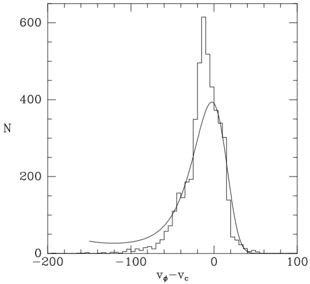


Figure 4.17 The distribution of v_ϕ components of 4787 F and G stars that have space velocities in Nordström et al. (2004). Stars with a high probability of having variable radial velocities are excluded. The smooth curve shows the distribution predicted by the Schwarzschild DF for a population with the same value of $\overline{v_R^2}^{1/2} = 34\,\mathrm{km\,s^{-1}}$.

The epicyclic approximation:

$$\frac{\overline{[v_\phi - v_c(R_0)]^2}}{\overline{v_R^2}} \simeq \frac{-B}{A-B} = -\frac{B}{\Omega_0} = \frac{K^2}{4\Omega^2} \simeq 0.5$$

Application of Jeans equations

Asymmetric drift

The velocity of the asymmetric drift

$$v_a \equiv v_c - \bar{v}_\phi$$

Jeans tells us that

$$\frac{\partial(\nu \bar{v}_R)}{\partial t} + \frac{\partial(\nu \overline{v_R^2})}{\partial R} + \frac{\partial(\nu \overline{v_R v_z})}{\partial z} + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) = 0$$

We assume

- The Galactic disk is in the steady state
- The Sun lies sufficiently close to the equator, at $z = 0$
- The disk is symmetric with respect to z and hence $\partial\nu/\partial z = 0$

So,

$$\frac{R}{\nu} \frac{\partial(\nu \overline{v_R^2})}{\partial R} + R \frac{\partial(\overline{v_R v_z})}{\partial z} + \overline{v_R^2} - \overline{v_\phi^2} + R \frac{\partial \Phi}{\partial R} = 0 \qquad (5.33)$$

Application of Jeans equations

Asymmetric drift

$$\boxed{\frac{R}{\nu}\frac{\partial(\nu\overline{v_R^2})}{\partial R} + R\frac{\partial(\overline{v_R v_z})}{\partial z} + \overline{v_R^2} - \overline{v_\phi^2} + R\frac{\partial\Phi}{\partial R} = 0}$$

Define

$$\sigma_\phi^2 = \overline{v_\phi^2} - \overline{v_\phi}^2$$

Remember that

$$v_c^2 = R\frac{\partial\Phi}{\partial R}$$

Therefore

$$\begin{aligned}\sigma_\phi^2 - \overline{v_R^2} \quad &- \quad \frac{R}{\nu}\frac{\partial(\nu\overline{v_R^2})}{\partial R} - R\frac{\partial(\overline{v_r v_z})}{\partial z} = v_c^2 - \overline{v_\phi}^2 \qquad (5.34)\\ &= \quad (v_c - \overline{v}_\phi)(v_c + \overline{v}_\phi) = v_a(2v_c - v_a)\end{aligned}$$

If we neglect v_a compared to $2v_c$

$$v_a \simeq \frac{\overline{v_R^2}}{2v_c}\left[\frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial\ln(\nu\overline{v_R^2})}{\partial\ln R} - \frac{R}{\overline{v_R^2}}\frac{\partial(\overline{v_r v_z})}{\partial z}\right] \qquad (5.35)$$

Application of Jeans equations

Asymmetric drift

This is Stromberg's **asymmetric drift equation**

$$v_a \simeq \frac{\overline{v_R^2}}{2v_c}\left[\frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu \overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}}\frac{\partial(\overline{v_r v_z})}{\partial z}\right]$$

- $\sigma_\phi^2/\overline{v_R^2} = 0.35$
- ν and $\overline{v_R^2}$ are both $\propto e^{-R/Rd}$ with $R_0/Rd = 3.2$

First three terms sum up to 5.8

- The last term is tricky, as it requires measuring the velocity ellipsoid outside the plane of the Galaxy, it averages to between 0 and -0.8

Averaging over, the value in the brackets is 5.4 ± 0.4, so

$$v_a \simeq \overline{v_R^2}/(82 \pm 6)\mathrm{kms}^{-1}$$

Application of Jeans equations

Asymmetric drift

But, what is measured?

Application of Jeans equations

Asymmetric drift

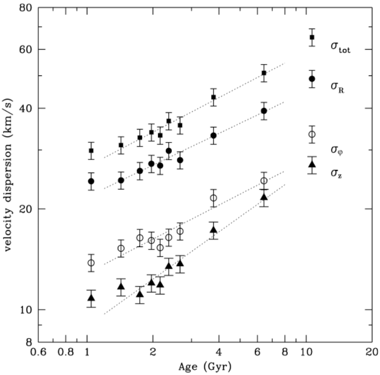


Figure 8.11 The velocity dispersion of stars in the solar neighborhood as a function of age, from Nordström et al. (2004). From bottom to top, the plots show the vertical dispersion σ_z, the azimuthal dispersion σ_ϕ, the radial dispersion σ_R, and the RMS velocity $(\sigma_R^2+\sigma_\phi^2+\sigma_z^2)^{1/2}$. The lines show fits of the form $\sigma_i \propto t^\alpha$ where t is the age; from bottom to top the best-fit exponents α are 0.47, 0.34, 0.31, and 0.34.

Application of Jeans equations

Asymmetric drift

Something has been heating the disk! Curious what that might be.

- Heating by MACHOs
- Scattering of disk stars by molecular clouds
- Scattering by spiral arms

Application of Jeans equations

Asymmetric drift

...MACHOs???

MACHO = MAssive Compact Halo Object.

This was the primary candidate for the **baryonic** Dark Matter (as considered only 10-15 years ago).

Anything dark, massive and not fuzzy goes:

- black holes
- neutron stars
- very old white dwarfs = black dwarfs?
- brown dwarfs
- rogue planets

Application of Jeans equations

Asymmetric drift

Unfortunately, any significant contribution of MACHOs to the Galaxy's mass budget is **ruled out**, due to

- they are too efficient in heating the disk and predict the amplitude of the effect to grow faster with time than observed
- can be detected directly through observations of gravitational microlensing effect. While the first claims put $f_{\rm MACHO} \sim 20\%$, it is consistent with zero.

Application of Jeans equations

Asymmetric drift

We **know** that the irregularities in the Galaxy's gravitational potential heat the disk and (re)shape the velocity distribution of the disk stars.

We **do not know** exactly which phenomenon is the primary source of heating

Most likely, it is the combined effects of spiral transients and molecular clouds

Application of Jeans equations

Asymmetric drift

We predicted $v_a \simeq \overline{v_R^2}/(82 \pm 6)\mathrm{kms}^{-1}$

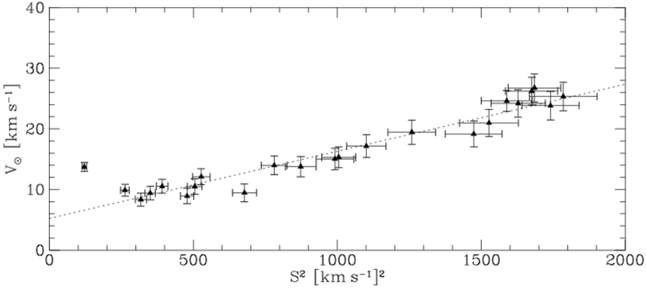


Figure 4.21 The asymmetric drift v_a for different stellar types is a linear function of the random velocity S^2 of each type. The vertical coordinate is actually $v_a + \tilde{v}_{\phi,\odot}$ where $\tilde{v}_{\phi,\odot}$ is the azimuthal velocity of the Sun relative to the LSR (after Dehnen & Binney 1998b).

The measured value from above:

$$v_a = \overline{v_R^2}/(80 \pm 5)\mathrm{kms}^{-1}$$

Application of Jeans equations

Local mass density

The mass density in the solar neighborhood.
Equation (5.32) can be written as

$$\frac{\partial(\nu \overline{v}_z)}{\partial t} + \frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \nu\frac{\partial\Phi}{\partial z} = 0$$

Take this equation and assume a steady state so $\frac{\partial}{\partial t} = 0$, so have

$$\frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} = -\nu\frac{\partial\Phi}{\partial z}$$

Application of Jeans equations

Local mass density

We are interested in the density in a thin disk, where the density falls off much faster in z than in R. Typically disk a few 100pc thick, with a radial scale of a few kpc, so

$$\frac{\partial}{\partial z} \sim 10 \frac{\partial}{\partial R} \sim 10 \frac{1}{R}$$

so neglect $\frac{\partial}{\partial R}$ term. So

$$\frac{1}{\nu}\frac{\partial}{\partial z}(\nu \overline{v_z^2}) = -\frac{\partial \Phi}{\partial z}$$

i.e.vertical pressure balances vertical gravity. This is the Jeans equation for one-dimensional slab.
Also can show that Poisson's equation in a thin disk approximation is

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho$$

where ρ is the total mass density.
So have

$$\frac{\partial}{\partial z}\frac{1}{\nu}\frac{\partial}{\partial z}(\nu \overline{v_z^2}) = -4\pi G \rho.$$

Application of Jeans equations

Local mass density

Note that by f we do not necessarily mean all stars, it could be any well-defined subset, such as all G stars (say).

The ν is the number density of G stars or whatever type is chosen. We have not linked ν and Φ (or ν and ρ) as was done in the previous example of a self-consistent spherical model.

Thus if for any population of stars we can measure $\overline{v_z^2}$ and ν as a function of height z we can calculate the **total local density** ρ. This involves differentiation of really noisy data, so the results are very uncertain.
Using this technique for F stars + K giants Oort found

$$\rho_0 = \rho(R_0, z = 0) = 0.15\ \mathrm{M_\odot\ pc^{-3}} = \text{ Oort limit.}$$

Application of Jeans equations

Local mass density

Note that one can determine instead

$$\Sigma(z) = \int_{-z}^{z} \rho dz' = -\frac{1}{2\pi G\nu}\frac{\partial}{\partial z}(\nu\overline{v_z^2})$$

more accurately (since there is one less difference, or differential, involved).

Oort: $\Sigma(700\mathrm{pc}) \simeq 90\ \mathrm{M}_\odot\ \mathrm{pc}^{-2}$

This compares with the observable mass:
$\Sigma(1.1\mathrm{kpc}) \simeq 71 \pm 6\ \mathrm{M}_\odot\ \mathrm{pc}^{-2}$ (Kuijken & Gilmore, 1991)

The baryons account for $\Sigma(\mathrm{stars\ plus\ gas}) \simeq 41 \pm 15\ \mathrm{M}_\odot\ \mathrm{pc}^{-2}$ (Binney & Evans, 2001)

Application of Jeans equations

Local mass density

Or we can estimate Dark Matter halo's contribution to Σ by supposing that

- the halo is spherical
- the circular speed $v_c = v_0 = constant$
- without the halo, $v_c = (GM_d/r)^{1/2}$

Then, the halo mass $M(r)$ satisfies $G[M(r) + M_d] = rv_0^2$

The halo's density:

$$\rho_h = \frac{1}{4\pi r^2}\frac{dM}{dr} = \frac{v_0^2}{4\pi G r^2} = 0.014 M_\odot \mathrm{pc}^{-3}\left(\frac{v_0}{200\mathrm{kms}^{-1}}\right)^2\left(\frac{R_0}{8\mathrm{kpc}}\right)^{-2}$$

The halo's contribution $\Sigma^h_{1.1} = 2.2\ \mathrm{kpc} \times \rho_h = 30.6 M_\odot \mathrm{pc}^{-2}$

So local dark matter is relatively tightly constrained, and the Sun lies in transition region in which both disk and halo contribute significant masses.

Application of Jeans equations

Mass profile of the Galaxy

Jeans equation for spherical systems:

$$\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r} + \nu\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r}\right) = 0 \tag{5.36}$$

For the stationary and spherically symmetric Galactic halo, the radial velocity dispersion $\sigma_{r,*}$ of stars with density ρ_* obeys the above Jeans equation (albeit modified slightly):

$$\frac{1}{\rho_*}\frac{\mathrm{d}(\rho_*\sigma_{r,*}^2)}{\mathrm{d}r} + \frac{2\beta\sigma_{r,*}^2}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{v_c^2}{r} \tag{5.37}$$

where the velocity anisotropy parameter is

$$\beta \equiv 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\overline{v_r^2}} \tag{5.38}$$

Thus, the Jeans equation allows us to determine a unique solution for the mass profile if we know $\sigma_{r,*}^2$, ρ_* and $\beta(r)$.

Application of Jeans equations

Mass profile of the Galaxy

The expected radial velocity dispersion for a tracer population is derived by integrating the Jeans equation:

$$\sigma_{r,*}^2 = \frac{1}{\rho_* e^{\int 2\beta \mathrm{d}x}} \int_x^{\infty} \rho_* v_c^2 e^{\int 2\beta \mathrm{d}x''} \mathrm{d}x', \quad x = \ln r \tag{5.39}$$

However, the proper motions are not available for the majority of the tracers, therefore we can only measure the line-of-sight velocity dispersion:

$$\sigma_{\mathrm{GSR},*}(r) = \sigma_{r,*}(r)\sqrt{1 - \beta H(r)} \tag{5.40}$$

Where

$$H(r) = \frac{r^2 + R_\odot^2}{4r^2} - \frac{(r^2 - R_\odot^2)^2}{8r^3 R_\odot} \ln \frac{r + R_\odot}{r - R_\odot} \tag{5.41}$$

Application of Jeans equations

Mass profile of the Galaxy

Alternatively,

$$M(r) = -\frac{r\sigma_r^2}{G}\left[\frac{d\ln\nu}{d\ln r} + \frac{d\ln\sigma_r^2}{d\ln r} + 2\beta(r)\right] \qquad (5.42)$$

Application of Jeans equations

Mass profile of the Galaxy

Still, there are further complications. Namely, the two ingredients are uncertain

- the behavior of the stellar velocity anisotropy
- stellar halo density profile at large radii

Application of Jeans equations

Mass profile of the Galaxy

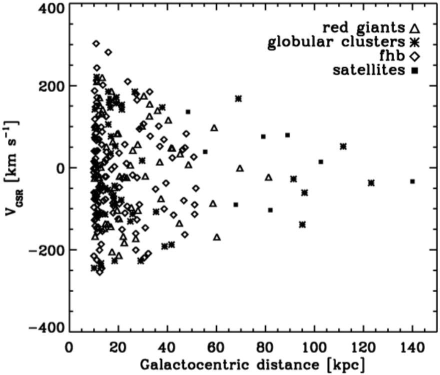


Figure 2. Heliocentric line-of-sight velocities corrected for the Solar motion and the LSR motion (V_{GSR}) for the sample used in this work (triangles, red giants; asterisks, globular clusters; diamonds, field horizontal branch stars; filled squares, satellite galaxies).

from Battaglia et al, 2005

Application of Jeans equations

Mass profile of the Galaxy

With constant velocity anisotropy

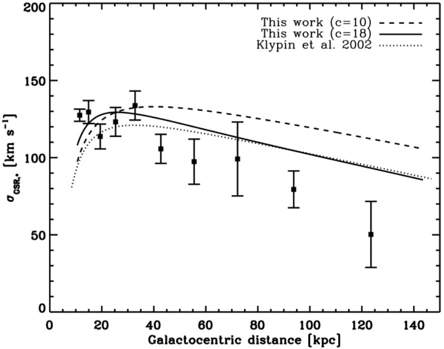


Figure 1. Observed radial velocity dispersion (squares with error bars) overlaid on two of the best-fitting models for the NFW mass distributions (dashed line: $c = 10$; solid line: $c = 18$). The dotted curve corresponds to the Galactocentric radial velocity dispersion profile obtained using the preferred model (B1) of Klypin et al. (2002). This figure replaces the bottom panel of fig. 4 in the original manuscript.

Application of Jeans equations

Mass profile of the Galaxy

Letting velocity anisotropy vary with radius

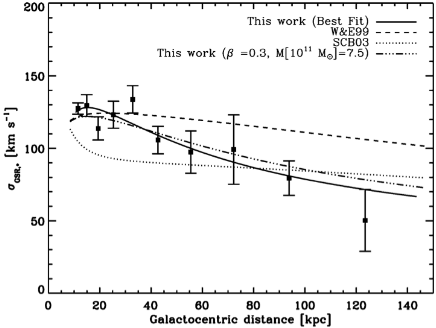


Figure 2. Observed radial velocity dispersion (squares with error bars) overlaid on the best-fitting model for the TF mass distribution (solid line). The dashed line shows the Galactocentric radial velocity dispersion obtained using the best-fitting parameters from Wilkison & Evans (1999) and the dotted line using the best-fitting parameters from Sakamoto et al. (2003). The dashed–double-dotted line shows $\sigma_{GSR,*}$ for a TF model with mass equal to the upper 1σ value from our best fit and a velocity anisotropy equal to the lower 1σ β. This figure replaces the right-hand panel of fig. 5 in the original manuscript.

The Virial Theorem

We have obtained the first moment of CBE by multiplying it through by v_j and integrating over all velocities. This allowed us to reduce an equation for 6D distribution function f to an equation for 3D density ν and the velocity moments:

$$\frac{\partial}{\partial t}(\nu \overline{v}_j) + \frac{\partial}{\partial x_i}(\nu \overline{v_i v_j}) + \nu \frac{\partial \Phi}{\partial x_j} = 0 \tag{5.43}$$

Now, let us multiply the above equation ↑ by x_k and integrate over all positions, converting these differential 1st moment equations into a tensor equation relating the global properties of the galaxy such as kinetic energy.

$$\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho \overline{v}_j)}{\partial t} = -\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho \overline{v_i v_j})}{\partial x_i} - \int \mathrm{d}^3\mathbf{x} \rho x_k \frac{\partial \Phi}{\partial x_j} \tag{5.44}$$

The Virial Theorem

Potential-energy tensor

$$\boxed{\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho \overline{v}_j)}{\partial t} = -\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho \overline{v_i v_j})}{\partial x_i} - \int \mathrm{d}^3\mathbf{x} \rho x_k \frac{\partial \Phi}{\partial x_j}}$$

By definition, the Chandrasekhar potential-energy tensor:

$$W_{jk} \equiv -\int \mathrm{d}^3\mathbf{x} \rho(\mathbf{x}) x_j \frac{\partial \Phi}{\partial x_k} \quad (5.45)$$

Also, by definition:

$$\Phi(\mathbf{x}) \equiv -G \int \mathrm{d}^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \quad (5.46)$$

Which makes $\mathbf{W}$ on substituting Φ:

$$W_{jk} = G \int \mathrm{d}^3\mathbf{x} \rho(\mathbf{x}) x_j \frac{\partial}{\partial x_k} \int \mathrm{d}^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \quad (5.47)$$

The Virial Theorem

Potential-energy tensor

$$\boxed{\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho \overline{v}_j)}{\partial t} = -\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho \overline{v_i v_j})}{\partial x_i} - \int \mathrm{d}^3\mathbf{x} \rho x_k \frac{\partial \Phi}{\partial x_j}}$$

Taking the differentiation inside the integral, re-labeling the dummy variables $\mathbf{x}$ and $\mathbf{x}'$ and writing W_{jk} twice, we get:

$$W_{jk} = -\frac{1}{2} G \int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{x}' \rho(\mathbf{x})\rho(\mathbf{x}') \frac{(x_j' - x_j)(x_k' - x_k)}{|\mathbf{x}' - \mathbf{x}|^3} \qquad (5.48)$$

Therefore, $\mathbf{W}$ is symmetric, i.e. $W_{jk} = W_{kj}$. Taking the trace:

$$\begin{aligned} \mathrm{trace}(\mathbf{W}) \equiv \sum_{j=1}^{3} W_{jj} &= -\frac{1}{2} G \int \mathrm{d}^3\mathbf{x}\, \rho(\mathbf{x}) \int \mathrm{d}^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \\ &= \frac{1}{2} \int \mathrm{d}^3\mathbf{x}\, \rho(\mathbf{x})\, \Phi(\mathbf{x}) \end{aligned} \qquad (5.49)$$

This is the total potential energy of the body W,

$$W = -\int \mathrm{d}^3\mathbf{x}\, \rho\, \mathbf{x}\, \nabla \mathbf{\Phi}. \qquad (5.50)$$

The Virial Theorem
Kinetic-energy tensor

$$\boxed{\int d^3\mathbf{x} x_k \frac{\partial(\rho\overline{v}_j)}{\partial t} = -\int d^3\mathbf{x} x_k \frac{\partial(\rho\overline{v_i v_j})}{\partial x_i} - \int d^3\mathbf{x} \rho x_k \frac{\partial\Phi}{\partial x_j}}$$

With the help of divergence theorem:

$$\int d^3\mathbf{x} x_k \frac{\partial(\rho\overline{v_i v_j})}{\partial x_i} = -\int d^3\mathbf{x} \delta_{ki} \rho \overline{v_i v_j} = -2K_{kj} \tag{5.51}$$

Here we have defined the **kinetic-energy tensor**:

$$K_{jk} \equiv \frac{1}{2}\int d^3\mathbf{x} \rho \overline{v_j v_k} \tag{5.52}$$

Remembering that $\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v_i}\ \overline{v_j}$, contributions from ordered $\mathbf{T}$ and random $\mathbf{\Pi}$ motion:

$$K_{jk} = T_{jk} + \frac{1}{2}\Pi_{jk}, \quad T_{jk} \equiv \frac{1}{2}\int d^3\mathbf{x} \rho \overline{v}_j \overline{v}_k, \quad \Pi_{jk} \equiv \int d^3\mathbf{x} \rho \sigma_{jk}^2 \tag{5.53}$$

The Virial Theorem

$$\boxed{\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho\overline{v}_j)}{\partial t} = -\int \mathrm{d}^3\mathbf{x} x_k \frac{\partial(\rho\overline{v_i v_j})}{\partial x_i} - \int \mathrm{d}^3\mathbf{x} \rho x_k \frac{\partial\Phi}{\partial x_j}}$$

Taking the time derivative outside and averaging the (k, j) and the (j, k) components of the above equation ↑

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\mathbf{x}\rho(x_k\overline{v}_j + x_j\overline{v}_k) = 2T_{jk} + \Pi_{jk} + W_{jk} \tag{5.54}$$

where we have taken advantage of the symmetry of $\mathbf{T}, \mathbf{\Pi}, \mathbf{W}$ under exchange of indices

If we define moment of inertia tensor

$$I_{jk} \equiv \int \mathrm{d}^3\mathbf{x}\rho x_j x_k \quad \text{and} \quad \frac{\mathrm{d}I_{jk}}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\rho(x_k\overline{v}_j + x_j\overline{v}_k) \tag{5.55}$$

Tensor Virial Theorem

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2T_{jk} + \Pi_{jk} + W_{jk} \tag{5.56}$$

The Virial Theorem

- The theorem is derived for collisionless systems, but can be proven for self-gravitating collisional systems too.
- This is the equation of energy balance in systems in equilibrium under gravity.
- Can be extended to include energy from turbulence and convective motions, magnetic energy etc

$$\boxed{\frac{1}{2}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2T_{jk} + \Pi_{jk} + W_{jk}}$$

In a steady state $\ddot{\mathbf{I}} = 0$, the trace of the Tensor Virial Theorem equation above is:

Scalar Virial Theorem

$$2K + W = 0 \tag{5.57}$$

where

$$K \equiv \mathrm{trace}(\mathbf{T}) + \frac{1}{2}\mathrm{trace}(\mathbf{\Pi}) \tag{5.58}$$

The Virial Theorem

Curiously, if E is the energy of the system then

$$E = K + W = -K = \frac{1}{2}W \tag{5.59}$$

The Virial Theorem

The kinetic energy of a stellar system with mass M where stars move at mean-square speed $\langle v^2 \rangle$ is

$$K = \frac{1}{2} M \langle v^2 \rangle \tag{5.60}$$

The virial theorem states that:

$$\langle v^2 \rangle = \frac{|W|}{M} = \frac{GM}{r_g} \tag{5.61}$$

This is the fastest way to get the mass of the system! Here the gravitational radius r_g

$$r_g \equiv \frac{GM^2}{|W|} \tag{5.62}$$

For example, for a homogeneous sphere of radius a and density ρ, the potential energy:

$$W = -\frac{16\pi^2}{3} G\rho^2 \int_0^a dr r^4 = -\frac{16}{15}\pi^2 G\rho^2 a^5 = -\frac{3}{5}\frac{GM^2}{a} \tag{5.63}$$

And $r_g = \frac{5}{3} a$

The Virial Theorem

Applications Virial Theorem

- Despite the elegance of the Virial Theorem, its applications are not straightforward.
- This is because neither $\langle v^2 \rangle$ or r_g are readily available for most systems.
- Instead of $\langle v^2 \rangle$, the line of sight velocity dispersion $\langle v_{||}^2 \rangle$ is used.
- And isotropy is assumed (not going to work in many situations)

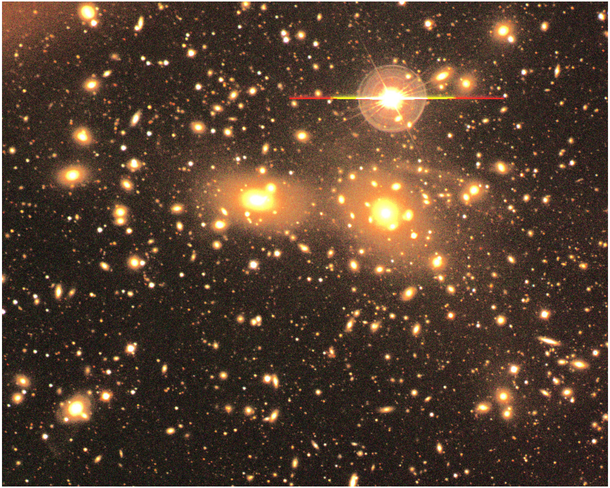
$$\langle v^2 \rangle = 3 \langle v_{||}^2 \rangle$$

- Instead of gravitational radius r_g the rough extent of the system is used
- or use the so-called half-mass radius r_h obtained by integrating light and assuming mass/light ratio. It can be shown that for variety of systems $r_h / r_g \sim \frac{1}{2}$

See Eddington (1916). Einstein (1921) used the Virial Theorem to estimate the mass of globular clusters.

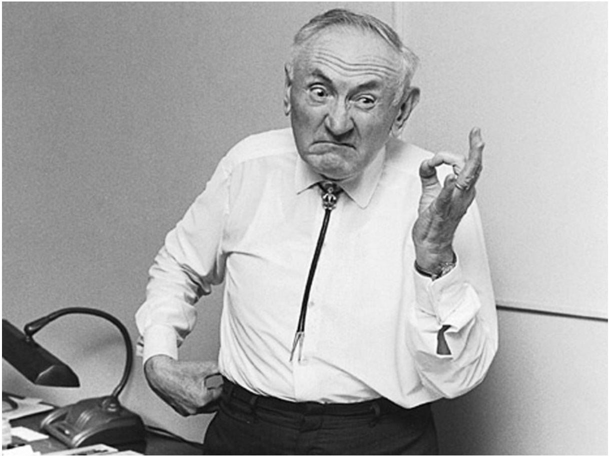
The Virial Theorem

Coma Cluster



AKA Abel 1656, $D \sim 100$ Mpc, $N > 1000$ galaxies

The Virial Theorem
Fritz Zwicky and the Coma Cluster



The Virial Theorem

Fritz Zwicky and the Coma Cluster

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ON THE MASSES OF NEBULAE AND OF CLUSTERS OF NEBULAE

F. ZWICKY

ABSTRACT

Present estimates of the masses of nebulae are based on observations of the *luminosities* and *internal rotations* of nebulae. It is shown that both these methods are unreliable; that from the observed luminosities of extragalactic systems only lower

The Virial Theorem

Fritz Zwicky and the Coma Cluster

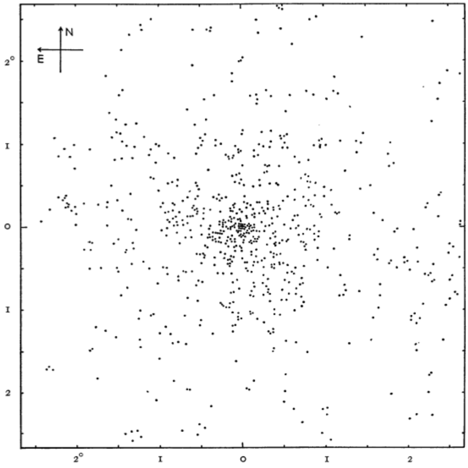

FIG. 3.—The Coma cluster of nebulae

The Virial Theorem

Fritz Zwicky and the Coma Cluster

as yet unknown masses. The mass $\mathscr{M}$, as obtained from the virial theorem, can therefore be regarded as correct only in order of magnitude.

Combining (33) and (34), we find

$$\mathscr{M} > 9 \times 10^{46}\text{gr} . \tag{35}$$

The Coma cluster contains about one thousand nebulae. The average mass of one of these nebulae is therefore

$$\overline{M} > 9 \times 10^{43}\text{ gr} = 4.5 \times 10^{10}\ M_{\odot} . \tag{36}$$

Inasmuch as we have introduced at every step of our argument inequalities which tend to depress the final value of the mass $\mathscr{M}$, the foregoing value (36) should be considered as the lowest estimate for the average mass of nebulae in the Coma cluster. This result is somewhat unexpected, in view of the fact that the luminosity of an average nebula is equal to that of about 8.5×10^7 suns. According to (36), the conversion factor γ from luminosity to mass for nebulae in the Coma cluster would be of the order

$$\gamma = 500 , \tag{37}$$

as compared with about $\gamma' = 3$ for the local Kapteyn stellar system.