Stellar Dynamics and Structure of Galaxies

Circular and nearly circular orbits

Martin Haehnelt *
haehnelt@ast.cam.ac.uk

Institute of Astronomy

Michaelmas Term 2018

* based on slides prepared by Vasily Belokurov and lecture notes by Jim Pringle

Outline I

Circular and Nearly Circular Orbits

Rotation in a disk galaxy is the obvious example of such orbit. Given a central force f_r due to a fixed potential Φ, we have

$$\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr} \tag{3.1}$$

$$r^2\dot{\phi} = h = \text{constant} \tag{3.2}$$

Circular and Nearly Circular Orbits

$\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr}$ 3.1

$r^2\dot{\phi} = h = \text{constant}$ 3.2

For a circular orbit $r = R$ =constant and $\dot{\phi} = \Omega$ =constant.
Then (3.2) is satisfied trivially, and (3.1)⇒

$$R\Omega^2 = -f_r = \left.\frac{d\Phi}{dr}\right|_{r=R} \quad (3.3)$$

so if $\Phi = -\frac{GM}{r}$, then

$$R\Omega^2 = \frac{GM}{R^2} \Rightarrow \Omega = \left(\frac{GM}{R^3}\right)^{\frac{1}{2}}$$

and the period

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{R^3}{GM}}$$

From the earlier Keplerian orbit discussion, $R = a =$ the radius of the orbit, or the separation between the two stars for a binary system with circular orbits.

Circular and Nearly Circular Orbits

Now consider an orbit which is nearly circular, so we take

$$r = R + \varepsilon(t) \quad \text{with } \varepsilon << R$$

and

$$\dot{\phi} = \Omega + \omega(t) \quad \text{with } \omega << \Omega$$

If we choose to characterize orbits by their angular momentum, we keep the angular momentum unchanged, and the (3.2)⇒

$$\begin{aligned} h = R^2\Omega &= (R+\varepsilon)^2(\Omega+\omega) \\ &= (R^2+2R\varepsilon)(\Omega+\omega) \\ &= R^2\Omega + 2R\varepsilon\Omega + R^2\omega \end{aligned} \tag{3.4}$$

if we retain only terms to first order. Therefore

$$R\omega = -2\varepsilon\Omega \tag{3.5}$$

Circular and Nearly Circular Orbits

$\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr}$ 3.1 $R\Omega^2 = -f_r = \frac{d\Phi}{dr}\Big|_{r=R}$ 3.3

Now, using (3.1) and retaining only terms to first order, the perturbation's behaviour is described by:

$$\ddot{\varepsilon} - (R+\varepsilon)(\Omega^2 + 2\Omega\omega) = f(R+\varepsilon) \quad (3.6)$$

$$\ddot{\varepsilon} - R\Omega^2 - \varepsilon\Omega^2 - 2R\Omega\omega = f(R) + \varepsilon f'(R) \quad (3.7)$$

$R\Omega^2 = -f(R)$ from (3.3), and using (3.5) $-2R\Omega\omega = 4\varepsilon\Omega^2$, so we have

$$\ddot{\varepsilon} + 3\varepsilon\Omega^2 = \varepsilon f'(R) \quad (3.8)$$

$$\text{or} \quad \ddot{\varepsilon} + \left(3\Omega^2 - f'(R)\right)\varepsilon = 0 \quad (3.9)$$

This is stable simple harmonic motion if $\Omega_R^2 = 3\Omega^2 - f'(R) > 0$ so, using (3.3), if

$$f'(R) + 3\frac{f(R)}{R} < 0 \Leftrightarrow \frac{d}{dR}(R^3 f) < 0$$

e.g. $f(R) \propto -R^{-n}$ is stable only if $n < 3$ i.e. unstable if potential is steep.

Circular and Nearly Circular Orbits

$\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr}$ 3.1

$R\Omega^2 = -f_r = \frac{d\Phi}{dr}\Big|_{r=R}$ 3.3

Now, using (3.1) and retaining only terms to first order, the perturbation's behaviour is described by:

$$\ddot{\varepsilon} - (R+\varepsilon)(\Omega^2 + 2\Omega\omega) = f(R+\varepsilon) \quad (3.6)$$

$$\ddot{\varepsilon} - R\Omega^2 - \varepsilon\Omega^2 - 2R\Omega\omega = f(R) + \varepsilon f'(R) \quad (3.7)$$

$R\Omega^2 = -f(R)$ from (3.3), and using (3.5) $-2R\Omega\omega = 4\varepsilon\Omega^2$, so we have

$$\ddot{\varepsilon} + 3\varepsilon\Omega^2 = \varepsilon f'(R) \quad (3.8)$$

$$\text{or} \quad \ddot{\varepsilon} + \left(3\Omega^2 - f'(R)\right)\varepsilon = 0 \quad (3.9)$$

This is stable simple harmonic motion if $\Omega_R^2 = 3\Omega^2 - f'(R) > 0$ so, using (3.3), if

$$f'(R) + 3\frac{f(R)}{R} < 0 \Leftrightarrow \frac{d}{dR}(R^3 f) < 0$$

e.g. $f(R) \propto -R^{-n}$ is stable only if $n < 3$ i.e. unstable if potential is steep.

Precession

To a first approximation, a particle circles the origin with a period $T = 2\pi/\Omega$.
It executes radial motion with a period $T_r = 2\pi/\Omega_R$ where $\Omega_R^2 = 3\Omega^2 - f'(R)$.
In general $\Omega_R \neq \Omega$, so the orbit is not closed.
The orbit is like an ellipse which rotates (or precesses) with a period $2\pi/\Omega_p$ where $\Omega_p = \Omega - \Omega_R$

In general for galaxies precession is retrograde (i.e. opposite to the rotation direction of the stars) since T_r is usually less than T_ϕ. We'll see why later, but the basic results are for a harmonic (uniform density) model $\Delta\phi = \pi$ in one radial period, and for Keplerian orbits $\Delta\phi = 2\pi$ in one radial period, and real galaxies fall between these extremes

For Keplerian potential $f(R) = -\frac{GM}{R^2}$, $\Omega^2 = \frac{GM}{R^3}$ and $f'(R) = \frac{2GM}{R^3}$, so $\Omega_R^2 = 3\Omega^2 - f'(R) = \frac{GM}{R^3} = \Omega^2$, so the orbits are closed.
Note: Often Ω_R^2 is written K^2, and K called the epicyclic frequency.

Epicyclic approximation

Move to a frame in which the unperturbed particle is at rest, with the coordinates in the direction of rotation and in the radial direction. This is necessarily a rotating frame.

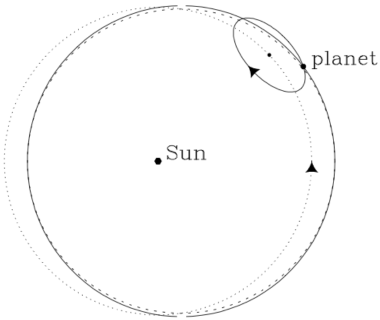


Figure 3.7 An elliptical Kepler orbit (dashed curve) is well approximated by the superposition of motion at angular frequency κ around a small ellipse with axis ratio $\frac{1}{2}$, and motion of the ellipse's center in the opposite sense at angular frequency Ω around a circle (dotted curve).

Epicyclic approximation

$$r = R + y$$
$$R\dot{\phi} = R\Omega + \dot{x}$$

so

$$y = \varepsilon$$
$$\dot{x} = R\omega = -2\varepsilon\Omega$$

The second equality from the conservation of angular momentum $R\omega = -2\varepsilon\Omega$.
So can use relation $\ddot{\varepsilon} + \left(3\Omega^2 - f'(R)\right)\varepsilon = 0$, which becomes

$$\ddot{y} + K^2 y = 0$$

Epicyclic approximation

so if we take $y = -b\cos(Kt)$, $\dot{x} = 2\Omega b\cos(Kt)$, so

$$x = \frac{2\Omega b}{K}\sin(Kt) = a\sin(Kt)$$

defines a, and then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

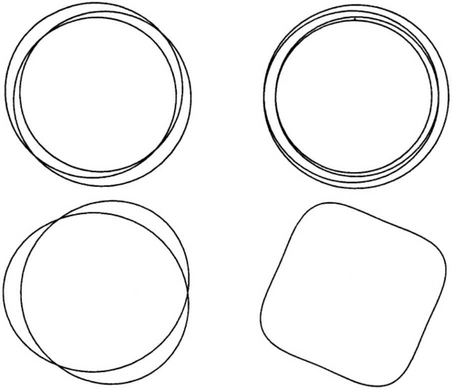
$\Rightarrow$ motion is an ellipse which moves retrograde at frequency K and is such that $b = \frac{K}{2\Omega}a$

For Keplerian potential $K = \Omega$ so $b = a/2$

[For harmonic potential (to come) $K = 2\Omega$ so $b = a$]

In general epicycle is elongated along tangential direction.

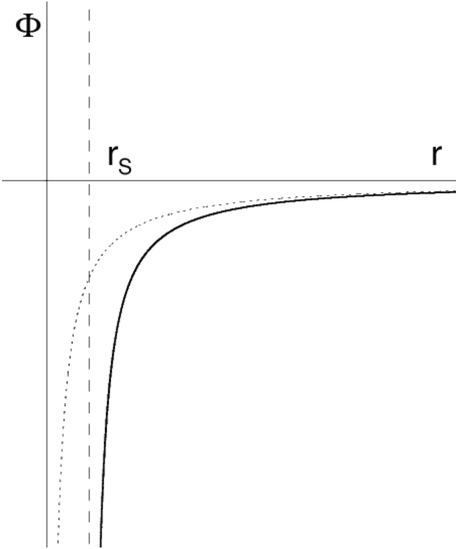
Epicyclic approximation



Quasi-circular orbits when the ratio of angular to radial frequency is rational (3/2, upper left; 2/3 lower left; 4, upper right; 1/4, lower right).

Epicyclic approximation

Example: pseudo black hole potential



Epicyclic approximation

Example: pseudo black hole potential

$$\Phi(r) = -\frac{GM}{r - R_s}$$

$$f(r) = -\frac{d\Phi}{dr} = -\frac{GM}{(r - R_s)^2}$$

For a circular orbit $\Omega_c^2 = -\frac{f(R)}{R}$ so

$$\Omega_c^2 = \frac{GM}{R\,(R - R_s)^2}$$

Also

$$f'(R) = \frac{2GM}{(R - R_s)^3}$$

so

$$K^2 = 3\Omega^2 - f'(R) = \frac{3GM}{R\,(R - R_s)^2} - \frac{2GM}{(R - R_s)^3}$$

Epicyclic approximation

Example: pseudo black hole potential

Stable circular orbits are those for which $K^2 > 0$, so require

$$3\left(R - R_s\right)^3 > 2R\left(R - R_s\right)^2$$

so for $R \neq R_s$

$$3\left(R - R_s\right) > 2R$$

or

$$R > 3R_s$$

This is reminiscent of a Schwarzschild black hole: $R_s = \frac{2GM}{c^2}$.

More general potentials

Axisymmetric Potentials

In most of the things we are interested in, the density distribution is not always (or even often) spherically symmetric, but it may be approximately axisymmetric. In such cases we use cylindrical polar coordintes (R, ϕ, z).

If $\rho = \rho(R, z)$, then $\Phi(\mathbf{r}) = \Phi(R, z)$.

Often also have plane symmetry, where $\rho(R, z) = \rho(R, -z)$ (with choice of origin in the plane of symmetry of course).

e.g. Spheroidal galaxy, or central bulge in a spiral
thin disk
and so, by addition, get the full galaxy potential
or fast rotating planet (Jupiter, Saturn) has equatorial bulge
or even the time averaged potential of the moon (for the study of long timescale effects)

More general potentials

Axisymmetric Potentials

So we have to consider orbits in axisymmetric potentials, where there is no ϕ-dependence so $\Phi(R, \phi, z) = \Phi(R, z)$.

The force

$$\mathbf{F} = \left(-\frac{\partial \Phi}{\partial R}, 0, -\frac{\partial \Phi}{\partial z} \right)$$

Since there is no force in the ϕ direction, the angular momentum about the z-axis L_z is constant, so the equation of motion becomes

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial \Phi}{\partial R} \tag{3.10}$$

$$R^2\dot{\phi} = L_z \tag{3.11}$$

$$\ddot{z} = -\frac{\partial \Phi}{\partial z} \tag{3.12}$$

More general potentials

Axisymmetric Potentials

We can remove the $\dot{\phi}$ term from the first two to obtain

$$\ddot{R} = -\frac{\partial \Phi}{\partial R} + \frac{L_z^2}{R^3} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial R} \tag{3.13}$$

where

$$\Phi_{\mathrm{eff}} = \Phi + \frac{L_z^2}{2R^2}$$

and since $\frac{L_z}{2R^2}$ is independent of z,

$$\ddot{z} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial z} \tag{3.14}$$

So we have reduced a 3D problem to a 2D one.
In astronomical situations we also have plane symmetry, so $\Phi(R, z) = \Phi(R, -z)$.
General orbits are complicated, and beyond the scope of this course (but see Part III). We will deal with circular and nearly circular orbits close to the $z = 0$ plane.

More general potentials

Circular orbits in the $z = 0$ plane

Look for solution $z = 0$, $R = R_c$ =constant, $\dot{\phi} = \Omega$ =constant.
Equation (3.14) is satisfied because $\frac{\partial \Phi}{\partial z} = 0$ at $z = 0$, from the plane symmetry condition.
Equation (3.13)⇒

$$\frac{L_z^2}{R^3} = \frac{\partial \Phi}{\partial R}$$

Since $R_c^2 \Omega_c = L_z$, then

$$\Omega_c^2 = \frac{1}{R} \left. \frac{\partial \Phi}{\partial R} \right|_{R=R_c}$$

as before.

More general potentials

Nearly circular orbits close to the $z = 0$ plane

Stars on orbits in the plane in a flattened potential have no way of perceiving that the potential they are moving in is not spherically symmetric. Therefore our deductions apply: star oscillates between two extrema in the radial coordinate.

What happens to stars whose orbits carry them out of the plane?

$R = R_c + x$, and $z = z$, with $x, z << R_c$.
At $z = x = 0$, we have
$\frac{\partial \Phi_{\rm eff}}{\partial z} = 0$ from symmetry, and
$\frac{\partial \Phi_{\rm eff}}{\partial R} = 0$ since $\ddot{R} = 0 = \frac{\partial \Phi_{\rm eff}}{\partial R}$

More general potentials

Nearly circular orbits close to the $z = 0$ plane

We can expand the function $\Phi_{\rm eff}$ about $z = x = 0$ to obtain

$$\begin{aligned}\Phi_{\rm eff}(R_c + x, z) \quad = \quad & \Phi_{\rm eff}(R_c, 0) + x \left.\frac{\partial \Phi_{\rm eff}}{\partial R}\right|_{(R_c,0)} + z \left.\frac{\partial \Phi_{\rm eff}}{\partial z}\right|_{(R_c,0)} \\ & + \frac{x^2}{2!} \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|_{(R_c,0)} + \frac{2xz}{2!} \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R \partial z}\right|_{(R_c,0)} \\ & + \frac{z^2}{2!} \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right|_{(R_c,0)} \qquad (3.15)\end{aligned}$$

The linear terms are zero from the considerations above, and the cross term (xz) coefficient is also zero from the plane symmetry.

More general potentials

Nearly circular orbits close to the $z = 0$ plane

Thus, from (3.13) ($\ddot{R} = -\frac{\partial \Phi_{\rm eff}}{\partial R}$)

$$\ddot{x} = -\frac{\partial \Phi_{\rm eff}}{\partial x} = -x \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|_{(R_c,0)}$$

and from(3.14)

$$\ddot{z} = -\frac{\partial \Phi_{\rm eff}}{\partial z} = -z \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right|_{(R_c,0)}$$

More general potentials

Nearly circular orbits close to the $z = 0$ plane

Therefore the equations become

$$\ddot{x} = -K^2 x$$

- the epcyclic frequency, and

$$\ddot{z} = -\mathcal{V}^2 z$$

- the vertical frequency.

Here

$$\mathcal{V}^2 = \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{(R_c,0)}$$

and

$$K^2 = \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{(R_c,0)} + \frac{3L_z^2}{R_c^4}$$

More general potentials

Nearly circular orbits close to the $z = 0$ plane

But

$$\Omega_c^2(R) = \frac{1}{R}\frac{\partial \Phi}{\partial R} = \frac{L_z^2}{R_c^4}$$

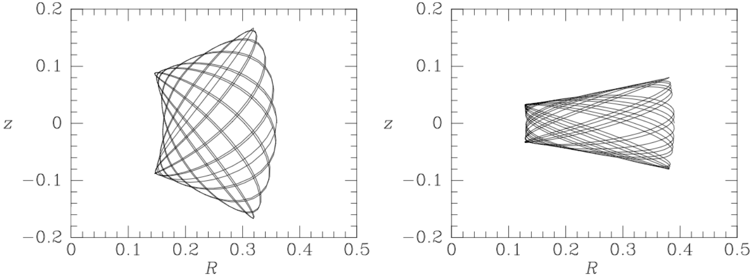
$\Rightarrow$

$$K^2 = \left(R\frac{\partial \Omega^2}{\partial R} + 4\Omega^2 \right)\Bigg|_{(R_c,0)}$$

[See example sheet 2].
Thus there are two types of precession - radial precession (or rotation of pericentre, as before) $\Omega_p = \Omega - K$, and vertical or nodal precession $\Omega_z = \Omega - \mathcal{V}$. The orbit is in a tilted plane which rotates at rate Ω_z.
A **node** is the place where the orbit crosses the $z = 0$ plane upwards (by convention, also called the ascending node).

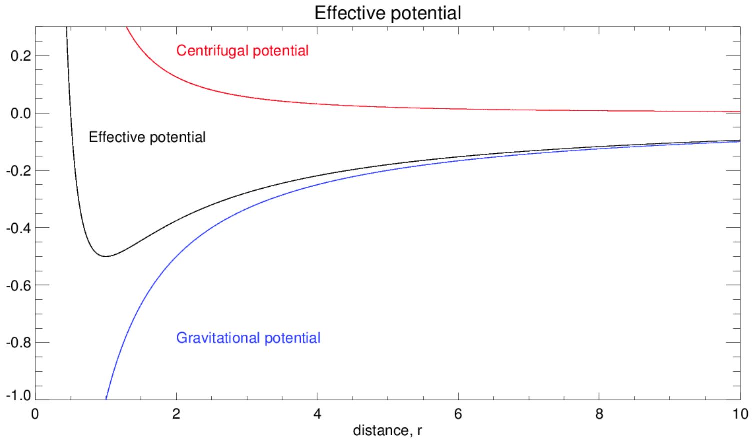
More general potentials

Nearly circular orbits close to the $z = 0$ plane

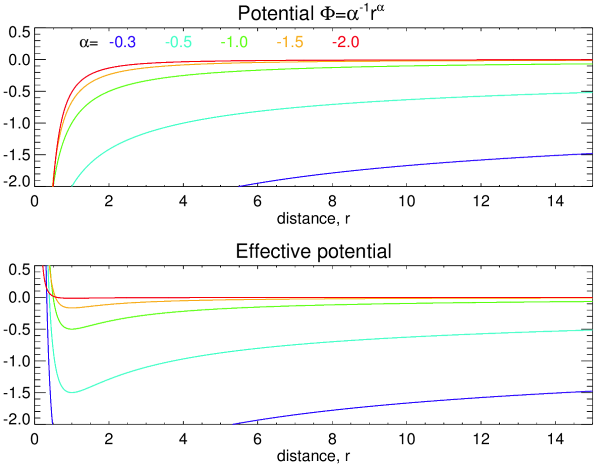


Binney and Tremain, Fig 3.4 Orbits in axisymmetric potential.

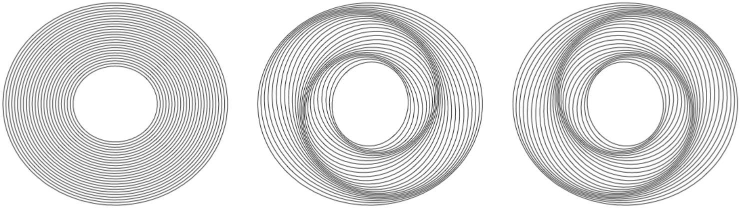
Another look at circular orbit stability



Another look at circular orbit stability



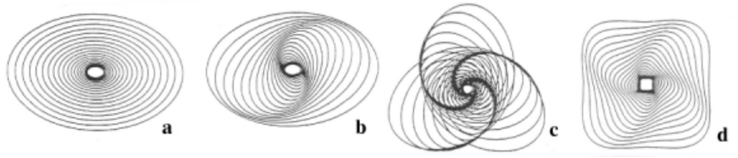
Special arrangements of epicycles



(a) (b) (c)

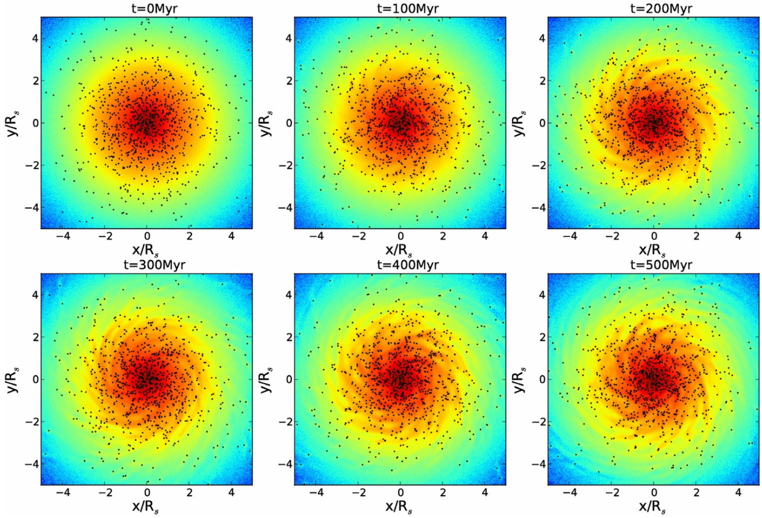
Figure 6.12 Arrangement of closed orbits in a galaxy with $\Omega - \frac{1}{2}\kappa$ independent of radius, to create bars and spiral patterns (after Kalnajs 1973b).

Special arrangements of epicycles



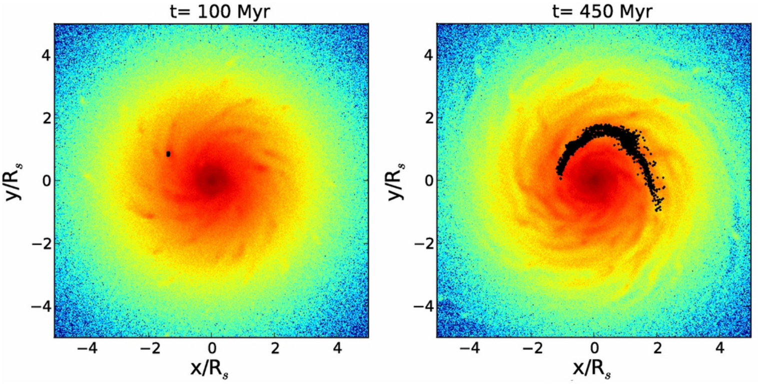
a - bar (aligned azimuthal/radial = 1/2 resonance)
b - 2 arm spiral (offset 1/2 resonance)
c - 3 arm spiral (offset 2/3 resonance)
d - 4 arm spiral (offset 1/4 resonance)

Molecular clouds as perturbers



from D'Onghia et al 2013

Molecular clouds as perturbers



from D'Onghia et al 2013

Swing amplification

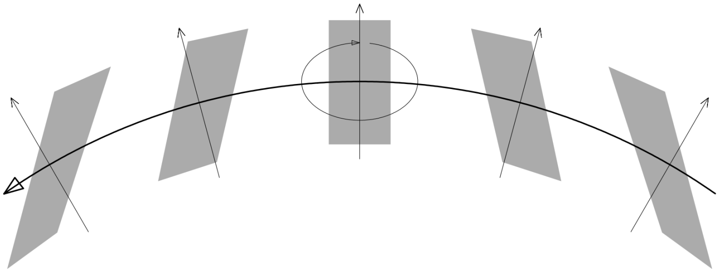


Figure 15.1: Evolution of an overdense perturbation in a shearing disk. The disk rotates counter-clockwise, as indicated by the heavy arc; a typical star moves around an elliptical epicycle in a clockwise direction. The perturbation (grey patch) initially has the form of a *leading* spiral (right), but is sheared into a *trailing* spiral (left) by the differential rotation of the disk. The epicycle and the perturbation rotate in the same direction, so stars stay in the perturbation longer than they would under other conditions.