Stellar Dynamics and Structure of Galaxies

Derivation of potential from density distribution

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* based on slides prepared by Vasily Belokurov and lecture notes by Jim Pringle

Outline I

Potentials from density distribution

Poisson's Equation

Poisson's equation relates $\rho(\mathbf{r})$ to $\Phi(\mathbf{r})$.

Also covered in the Astrophysical Fluid Dynamics course

To determine the force due to a given density distribution $\rho(r')$ we split it into many point masses of size

$$dm' = \rho(r')d^3\mathbf{r}' \ \text{ at } \mathbf{r}'$$

Newtonian gravity is linear, so just add up the forces

$$\mathbf{f}(\mathbf{r}) = -\int \frac{G dm'}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')$$

<u>or</u> since we want the total <u>potential</u> add up the individual contributions

$$\Phi(\mathbf{r}) = \int\int\int \frac{G\rho(\mathbf{r}')d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

As an exercise, show that $\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}'-\mathbf{r}|} = \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}$, and hence $\mathbf{f}(\mathbf{r}) = -\nabla\Phi$

Potentials from density distribution
Poisson's Equation

Consider

$$\nabla^2\Phi(\mathbf{r}) = -\int\int\int G\rho(\mathbf{r}')\nabla^2_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) d^3\mathbf{r}'$$

$\Rightarrow$ need $\nabla^2_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)$.

To keep the algebra simple move the origin to $\mathbf{r}'$ (and move back later)

for those who want everything in full generality, see Binney & Tremaine

So we need $\nabla^2(\frac{1}{r})$. For $r \neq 0$,

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right] = 0 \text{ trivially}$$

Potentials from density distribution

Poisson's Equation

But at $r = 0$ $\nabla^2(\frac{1}{r})$ is undefined.

You've seen that sort of thing before. Recall that the Dirac δ-function $\delta(x)$ satisfies $\int_{-\epsilon}^{\epsilon} \delta(x) dx = 1$

So now ask: what is the volume integral of $\nabla^2(\frac{1}{r})$ over a small volume V containing the origin?

$$\begin{aligned} \int\int\int_V \nabla^2\left(\frac{1}{r}\right) d^3V &= \int\int\int_V \nabla\cdot\left[\nabla\left(\frac{1}{r}\right)\right] d^3V \quad \text{by definition} \\ &= \int\int_S \hat{\mathbf{n}}\cdot\left[\nabla\left(\frac{1}{r}\right)\right] d^2S \qquad (2.1) \end{aligned}$$

Divergence theorem ($\hat{\mathbf{n}}$ - outward normal) $\int_V d^3\mathbf{x}\nabla\mathbf{F} = \int_S \hat{\mathbf{n}}\mathbf{F}$

Potentials from density distribution

Poisson's Equation

Take V to be a sphere, so $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, $d^2S = r^2 \sin\theta d\theta d\phi$, and have $\nabla(1/r) = -\frac{1}{r^2}\hat{\mathbf{r}}$. Then

$$\int\int\int_V \nabla^2 \left(\frac{1}{r}\right) d^3V = -\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = -4\pi \quad (2.2)$$

Since the integral is -4π, and is non-zero only at $r = 0$, we must therefore have

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r})$$

or, going back to the general origin,

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -4\pi\delta(\mathbf{r} - \mathbf{r}')$$

Potentials from density distribution
Poisson's Equation

Hence

$$\begin{aligned}\nabla^2\Phi(\mathbf{r}) &= -G\int\int\int \rho(\mathbf{r}')\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\, d^3\mathbf{r}' \\ &= 4\pi G\int\int\int \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\, d^3\mathbf{r}' \\ &= 4\pi G\rho(\mathbf{r}) \qquad (2.3)\end{aligned}$$

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$$

Poisson's Equation

Gauss's Theorem

Application of the Divergence Theorem to the Poisson's Equation

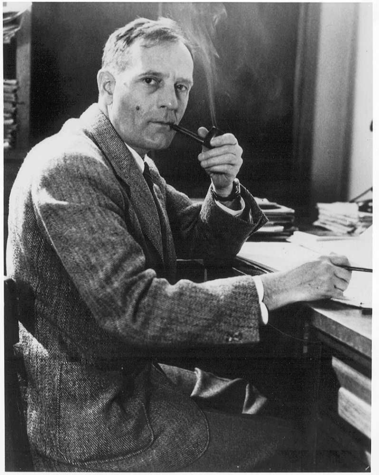
"The integral of the normal component of $\nabla\Phi$ over any closed surface equals $4\pi G$ times the mass enclosed within that surface"

To prove this simply take Poisson's equation and integrate over a volume V containing a mass M.

$$\begin{aligned} 4\pi G \int \rho d^3\mathbf{r} = 4\pi GM &= \int \nabla^2\Phi \; d^3\mathbf{r} \\ &= \int \nabla \cdot \nabla\Phi \; d^3\mathbf{r} \\ &= \int \nabla\Phi \cdot \hat{\mathbf{n}} \; d^2S \qquad (2.4) \end{aligned}$$

where the last step follows from the divergence theorem.

Edwin Hubble's classification of galaxies



EXTRA-GALACTIC NEBULAE[1]

By EDWIN HUBBLE

ABSTRACT

This contribution gives the results of a statistical investigation of 400 extra-galactic nebulae for which Holetschek has determined total visual magnitudes. The list is complete for the brighter nebulae in the northern sky and is representative to 12.5 mag. or fainter.

The classification employed is based on the forms of the photographic images. About 3 per cent are irregular, but the remaining nebulae fall into a sequence of type forms characterized by rotational symmetry about dominating nuclei. The sequence is composed of two sections, the elliptical nebulae and the spirals, which merge into each other.

Luminosity relations.—The distribution of magnitudes appears to be uniform throughout the sequence. For each type or stage in the sequence, the total magnitudes are related to the logarithms of the maximum diameters by the formula,

$$m_T = C - 5 \log d\,,$$

Astrophysical Journal, 64, 321-369 (1926)

Edwin Hubble's classification of galaxies

II. Extra-galactic nebulae:

A. Regular:

1. Elliptical . . . . E*n*
   (*n* = 1, 2, ... , 7 indicates the ellipticity of the image without the decimal point)

N.G.C. 3379	E0
221	E2
4621	E5
2117	E7

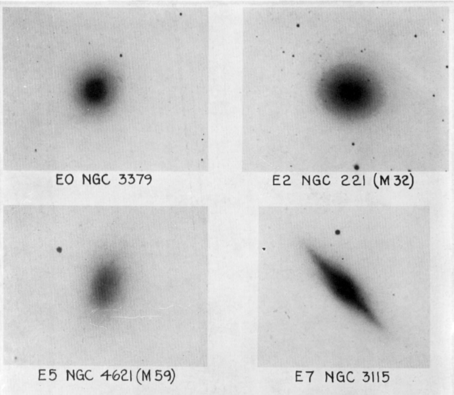
2. Spirals:

	Symbol	Example
a) Normal spirals	S	. . . .
(1) Early	Sa	N.G.C. 4594
(2) Intermediate	Sb	2841
(3) Late	Sc	5457
b) Barred spirals	SB	. . . .
(1) Early	SBa	N.G.C. 2859
(2) Intermediate	SBb	3351
(3) Late	SBc	7479

B. Irregular . . . . Irr N.G.C. 4449

Edwin Hubble's classification of galaxies

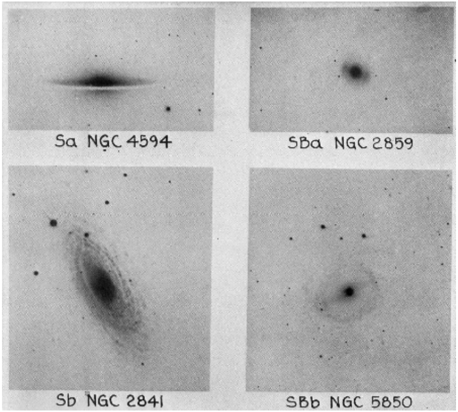
Early Types



Fundamental plane exists that ties surface brightness, size and LOS velocity dispersion

Edwin Hubble's classification of galaxies

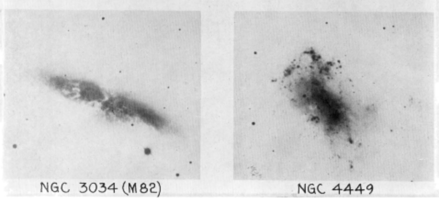
Spirals



Tully-Fisher law exists that ties together circular speed and luminosity

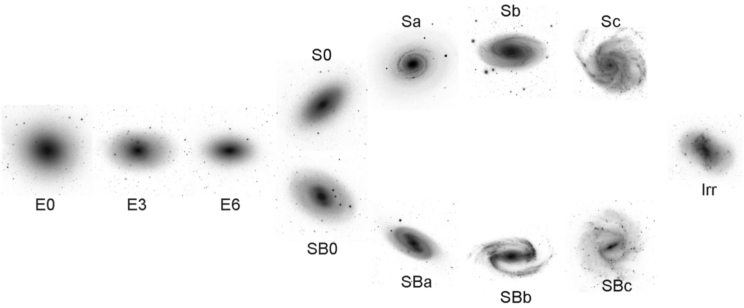
Edwin Hubble's classification of galaxies

Irregulars



Edwin Hubble's classification of galaxies

The Tuning Fork



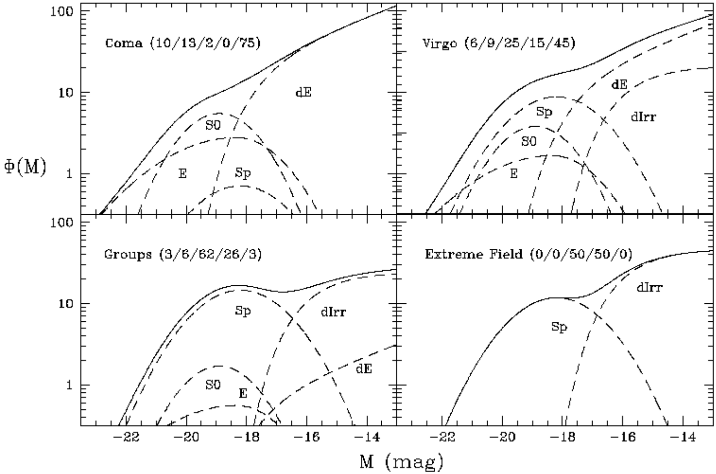
Edwin Hubble's classification of galaxies

The Three Pioneers

Albert Einstein, Edwin Hubble, and Walter Adams in 1931 at the Mount Wilson Observatory 100" telescope, in the San Gabriel Mountains of southern California.

Edwin Hubble's classification of galaxies

Galaxy Luminosity Function



In any environment, dwarfs dominate!

Deriving potentials of spherical systems

we can take $\rho(\mathbf{r}) = \rho(r)$

In spherical polars

$$\nabla^2\Phi(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\Phi\right) = \frac{1}{r}\frac{d^2}{dr^2}(r\Phi)$$

Exercise: show the last equality is true

So

$$\nabla^2\Phi = 4\pi G\rho$$

becomes

$$\frac{1}{r}\frac{d^2}{dr^2}(r\Phi) = 4\pi G\rho,$$

and, given ρ we can solve for $\Phi(r)$.

Deriving potentials of spherical systems

Homogeneous Sphere

(**a**) Homogeneous sphere: $\rho(r) = \rho_0$ for $0 < r < r_0$, and $\rho(r) = 0$ for $r > r_0$.
So for $r < r_0$, have

$$\begin{aligned}
\frac{1}{r}\frac{d^2}{dr^2}(r\phi) &= 4\pi G\rho_0 \\
\frac{d^2}{dr^2}(r\phi) &= 4\pi G\rho_0 r \\
\frac{d}{dr}(r\phi) &= 2\pi G\rho_0 r^2 + A \\
r\Phi &= \frac{2}{3}\pi G\rho_0 r^3 + Ar + B \\
\Phi(r) &= \frac{2}{3}\pi G\rho_0 r^2 + A + \frac{B}{r} \qquad (2.5)
\end{aligned}$$

Deriving potentials of spherical systems

Homogeneous Sphere

$$\Phi(r) = \frac{2}{3}\pi G\rho_0 r^2 + A + \frac{B}{r}$$

Require that Φ is finite at $r = 0$, else there is a point mass there, and so $B = 0$.
$\Rightarrow \Phi(r) = \frac{2}{3}\pi G\rho_0 r^2 + A$ for $0 < r < r_0$.

For $r > r_0$ have

$$\frac{1}{r}\frac{d^2}{dr^2}(r\Phi) = 0$$
$$\Rightarrow r\Phi = Cr + D$$
$$\Phi(r) = C + \frac{D}{r}$$

WLOG [1] let $\Phi \to 0$ as $r \to \infty$ (this is just choosing the zero point of the potential).

$$\Rightarrow \Phi(r) = \frac{D}{r} \quad \text{for } r_0 < r$$

[1] WLOG=Without Loss Of Generality

Deriving potentials of spherical systems

Homogeneous Sphere

$\Phi(r) = \frac{2}{3}\pi G\rho_0 r^2 + A$ for $0 < r < r_0$

$\Phi(r) = \frac{D}{r}$ for $r_0 < r$

Also require Φ to be continuous at $r = r_0$, since $\nabla\Phi$=force is finite there, and $\frac{d\Phi}{dr}$ also continuous (else $\nabla^2\Phi = 4\pi G\rho$ is infinite there).

$$\Rightarrow \frac{2}{3}\pi G\rho_0 r_0^2 + A = \frac{D}{r_0}$$

and

$$\frac{4}{3}\pi G\rho_0 r_0 = -\frac{D}{r_0^2}$$

$\Rightarrow$

$$D = -\frac{4}{3}\pi G\rho_0 r_0^3$$

and

$$A = -2\pi G\rho_0 r_0^2$$

Deriving potentials of spherical systems

Homogeneous Sphere

Hence

Potential of a homogeneous sphere

$$\begin{aligned} \Phi(r) &= \frac{2}{3}\pi G\rho_0(r^2 - 3r_0^2) \quad 0 < r < r_0 \\ &= -\frac{4}{3}\pi G\rho_0 r_0^3/r \quad r_0 < r \end{aligned} \tag{2.6}$$

Note: Outside the sphere $\Phi = -\frac{GM}{r}$ as expected, where $M = \frac{4}{3}\pi\rho_0 r_0^3$.

Newton's 2nd theorem: "Outside a closed spherical shell of matter, the gravitational potential is as if all the mass were at a point at the centre"

Deriving potentials of spherical systems

Spherical Shell

(**b**) Spherical shell $\rho(r) = \rho_0$ for $r_1 < r < r_2$ and $\rho(r) = 0$ otherwise.
Newtonian gravity is linear, so this is the same as
(1) a uniform sphere density ρ_0, radius r_2

PLUS

(2) a uniform sphere density $-\rho_0$, radius r_1.
So we can write the answer down. It is

$$\begin{aligned}\Phi(r) &= \frac{2}{3}\pi G\rho_0(r^2 - 3r_2^2) - \frac{2}{3}\pi G\rho_0(r^2 - 3r_1^2) \quad 0 < r < r_1 \\ &= \frac{2}{3}\pi G\rho_0(r^2 - 3r_2^2) + \frac{4}{3}\pi G\rho_0 r_1^3/r \quad r_1 < r < r_2 \\ &= -\frac{4}{3}\pi G\rho_0 r_2^3/r + \frac{4}{3}\pi G\rho_0 r_1^3/r \quad r_2 < r \qquad (2.7)\end{aligned}$$

Deriving potentials of spherical systems

Spherical Shell

Notes:

(1) Inside the cavity $0 < r < r_1$:

$\Phi(r) = \frac{2}{3}\pi G\rho_0(r^2 - 3r_2^2) - \frac{2}{3}\pi G\rho_0(r^2 - 3r_1^2)$

Φ =constant since the r^2 terms cancel. Therefore there is no force due to an external spherically symmetric mass distribution

Newton's first theorem

(2) Outside the shell $r > r_2$: $\Phi(r) = -\frac{4}{3}\pi G\rho_0 r_2^3/r + \frac{4}{3}\pi G\rho_0 r_1^3/r$

$$\Phi = -\frac{GM_{\rm shell}}{r}$$

where $M_{\rm shell} = \frac{4}{3}\pi\rho_0(r_2^3 - r_1^3)$ is the mass in the shell

Deriving potentials of spherical systems

Spherical Shell

Notes:

(1) Inside the cavity $0 < r < r_1$:

$\Phi(r) = \frac{2}{3}\pi G\rho_0(r^2 - 3r_2^2) - \frac{2}{3}\pi G\rho_0(r^2 - 3r_1^2)$

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where $M_{\rm shell} = \frac{4}{3}\pi\rho_0(r_2^3 - r_1^3)$ is the mass in the shell

Deriving potentials of spherical systems

Shells Galore

Since Newtonian gravitational potentials add linearly, we can calculate the potential at r due to an arbitrary spherically symmetric $\rho(r)$ by adding contributions from shells inside and outside r.
Mass in shell of thickness dr' and radius r' is

$$4\pi {r'}^2 \rho(r') dr'$$

The potential inside a shell is constant, so we can evaluate it anywhere - easiest is just inside the shell, where

$$\Phi = -\frac{4\pi G {r'}^2 \rho(r') dr'}{r'}$$

(from $-GM/r$).

Deriving potentials of spherical systems

Shells Galore

Thus, at any r, we have:

$$\Phi(r) = -\frac{4\pi G}{r}\int_0^r {r'}^2\rho(r')dr' - 4\pi G\int_r^\infty r'\rho(r')dr'$$

where the first term is from shells inside r, and the second from shells outside r (to get $\Phi(\infty) = 0$).

Potential of an arbitrary spherical distribution

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r {r'}^2\rho(r')dr' + \int_r^\infty r'\rho(r')dr'\right] \qquad (2.8)$$

Profiles and potentials
Modified Hubble profile

If a galaxy has a spherical luminosity density

$$j(r) = j_0 \left(1 + \left(\frac{r}{a}\right)^2\right)^{-\frac{3}{2}} \tag{2.9}$$

then the surface brightness distribution is the projection of this on the plane of the sky

$$I(R) = 2\int_0^\infty j(z)dz \tag{2.10}$$

Now $r^2 = R^2 + z^2$, so

$$I(R) = 2j_0 \int_0^\infty \left[1 + \left(\frac{R}{a}\right)^2 + \left(\frac{z}{a}\right)^2\right]^{-\frac{3}{2}} \mathrm{d}z \tag{2.11}$$

Profiles and potentials

Modified Hubble profile

Let $y = z/\sqrt{a^2 + R^2}$, and then

$$1 + \left(\frac{R}{a}\right)^2 + \left(\frac{z}{a}\right)^2 = \frac{1}{a^2}\left(a^2 + R^2 + z^2\right) = \frac{\left(a^2 + R^2\right)}{a^2}\left(1 + y^2\right) \tag{2.12}$$

$$\Rightarrow I(R) = 2j_0 \left(\frac{a^2}{a^2 + R^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{\sqrt{a^2 + R^2}\, dy}{(1 + y^2)^{\frac{3}{2}}} \tag{2.13}$$

$$= 2j_0 \frac{a^3}{a^2 + R^2} \int_0^\infty \frac{dy}{(1 + y^2)^{\frac{3}{2}}} \tag{2.14}$$

Can be evaluated by setting $y = \tan x$, so $dy = \sec^2 x \; dx$, and the integral becomes

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \; dx}{(\sec^2 x)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{2}} \cos x \; dx = \sin\frac{\pi}{2} - \sin 0 = 1 \tag{2.15}$$

Profiles and potentials

Modified Hubble profile

and hence

$$I(R) = 2\int_0^\infty j(z)dz = 2j_0 \int_0^\infty \left[1 + \left(\frac{R}{a}\right)^2 + \left(\frac{z}{a}\right)^2\right]^{-\frac{3}{2}} \mathrm{d}z$$
$$= 2j_0 \frac{a^3}{a^2 + R^2} \int_0^\infty \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{2j_0 a}{1 + \left(\frac{R^2}{a^2}\right)} \quad (2.16)$$

This profile is quite a good fit to elliptical galaxies - it is similar to the Hubble profile.

Now ask: assuming a fixed mass-to-light ratio Υ, what is the potential?

Assume

$$\rho(r) = \frac{\rho_0}{\left[1 + \left(\frac{r}{a}\right)^2\right]^{\frac{3}{2}}} \quad (2.17)$$

where $\rho_0 = \Upsilon j_0$.

Profiles and potentials

Modified Hubble profile

Let's use Poisson's equation $\nabla^2\Phi = 4\pi G\rho \Rightarrow \frac{d^2}{dr^2} r\Phi = 4\pi G r\rho$

$$\frac{1}{4\pi G}\frac{d^2}{dr^2}(r\Phi) = \frac{\rho_0 r}{\left(1+\frac{r^2}{a^2}\right)^{\frac{3}{2}}}$$

$$\frac{1}{4\pi G}\frac{d}{dr}(r\Phi) = \rho_0 \int \frac{r\ dr}{\left(1+\frac{r^2}{a^2}\right)^{\frac{3}{2}}}$$

$$= \frac{\rho_0 a^2}{2}\int \frac{2r\ dr/a^2}{(1+r^2/a^2)^{\frac{3}{2}}} \qquad (2.18)$$

Let $u = 1 + r^2/a^2$, then $du = \frac{2r}{a^2} dr$

Profiles and potentials
Modified Hubble profile

$u = 1 + r^2/a^2$

$\rho(r) = \frac{\rho_0}{\left[1+\left(\frac{r}{a}\right)^2\right]^{\frac{3}{2}}}$

$\frac{d^2}{dr^2} r\Phi = 4\pi G r\rho$

And so

$$\frac{1}{4\pi G}\frac{d}{dr}(r\Phi) = \frac{\rho_0 a^2}{2}\int \frac{du}{u^{\frac{3}{2}}}$$

$$= -2\frac{\rho_0 a^2}{2}\left(1+\frac{r^2}{a^2}\right)^{-\frac{1}{2}} + A \qquad (2.19)$$

Then

$$\frac{r\Phi}{4\pi G} = Ar - \rho_0 a^3 \int \frac{dr}{\sqrt{a^2+r^2}} \qquad (2.20)$$

Then we have the fairly standard integral

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln(2\sqrt{a^2+x^2}+2x) \text{ or } \sinh^{-1}\left(\frac{x}{a}\right) \qquad (2.21)$$

Profiles and potentials

Modified Hubble profile

So

$$\frac{r\Phi}{4\pi G} = Ar + B - \rho_0 a^3 \ln(2\sqrt{a^2 + r^2} + 2r) \qquad (2.22)$$

$B = \rho_0 a^3 \ln(2a)$ as otherwise $1/r \to \infty$ as $r \to 0$ [i.e. no point mass at origin].

$$\Phi = 4\pi GA - 4\pi G\rho_0 a^3 \frac{\ln(2\sqrt{a^2 + r^2} + 2r) - \ln(2a)}{r} \qquad (2.23)$$

Note that we can choose $A = 0$, and then $\Phi \to 0$ as $r \to \infty$ (but more slowly than $\frac{1}{r}$ due to infinite total mass).
The total mass within r is

$$M(r) = \int_0^r \frac{4\pi\rho_0 r^2 dr}{\left(1 + \frac{r^2}{a^2}\right)^{\frac{3}{2}}} \qquad (2.24)$$

This is $\propto \ln r$ for large r, so diverges as $r \to \infty$.

Profiles and potentials

Power law density profile

$$\rho(r) = \rho_0 \left(\frac{a}{r}\right)^{\alpha} \tag{2.25}$$

$$\frac{d^2}{dr^2}(r\Phi) = 4\pi G \rho_0 a^{\alpha} r^{1-\alpha} \tag{2.26}$$

so

$$\frac{d}{dr}(r\Phi) = 4\pi G \rho_0 a^{\alpha} \frac{r^{2-\alpha}}{2-\alpha} + A \tag{2.27}$$

Profiles and potentials

Power law density profile

$$r\Phi = 4\pi G\rho_0 a^\alpha \frac{r^{3-\alpha}}{(2-\alpha)(3-\alpha)} + Ar + B \tag{2.28}$$

or

$$\Phi = -\frac{4\pi G\rho_0 a^\alpha r^{2-\alpha}}{(3-\alpha)(\alpha-2)} + A + \frac{B}{r} \tag{2.29}$$

$A = 0$ by setting zero, and $B = 0$ because no point mass at centre as usual.

Profiles and potentials

Power law density profile

$$\Phi = -\frac{4\pi G\rho_0 a^\alpha r^{2-\alpha}}{(3-\alpha)(\alpha-2)}$$

$$\rho(r) = \rho_0 \left(\frac{a}{r}\right)^\alpha$$

Notes:
(1) $\alpha < 3$ to get $M(r)$ finite at the origin (determine $\int 4\pi G\rho r^2 dr$ near origin).
(2) $\Phi \to 0$ at ∞ if $\alpha > 2$,

$$\Rightarrow 2 < \alpha < 3$$

$\alpha = 2$ gives spiral rotation curves (flat), from
$v_c^2/r = \frac{d\Phi}{dr}$ $(= -f_r) \Rightarrow v_c^2 \propto r^{2-\alpha}$.
[Circular motion $\Rightarrow \ddot{r}$ & $\dot{r} = 0$, so $\ddot{r} - r\dot{\phi}^2 = -\frac{d\Phi}{dr}$ becomes, with $v_c = r\dot{\phi}$, $\frac{v_c^2}{r} = -\frac{d\Phi}{dr}$. Then substituting Φ from equation (2.29) gives $v_c^2 \propto r^{2-\alpha}$.]
$\alpha = 3$ gives elliptical galaxy profiles (mod. Hubble profile)
but all these models have infinite mass, since $M(r)$ diverges at large r

Projected density → spherical density

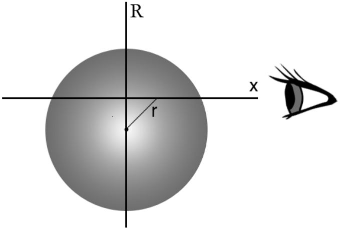
What we have done so far is to guess a luminosity density $j(r)$ (which we assume is proportional to the matter density $\rho(r)$) and formed the projected surface brightness $I(R)$ using the relation

$$I(R) = 2\int_R^\infty \frac{j(r)rdr}{\sqrt{r^2 - R^2}} \tag{2.30}$$

and then check that $I(R)$ is a reasonable approximation to what is seen for our guessed density distribution.

$$r^2 = x^2 + R^2$$

$$dx = \frac{rdr}{\sqrt{r^2 - R^2}}$$



Projected density → spherical density

OK, so

$$I(R) = 2\int_R^\infty \frac{j(r)rdr}{\sqrt{r^2 - R^2}}$$

In fact, if $I(R)$ is known, then the equation above may be inverted to yield $j(r)$ directly, to yield

$$j(r) = -\frac{1}{2\pi r}\frac{d}{dr}\int_r^\infty \frac{I(R)RdR}{\sqrt{R^2 - r^2}}. \qquad (2.31)$$

This is not quite pulled out of the air - it is a form of Abel's integral equation.

Projected density $\rightarrow$ spherical density

We can simplify the form a bit if we set $t = R^2$ and $x = r^2$, and then we have

$$I(t) = \int_t^\infty \frac{j(x)dx}{(x-t)^{\frac{1}{2}}}$$

and then the inverse relation quoted becomes

$$j(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^\infty \frac{I(t)dt}{(t-y)^{\frac{1}{2}}}$$

If we look just at the RHS, and call it $h(y)$ for the moment, this is

$$h(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^\infty \frac{dt}{(t-y)^{\frac{1}{2}}}\int_t^\infty \frac{j(x)dx}{(x-t)^{\frac{1}{2}}}.$$

or

$$h(y) = -\frac{1}{\pi}\frac{d}{dy}\int_{t=y}^\infty \int_{x=t}^\infty \frac{dtj(x)dx}{(t-y)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}}$$

Projected density → spherical density

We now switch the order of the integration, remembering when doing so to change the limits of the integration so that we are integrating over the same area in the (x, t)-plane.

$$h(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^{\infty} j(x)dx \int_y^x \frac{dt}{(t-y)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}}$$

The integral

$$\int_y^x \frac{dt}{(t-y)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}} = \pi$$

and so what we called $h(y)$ is then seen to be equal to $j(y)$. So the result follows.

Projected density → spherical density

[The statement that

$$S \equiv \int_y^x \frac{dt}{(t-y)^{\frac{1}{2}}(x-t)^{\frac{1}{2}}} = \pi$$

needs a bit more justification, or you can take it on trust.... For those who don't, we first change variables so the lower limit is zero, so $z = t - y$, and then

$$S = \int_0^{x-y} \frac{dz}{(x-y-z)^{\frac{1}{2}} z^{\frac{1}{2}}}$$

This invites yet another change of variables so that the upper limit is 1, i.e. $\zeta = \frac{z}{x-y} \Rightarrow z = (x-y)\zeta \Rightarrow x - y - z = (x-y)(1-\zeta) \Rightarrow$

$$S = \int_0^1 \frac{(x-y)d\zeta}{(x-y)^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}}(x-y)^{\frac{1}{2}}\zeta^{\frac{1}{2}}}$$

Projected density → spherical density

So

$$
\begin{aligned}
S &= \int_0^1 \frac{d\zeta}{(1-\zeta)^{\frac{1}{2}}\zeta^{\frac{1}{2}}} \\
&= \int_0^1 \frac{d\zeta}{(\zeta-\zeta^2)^{\frac{1}{2}}} \\
&= \int_0^1 \frac{d\zeta}{\left(\frac{1}{4}-(\zeta-\frac{1}{2})^2\right)^{\frac{1}{2}}} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{\sqrt{\frac{1}{4}-u^2}} \\
&= \int_{-1}^{1} \frac{\frac{1}{2}dv}{\sqrt{\frac{1}{4}-\frac{v^2}{4}}} \\
&= \int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}}
\end{aligned}
\tag{2.32}
$$

Projected density → spherical density

Then since we know

$$\frac{d}{d\xi}\mathrm{arcsin}\xi = \frac{1}{\sqrt{1-\xi^2}}$$

we have

$$\int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}} = \mathrm{arcsin} v|_{-1}^{1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

]