Stellar Dynamics and Structure of Galaxies

Orbits in a given potential

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Michaelmas Term 2018

* based on slides prepared by Vasily Belokurov and lecture notes by Jim Pringle

Outline I

Collisions

Do we have to worry about collisions?

Globular clusters look densest, so obtain a rough estimate of collision timescale for them

Collisions in globular clusters

The case of NGC 2808

$\rho_0 \sim 8 \times 10^4$ M$_\odot$ pc^{-3}
$M_* \sim 0.8$ M$_\odot$.
$\Rightarrow n_0 \sim 10^5$ pc^{-3} is the star number density.
We have $\sigma_r \sim 13$ km s^{-1} as the typical 1D speed of a star, so the 3D speed is $\sim \sqrt{3} \times \sigma_r$ $(= \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}) \sim 20$ km s^{-1}.
Since $M_* \propto R_*$ (see Fluids, or Stars, course notes), have $R_* \sim 0.8 R_\odot$.

Collisions in globular clusters

The case of NGC 2808

For a collision, need the volume $\pi(2R_*)^2\sigma t_{\rm coll}$ to contain one star, i.e.

$$n_0 = 1/\left(\pi(2R_*)^2\sigma t_{\rm coll}\right) \tag{1.1}$$

or

$$t_{\rm coll} = 1/\left(4\pi R_*^2\sigma n_0\right) \tag{1.2}$$

$R_* = 0.8R_\odot$, $n_0 = 10^5 pc^{-3}$, $\sigma = 20km/s$

Putting in the numbers gives $t_{\rm coll} \sim 10^{14}$ yr.
So direct collisions between stars are rare, but if you have $\sim 10^6$ stars then there is a collision every $\sim 10^8$ years, so they do happen.

Note that NGC 2808 is 10 times denser than typical

So, for now, ignore collisions, and we are left with stars orbiting in the potential from all the other stars in the system.

Model requirements

Model (e.g., a globular cluster) just as a self-gravitating collection of objects.
Have a gravitational potential well $\Phi(\mathbf{r})$, approximately smooth if the number of particles $>> 1$. Conventionally take $\Phi(\infty) = 0$.
Stars orbit in the potential well, with time per orbit (for a globular cluster) $\sim 2R_h/\sigma \sim 10^6$ years $<<$ age.
Stars give rise to $\Phi(\mathbf{r})$ by their mass, so for this potential in a steady state could average each star over its orbit to get $\rho(\mathbf{r})$.
The key problem is therefore self-consistently building a model which fills in the terms:

$$\Phi(\mathbf{r}) \rightarrow \text{ stellar orbits} \rightarrow \rho(\mathbf{r}) \rightarrow \Phi(\mathbf{r}) \tag{1.3}$$

Note that in most observed cases we only have $v_{\text{line of sight}}(R)$, so it is even harder to model real systems.

Self-consistent = orbits & stellar mass give ρ, which leads to Φ, which supports the orbits used to construct ρ

The law of attraction

Newton's laws of motion and Newtonian gravity

GR not needed, since

- $10\ \frac{\mathrm{km}}{\mathrm{s}} < \bar{v} < 10^3\ \frac{\mathrm{km}}{\mathrm{s}}$ is $<< c = 3 \times 10^5\ \frac{\mathrm{km}}{\mathrm{s}}$
- $\frac{GM}{r\,c^2} \ll 1$

The gravitational force per unit mass acting on a body due to a mass M at the origin is

$$\mathbf{f} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} \tag{1.4}$$

We can write this in terms of a potential Φ, using

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{\mathbf{r}} \tag{1.5}$$

The corresponding potential

So

$$\mathbf{f} = -\nabla\Phi \tag{1.6}$$

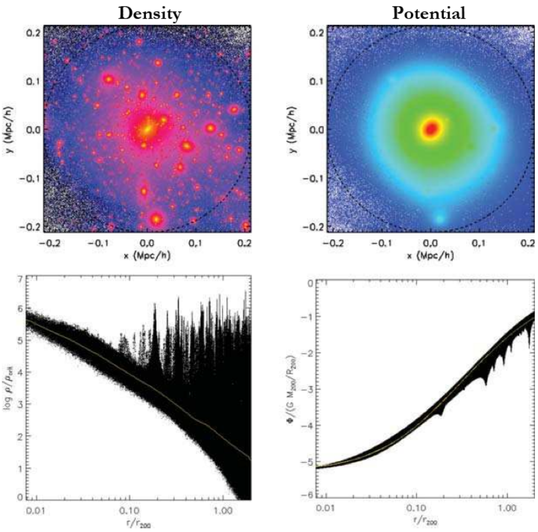
where Φ is a scalar,

$$\Phi = \Phi(r) = -\frac{GM}{r} \tag{1.7}$$

Hence the potential due to a point mass M at $\mathbf{r} = \mathbf{r}_1$ is

$$\Phi(\mathbf{r}) = -\frac{GM}{|\mathbf{r} - \mathbf{r}_1|} \tag{1.8}$$

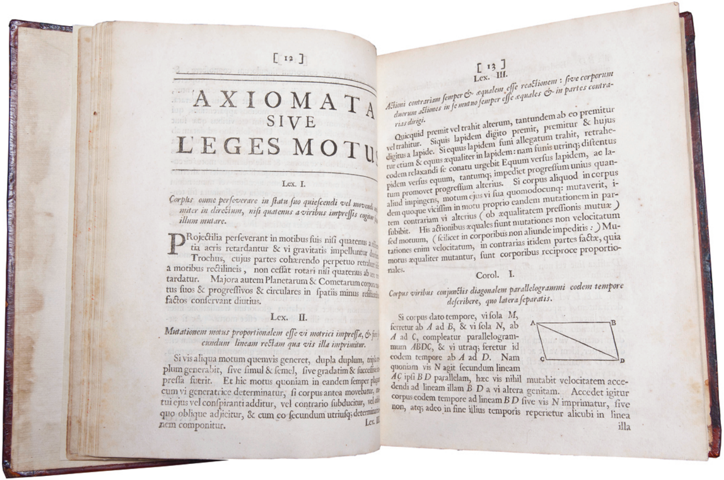
Density vs Potential



From Hayashi et al, "The shape of the gravitational potential in cold dark matter haloes"

Orbits

The law of motion



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AXIOMATA SIVE LEGES MOTU

Lex. I.

Corpus omne perfeverare in ftatu fuo quiefcendi vel movendi uniformiter in directum, nifi quatenus a viribus impreffis cogitur ftatum illum mutare.

Projectilia perfeverant in motibus fuis nifi quatenus a refiftentia aeris retardantur & vi gravitatis impelluntur deorfum. Trochus, cujus partes cohærendo perpetuo retrahunt fefe a motibus rectilineis, non ceffat rotari nifi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus fuos & progreffivos & circulares in fpatiis minus refiftentibus factos confervant diutius.

Lex. II.

Mutationem motus proportionalem effe vi motrici impreffæ, & fieri fecundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, five fimul & femel, five gradatim & fucceffive impreffa fuerit. Et hic motus quoniam in eandem femper plagam cum vi generatrice determinatur, fi corpus antea movebatur, motui ejus vel confpiranti additur, vel contrario fubducitur, vel obliquo oblique adjicitur, & cum eo fecundum utriufq; determinationem componitur.

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Lex. III.

Actioni contrariam femper & æqualem effe reactionem: five corporum duorum actiones in fe mutuo femper effe æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus æqualiter in lapidem: nam funis utrinq; diftentus eodem relaxandi fe conatu urgebit Equum verfus lapidem, ac lapidem verfus equum, tantumq; impediet progreffum unius quantum promovet progreffum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi fua quomodocunq; mutaverit, idem quoque viciffim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem preffionis mutuæ) fubibit. His actionibus æquales fiunt mutationes non velocitatum fed motuum, (fcilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, funt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore defcribere, quo latera feparatis.

Si corpus dato tempore, vi fola M, ferretur ab A ad B, & vi fola N, ab A ad C, compleatur parallelogrammum ABDC, & vi utraq; feretur id eodem tempore ab A ad D. Nam quoniam vis N agit fecundum lineam AC ipfi BD parallelam, hæc vis nihil mutabit velocitatem accedendi ad lineam illam BD a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam BD five vis N imprimatur, five non, atq; adeo in fine illius temporis reperietur alicubi in linea illa

Orbits

Particle of constant mass m at position $\mathbf{r}$ subject to a force $\mathbf{F}$. Newton's law:

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{F} \tag{1.9}$$

i.e.

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{1.10}$$

If $\mathbf{F}$ is due to a gravitational potential $\Phi(\mathbf{r})$, then

$$\mathbf{F} = m\mathbf{f} = -m\nabla\Phi \tag{1.11}$$

The angular momentum about the origin is $\mathbf{H} = \mathbf{r} \times (m\dot{\mathbf{r}})$.Then

$$\begin{aligned} \frac{d\mathbf{H}}{dt} &= \mathbf{r} \times (m\ddot{\mathbf{r}}) + m\dot{\mathbf{r}} \times \dot{\mathbf{r}} \\ &= \mathbf{r} \times \mathbf{F} \\ &\equiv \mathbf{G} \end{aligned} \tag{1.12}$$

where $\mathbf{G}$ is the torque about the origin.

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Energy

The kinetic energy

$$T = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \tag{1.13}$$

$$\frac{dT}{dt} = m \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}} \tag{1.14}$$

If $\mathbf{F} = -m\nabla\Phi$, then

$$\frac{dT}{dt} = -m\dot{\mathbf{r}} \cdot \nabla\Phi(\mathbf{r}) \tag{1.15}$$

But if Φ is independent of t, the rate of change of Φ along an orbit is

$$\frac{d}{dt}\Phi(\mathbf{r}) = \nabla\Phi \cdot \dot{\mathbf{r}} \tag{1.16}$$

from the chain rule

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from the chain rule

Energy

Hence

$$\frac{dT}{dt} = -m\frac{d}{dt}\Phi(\mathbf{r}) \tag{1.17}$$

$$\Rightarrow m\frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \Phi(\mathbf{r})\right) = 0 \tag{1.18}$$

$$\Rightarrow E = \frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \Phi(\mathbf{r}) \tag{1.19}$$

The total energy is constant for a given orbit

Energy

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The total energy is constant for a given orbit

Orbits in spherical potentials

$\Phi(\mathbf{r}) = \Phi(|\mathbf{r}|) = \Phi(r)$, so $\mathbf{f} = -\nabla\Phi = -\hat{\mathbf{r}}\frac{d\Phi}{dr}$.
The orbital angular momentum $\mathbf{H} = m\mathbf{r} \times \dot{\mathbf{r}}$, and

$$\frac{d\mathbf{H}}{dt} = \mathbf{r} \times m\mathbf{f} = -m\frac{d\Phi}{dr}\mathbf{r} \times \hat{\mathbf{r}} = 0. \tag{1.20}$$

So the angular momentum per unit mass $\mathbf{h} = \mathbf{H}/m = \mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector, and is perpendicular to $\mathbf{r}$ and $\dot{\mathbf{r}}$

$\Rightarrow$ the particle stays in a plane through the origin

which is perpendicular to $\mathbf{h}$

Check: $\mathbf{r} \perp \mathbf{h}$, $\mathbf{r} + \delta\mathbf{r} = \mathbf{r} + \dot{\mathbf{r}}\delta t \perp \mathbf{h}$ since both $\mathbf{r}$ and $\dot{\mathbf{r}} \perp \mathbf{h}$, so particle remains in the plane

Thus the problem becomes a two-dimensional one to calculate the orbit use 2-D cylindrical coordinates (R, ϕ, z) at $z = 0$, or spherical polars (r, θ, ϕ) with $\theta = \frac{\pi}{2}$.
So, in 2D, use (R, ϕ) and (r, ϕ) interchangeably.

Equation of motion in two dimensions

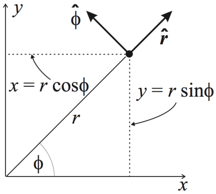
The equation of motion in two dimensions can be written in radial and angular terms, using $\mathbf{r} = r\hat{\mathbf{r}} = r\hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\phi$, so $\mathbf{r} = (r, 0)$.
We know that

$$\frac{d}{dt}\hat{\mathbf{e}}_r = \dot{\phi}\hat{\mathbf{e}}_\phi \tag{1.21}$$

and

$$\frac{d}{dt}\hat{\mathbf{e}}_\phi = -\dot{\phi}\hat{\mathbf{e}}_r \tag{1.22}$$

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \cos(\phi)\hat{\mathbf{e}}_x + \sin(\phi)\hat{\mathbf{e}}_y \\
\hat{\mathbf{e}}_\phi &= -\sin(\phi)\hat{\mathbf{e}}_x + \cos(\phi)\hat{\mathbf{e}}_y \\
\frac{d}{dt}\hat{\mathbf{e}}_r &= -\sin(\phi)\dot{\phi}\hat{\mathbf{e}}_x + \cos(\phi)\dot{\phi}\hat{\mathbf{e}}_y \\
\frac{d}{dt}\hat{\mathbf{e}}_\phi &= -\cos(\phi)\dot{\phi}\hat{\mathbf{e}}_x - \sin(\phi)\dot{\phi}\hat{\mathbf{e}}_y
\end{aligned}$$



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\tfrac{d}{dt}\hat{\mathbf{e}}_r &= -\sin(\phi)\dot{\phi}\hat{\mathbf{e}}_x + \cos(\phi)\dot{\phi}\hat{\mathbf{e}}_y \\
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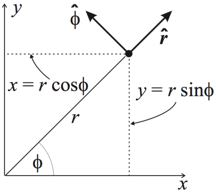

Equation of motion in two dimensions

Hence

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\phi}\hat{\mathbf{e}}_\phi \tag{1.23}$$

or $\dot{\mathbf{r}} = \mathbf{v} = (\dot{r}, r\dot{\phi})$

and so

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\dot{\phi}\hat{\mathbf{e}}_\phi + \dot{r}\dot{\phi}\hat{\mathbf{e}}_\phi + r\ddot{\phi}\hat{\mathbf{e}}_\phi - r\dot{\phi}^2\hat{\mathbf{e}}_r \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{e}}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\mathbf{e}}_\phi \\ &= \mathbf{a} = [\ddot{r} - r\dot{\phi}^2, \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)] \end{aligned} \tag{1.24}$$

In general $\mathbf{f} = (f_r, f_\phi)$, and then $f_r = \ddot{r} - r\dot{\phi}^2$, where the second term is the centrifugal force, since we are in a rotating frame, and the torque $rf_\phi = \frac{d}{dt}\left(r^2\dot{\phi}\right)$ $(= \mathbf{r} \times \mathbf{f})$.
In a spherical potential $f_\phi = 0$, so $r^2\dot{\phi}$ is constant.

Path of the orbit

To determine the shape of the orbit we need to remove t from the equations and find $r(\phi)$. It is simplest to set $u = 1/r$, and then from $r^2\dot{\phi} = h$ obtain

$$\dot{\phi} = hu^2 \tag{1.25}$$

Then

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\phi}\dot{\phi} = -h\frac{du}{d\phi} \tag{1.26}$$

and

$$\ddot{r} = -h\frac{d^2u}{d\phi^2}\dot{\phi} = -h^2u^2\frac{d^2u}{d\phi^2}. \tag{1.27}$$

Path of the orbit

So the radial equation of motion

$$\ddot{r} - r\dot{\phi}^2 = f_r$$

becomes

$$-h^2 u^2 \frac{d^2 u}{d\phi^2} - \frac{1}{u} h^2 u^4 = f_r \tag{1.28}$$

$$\Rightarrow \frac{d^2 u}{d\phi^2} + u = -\frac{f_r}{h^2 u^2} \tag{1.29}$$

The orbit equation in spherical potential

Path of the orbit

Since f_r is just a function of r (or u) this is an equation for $u(\phi)$, i.e. $r(\phi)$ - the path of the orbit. Note that it does not give $r(t)$, or $\phi(t)$ - you need one of the other equations for those.
If we take $f_r = -\frac{GM}{r^2} = -GMu^2$, then

$$\frac{d^2u}{d\phi^2} + u = GM/h^2 \tag{1.30}$$

(which is something you will have seen in the Relativity course).

Kepler orbits

Solution to the equation of motion

The solution to this equation is

$$\frac{\ell}{r} = \ell u = 1 + e\cos(\phi - \phi_0) \tag{1.31}$$

which you can verify simply by putting it in the differential equation. Then

$$-\frac{e\cos(\phi-\phi_0)}{\ell} + \frac{1 + e\cos(\phi-\phi_0)}{\ell} = \frac{GM}{h^2}$$

so $\ell = h^2/GM$ and e and ϕ_0 are constants of integration.

Kepler orbits
Bound orbits

$$\frac{1}{r} = \frac{1 + e\cos(\phi - \phi_0)}{\ell}$$

Note that if $e < 1$ then $1/r$ is never zero, so r is bounded in the range $\frac{\ell}{1+e} < r < \frac{\ell}{1-e}$. Also, in all cases the orbit is symmetric about $\phi = \phi_0$, so we take $\phi_0 = 0$ as defining the reference line for the angle ϕ. ℓ is the distance from the origin for $\phi = \pm\frac{\pi}{2}$ (with ϕ measured relative to ϕ_0).

Kepler orbits
Bound orbits

We can use different parameters. Knowing that the point of closest approach (perihelion for a planet in orbit around the Sun, periastron for something about a star) is at $\ell/(1+e)$ when $\phi = 0$ and the aphelion (or whatever) is at $\ell/(1-e)$ when $\phi = \pi$, we can set the distance between these two points ($=$ major axis of the orbit)$=2a$. Then

$$\frac{\ell}{1+e} + \frac{\ell}{1-e} = 2a \Rightarrow \ell(1-e) + \ell(1+e) = 2a(1-e^2) \qquad (1.32)$$

$$\Rightarrow \ell = a(1-e^2) \qquad (1.33)$$

$\Rightarrow r_P = a(1-e)$ is the perihelion distance from the gravitating mass at the origin, and $r_a = a(1+e)$ is the aphelion distance. The distance of the Sun from the midpoint is ae, and the angular momentum $h^2 = GM\ell = GMa(1-e^2)$.

Energy per unit mass

The energy per unit mass

$$E = \frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \Phi(\mathbf{r}) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r} \tag{1.34}$$

$r_p = a(1-e)$

This is constant along the orbit, so we can evaluate it anywhere convenient - e.g. at perihelion where $\dot{r} = 0$. Then $\dot{\phi} = \frac{h}{r_P^2}$ and so

$$\begin{aligned} E &= \frac{1}{2}\frac{GMa(1-e^2)}{a^2(1-e)^2} - \frac{GM}{a(1-e)} \\ &= \frac{GM}{a}\left[\frac{1}{2}\left(\frac{1+e}{1-e}\right) - \frac{1}{1-e}\right] \\ &= -\frac{GM}{2a} \end{aligned} \tag{1.35}$$

This is < 0 for a bound orbit, and depends only on the semi-major axis a (and not e).

Kepler's Laws

... deduced from observations, and explained by Newtonian theory of gravity.

Kepler's Laws

1 Orbits are ellipses with the Sun at a focus.

2 Planets sweep out equal areas in equal time

$$\delta A = \frac{1}{2} r^2 \delta\phi \quad [= \frac{1}{2} r (r \delta\phi)] \tag{1.36}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \frac{h}{2} = \text{constant} \tag{1.37}$$

$\Rightarrow$ Kepler's second law is a consequence of a central force, since this is why h is a constant.

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Kepler's Laws
3rd Law

3 $(\text{Period})^2 \propto (\text{size of orbit})^3$

In one period T, the area swept out is $A = \frac{1}{2}hT = \left(\int_0^T \frac{dA}{dt}dt\right)$

But $A = \text{area of ellipse} = \pi ab = \pi a^2\sqrt{1-e^2}$

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$$A = \int_0^{2\pi} d\phi \int_0^r rdr$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 d\phi$$

$\frac{\ell}{r} = \ell u = 1 + e\cos(\phi - \phi_0)$

$$= \frac{\ell^2}{2}\int_0^{2\pi} \frac{d\phi}{(1+e\cos\phi)^2}$$

Have

$$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi}{a^2-b^2}\frac{a}{\sqrt{a^2-b^2}}$$

Kepler's Laws
3rd Law

so

$$A = 2\frac{\ell^2}{2}\frac{\pi}{1-e^2}\frac{1}{\sqrt{1-e^2}}$$

Since $\ell = a(1-e^2)$ this implies

$$A = \pi a^2\sqrt{1-e^2}$$

def: $e = \sqrt{1 - \frac{b^2}{a^2}}$

and since $b = a\sqrt{1-e^2}$,

$$A = \pi ab$$

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Kepler's Laws
3rd Law

Therefore

$$
\begin{aligned}
T &= \frac{2A}{h} \\
&= \frac{2\pi a^2\sqrt{1-e^2}}{h} \\
&= \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{GMa(1-e^2)}} \\
\text{since} \quad h^2 &= GMa(1-e^2) \\
T &= 2\pi\sqrt{\frac{a^3}{GM}} \\
\Rightarrow T^2 &\propto a^3 \qquad (1.38)
\end{aligned}
$$

where in this case M is the mass of the Sun.
Note: Since $E = -\frac{GM}{2a}$, the period $T = \frac{2\pi GM}{(-2E)^{\frac{3}{2}}}$.

Unbound orbits

What happens to $\frac{\ell}{r} = 1 + e\cos\phi$ when $e \geq 1$?

- If $e > 1$ then $1 + e\cos\phi = 0$ has solutions ϕ_∞ where $r = \infty$ $\rightarrow \cos\phi_\infty = -1/e$
  Then $-\phi_\infty \leq \phi \leq \phi_\infty$, and, since $\cos\phi_\infty$ is negative, $\frac{\pi}{2} < \phi_\infty < \pi$. The orbit is a hyperbola.
- If $e = 1$ then the particle just gets to infinity at $\phi = \pm\pi$ - it is a parabola.

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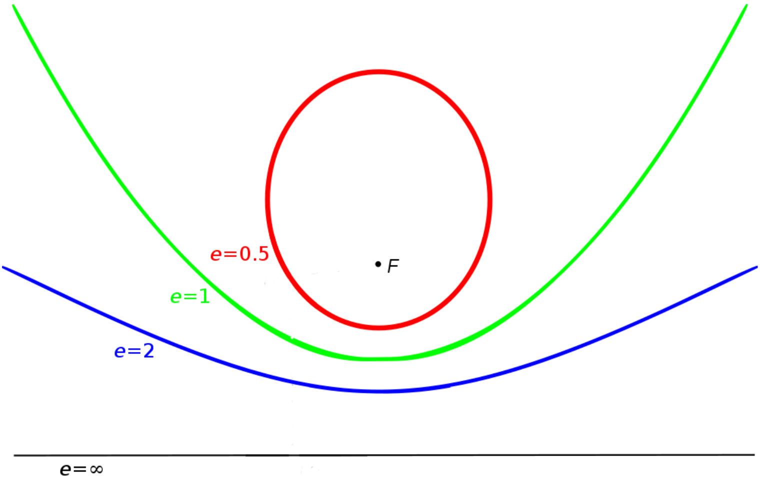
- If $e = 1$ then the particle just gets to infinity at $\phi = \pm\pi$ - it is a parabola.

Kepler orbits



Energies for these unbound orbits:

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{h^2}{r^2} - \frac{GM}{r}$$

$r^2\dot{\phi} = h$

So, as $r \to \infty$ $E \to \frac{1}{2}\dot{r}^2$

Unbound orbits

Recall

$$\frac{\ell}{r} = 1 + e\cos\phi$$

$\frac{d}{dt}$ of this $\Rightarrow$

$$-\frac{\ell}{r^2}\dot{r} = -e\sin\phi\ \dot{\phi}$$

and since $h = r^2\dot{\phi}$

$$\dot{r} = \frac{eh}{\ell}\sin\phi$$

As $r \to \infty$ $\cos\phi \to -1/e$

$$E \to \frac{1}{2}\dot{r}^2 = \frac{1}{2}\frac{e^2h^2}{\ell^2}\left(1 - \frac{1}{e^2}\right) = \frac{GM}{2\ell}(e^2 - 1)$$

(recalling that $h^2 = GM\ell$) Thus $E > 0$ if $e > 1$ and for parabolic orbits ($e = 1$) $E = 0$.

Escape velocity

We have seen that in a fixed potential $\Phi(\mathbf{r})$ a particle has constant energy $E = \frac{1}{2}\dot{\mathbf{r}}^2 + \Phi(\mathbf{r})$ along an orbit. If we adopt the usual convention and take $\Phi(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, then if at some point $\mathbf{r}_0$ the particle has velocity $\mathbf{v}_0$ such that

$$\frac{1}{2}\mathbf{v}_0^2 + \Phi(\mathbf{r}_0) > 0$$

then it is able to reach infinity. So at each point $\mathbf{r}_0$ we can define an escape velocity $v_{\rm esc}$ such that

$$v_{\rm esc} = \sqrt{-2\Phi(\mathbf{r}_0)}$$

Escape velocity

From the Solar neighborhood

The escape velocity from the Sun

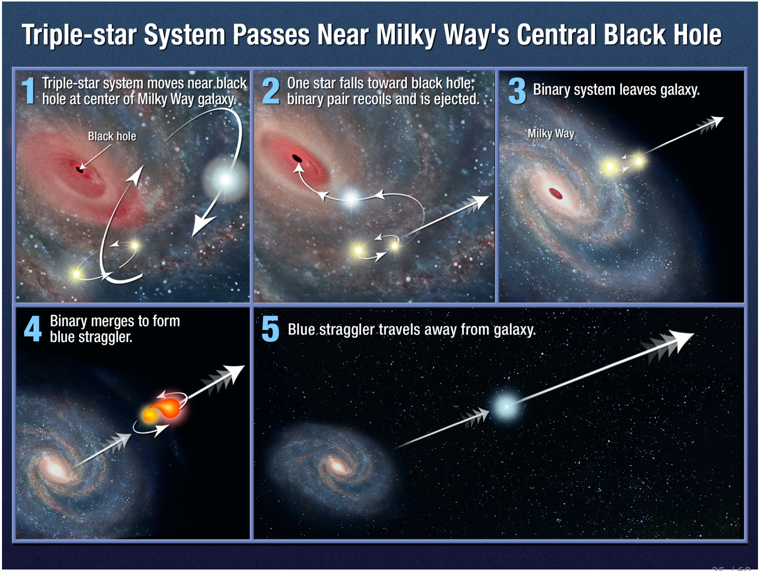
$$v_{\rm esc} = \left(\frac{2GM_\odot}{r_0}\right)^{\frac{1}{2}} = 42.2\left(\frac{r_0}{\rm a.u.}\right)^{-\frac{1}{2}}\ \rm km\ s^{-1}$$

Note: The circular velocity $v_{\rm circ}$ is such that $-r\dot{\phi}^2 = -\frac{GM}{r^2}$

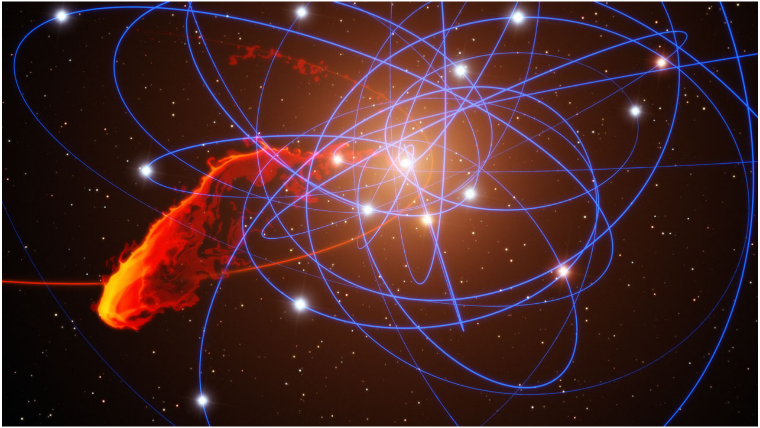
$$r\dot{\phi} = v_{\rm circ} = \sqrt{\frac{GM_\odot}{r_0}} = 29.8\left(\frac{r_0}{\rm a.u.}\right)^{-\frac{1}{2}}\ \rm km\ s^{-1}$$

($= 2\pi$ a.u./yr).
$v_{\rm esc} = \sqrt{2}v_{\rm circ}$ <u>for a point mass</u> source of the gravitational potential.

Escape velocity
From the Galaxy



Kepler orbits



Binary star orbits

- What we have done so far is assume a potential due to a fixed point mass which we take as being at the origin of our polar coordinates. We now wish to consider a situation in which we have two point masses, M_1 and M_2 both moving under the gravitational attraction of the other.
- This is a cluster of N stars where $N = 2$ and we can solve it exactly! Hooray!
- The potential is no longer fixed at origin

$$\Phi(\mathbf{r}) = -\frac{GM_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{GM_2}{|\mathbf{r} - \mathbf{r}_2|}$$

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Binary star orbits

Or the force acting on star 1, due to star 2 is

$$\mathbf{F_1} = \frac{GM_1 M_2}{|\mathbf{r_1} - \mathbf{r_2}|^2}$$

in the direction of $\mathbf{r_2} - \mathbf{r_1}$

$$\Rightarrow \mathbf{F_1} = \frac{GM_1 M_2}{|\mathbf{r_1} - \mathbf{r_2}|^3}(\mathbf{r_2} - \mathbf{r_1})$$

And by symmetry (or Newton's 3rd law)

$$\mathbf{F_2} = \frac{GM_1 M_2}{|\mathbf{r_1} - \mathbf{r_2}|^3}(\mathbf{r_1} - \mathbf{r_2})$$

Binary star orbits

Then we know

$$M_1\ddot{\mathbf{r}}_1 = -\frac{GM_1M_2}{d^2}\hat{\mathbf{d}} \tag{1.39}$$

and

$$M_2\ddot{\mathbf{r}}_2 = -\frac{GM_1M_2}{d^2}\left(-\hat{\mathbf{d}}\right) \tag{1.40}$$

where

$$\mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2 \tag{1.41}$$

is the vector from M_2 to M_1.
Using these two we can write for $\ddot{\mathbf{d}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2$

$$\ddot{\mathbf{d}} = -\frac{G(M_1 + M_2)}{d^2}\hat{\mathbf{d}} \tag{1.42}$$

Binary star orbits

$$\ddot{\mathbf{d}} = -\frac{G(M_1 + M_2)}{d^2}\hat{\mathbf{d}}$$

which is identical to the equation of motion of a particle subject to a fixed mass $M_1 + M_2$ at the origin.

So we know that the period

$$T = 2\pi\sqrt{\frac{a^3}{G(M_1 + M_2)}} \qquad (1.43)$$

where the size (maximum separation) of the relative orbit is $2a$.

Binary star orbits

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Binary star orbits

If we take the coordinates for the centre of mass

$$\mathbf{r}_{\rm CM} = \frac{M_1}{M_1 + M_2}\mathbf{r}_1 + \frac{M_2}{M_1 + M_2}\mathbf{r}_2 \qquad (1.44)$$

From equations (1.39) and (1.40) we know that

$$M_1\ddot{\mathbf{r}}_1 + M_2\ddot{\mathbf{r}}_2 = 0 \qquad (1.45)$$

and so

$$\frac{d}{dt}\left(M_1\dot{\mathbf{r}}_1 + M_2\dot{\mathbf{r}}_2\right) = 0 \qquad (1.46)$$

or

$$\left(M_1\dot{\mathbf{r}}_1 + M_2\dot{\mathbf{r}}_2\right) = \mathrm{constant} \qquad (1.47)$$

i.e.$\dot{\mathbf{r}}_{\rm CM}$ =constant.

We can choose an inertial frame in which the centre of mass has zero velocity

Binary star orbits

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Binary star orbits

Note that choosing $\mathbf{r}_{\mathrm{CM}} = 0 \Rightarrow M_1\mathbf{r}_1 = -M_2\mathbf{r}_2$, and so $\mathbf{r}_1 = \mathbf{d} + \mathbf{r}_2 = \mathbf{d} - \frac{M_1}{M_2}\mathbf{r}_1$

This $\Rightarrow \mathbf{r}_1 = \frac{M_2}{M_1+M_2}\mathbf{d}$ and similarly $\mathbf{r}_2 = -\frac{M_1}{M_1+M_2}\mathbf{d}$.

The angular momentum $\mathbf{J}$ (or $\mathbf{H}$ if you want) is

$$\begin{aligned} \mathbf{J} &= M_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + M_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 \\ &= \frac{M_1 M_2^2}{(M_1+M_2)^2}\mathbf{d} \times \dot{\mathbf{d}} + \frac{M_2 M_1^2}{(M_1+M_2)^2}\mathbf{d} \times \dot{\mathbf{d}} \\ &= \frac{M_1 M_2}{M_1+M_2}\mathbf{d} \times \dot{\mathbf{d}} \end{aligned} \tag{1.48}$$

So

$$\mathbf{J} = \mu\mathbf{h} \tag{1.49}$$

where μ is the reduced mass, and h is the specific angular momentum.

Binary star orbits

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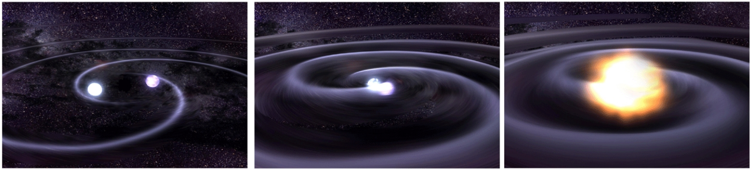
Binary star orbits

Momentum loss due to mass loss

Binary star orbits

Momentum loss due to Gravitational Radiation



Binary star orbits

Momentum loss due to Gravitational Radiation

Russell A. Hulse

Joseph H. Taylor Jr.

The Nobel Prize in Physics 1993 was awarded jointly to Russell A. Hulse and Joseph H. Taylor Jr. *"for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation"*

Question: predict the evolution of the pulsar's orbit.

Binary star orbits

Momentum loss due to Gravitational Radiation

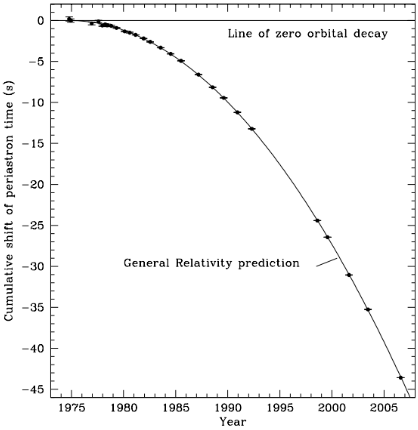


Figure 2. Orbital decay caused by the loss of energy by gravitational radiation. The parabola depicts the expected shift of periastron time relative to an unchanging orbit, according to general relativity. Data points represent our measurements, with error bars mostly too small to see.

Weisberg and Taylor 2010.

Binary star orbits

Gravitational waves now directly detected

The Nobel Prize in Physics 2017

Photo: Bryce Vickmark
Rainer Weiss
Prize share: 1/2

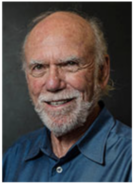


Photo: Caltech
Barry C. Barish
Prize share: 1/4

Photo: Caltech Alumni Association
Kip S. Thorne
Prize share: 1/4

The Nobel Prize in Physics 2017 was divided, one half awarded to Rainer Weiss, the other half jointly to Barry C. Barish and Kip S. Thorne *"for decisive contributions to the LIGO detector and the observation of gravitational waves"*.

Nobel Price 2017

Binary star orbits

Binary Super-massive Black holes

General orbit under radial force law

Remember the orbit equation?

$$\frac{d^2u}{d\phi^2} + u = -\frac{f\left(\frac{1}{u}\right)}{h^2u^2} \tag{1.50}$$

where $u \equiv \frac{1}{r}$ and $f_r = f$ for a spherical potential.
For f from a gravitational potential, we have

$$f\left(\frac{1}{u}\right) = -\frac{d\Phi}{dr} = u^2\frac{d\Phi}{du} \tag{1.51}$$

since gravity is conservative.
There are two types of orbit:

- Unbound: $r \to \infty$, $u \geq 0$ as $\phi \to \phi_\infty$
- Bound: r (and u) oscillate between finite limits.

General orbit under radial force law

Energy

If we take (1.50) $\times \frac{du}{d\phi}$:

$$\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + u\frac{du}{d\phi} + \frac{u^2}{h^2u^2}\frac{d\Phi}{du}\frac{du}{d\phi} = 0 \qquad (1.52)$$

$$\Rightarrow \frac{d}{d\phi}\left[\frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 + \frac{\Phi}{h^2}\right] = 0 \qquad (1.53)$$

and integrating over ϕ we have

$$\Rightarrow \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 + \frac{\Phi}{h^2} = \text{constant} = \frac{E}{h^2} \qquad (1.54)$$

General orbit under radial force law

Energy

and using $h = r^2\dot{\phi}$

$$\frac{E}{h^2} = \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 + \frac{\Phi}{h^2}$$

$$\begin{aligned} E &= \frac{r^4\dot{\phi}^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}r^2\dot{\phi}^2 + \Phi(r) \\ &= \frac{r^4}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}r^2\dot{\phi}^2 + \Phi(r) \\ &= \frac{r^4}{2}\left(\frac{du}{dr}\dot{r}\right)^2 + \frac{1}{2}r^2\dot{\phi}^2 + \Phi(r) \\ &= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \Phi(r) \qquad (1.55) \end{aligned}$$

i.e. we can show that the constant E we introduced is the energy per unit mass.

General orbit under radial force law

Peri and Apo

$$\frac{E}{h^2} = \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 + \frac{\Phi}{h^2}$$

For bound orbits, the limiting values of u (or r) occur where $\frac{du}{d\phi} = 0$, i.e. where

$$u^2 = \frac{2E - 2\Phi(u)}{h^2} \tag{1.56}$$

from (1.54).
This has two roots, $u_1 = \frac{1}{r_1}$ and $u_2 = \frac{1}{r_2}$

this is not obvious, since Φ is not defined

For $r_1 < r_2$, where r_1 is the pericentre, r_2 the apocentre

Orbital periods

Radial motion

The radial period T_r is defined as the time to go from $r_2 \rightarrow r_1 \rightarrow r_2$.
Now take (1.55) and re-write:

$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 + \Phi(r)$

$$\left(\frac{dr}{dt}\right)^2 = 2(E - \Phi(r)) - \frac{h^2}{r^2} \tag{1.57}$$

where we used $h = r^2\dot{\phi}$ to eliminate $\dot{\phi}$

So

$$\frac{dr}{dt} = \pm\sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}} \tag{1.58}$$

(two signs - $\dot{r}$ can be either > 0 or < 0, and $\dot{r} = 0$ at r_1 & r_2.
Then

$$T_r = \oint dt = 2\int_{r_1}^{r_2} \frac{dt}{dr}dr = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{h^2}{r^2}}} \tag{1.59}$$

Orbital periods
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Orbital periods

Azimuthal motion

If travelling from $r_2 \rightarrow r_1 \rightarrow r_2$ ϕ is increased by an amount

$$\Delta\phi = \oint d\phi = 2\int_{r_1}^{r_2} \frac{d\phi}{dr} dr = 2\int_{r_1}^{r_2} \frac{d\phi}{dt}\frac{dt}{dr} dr \tag{1.60}$$

so

$$\Delta\phi = 2h\int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{2(E-\Phi(r)) - \frac{h^2}{r^2}}} \tag{1.61}$$

Precession of the orbit

For a given orbit, the time taken to go around once (i.e. $0 \rightarrow 2\pi$) depends in general on where you start, so the azimuthal period is not well defined. Instead use the mean angular velocity $\bar{\omega} = \Delta\phi / T_r$ to obtain a mean azimuthal period T_ϕ, so

$$T_\phi = 2\pi/\bar{\omega} \Rightarrow T_\phi = \frac{2\pi}{\Delta\phi} T_r$$

is the mean time to go around once.
Note that unless $\Delta\phi/2\pi$ is a rational number the orbit is not closed.

Precession of the orbit

For Keplerian orbit $\Delta\phi = 2\pi \Rightarrow T_r = T_\phi$.
In one period T_r the apocentre (or pericentre) advances by an angle $\Delta\phi - 2\pi$. i.e.the orbit shifts in azimuth at an average rate given by the mean precession rate

$$\Omega_p = \frac{\Delta\phi - 2\pi}{T_r} \quad \mathrm{rad\ s^{-1}} \tag{1.62}$$

Thus the precession period is

$$T_p = \frac{2\pi}{\Omega_p} = \frac{T_r}{\frac{\Delta\phi}{2\pi} - 1} \tag{1.63}$$

This precession is in the sense opposite to the rotation of the star

In the special case of a Keplerian orbit $\Delta\phi = 2\pi \Rightarrow T_\phi = T_r$ and $\Omega_p = 0$, i.e. orbits are closed and do not precess. Otherwise general orbit is a rosette between r_1 & r_2.
This allows us to visualize how we can build a galaxy out of stars on different orbits.

Precession of the orbit

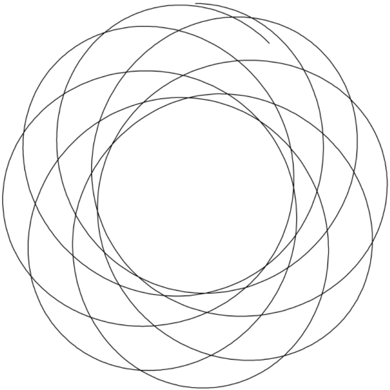

Figure 3.1 A typical orbit in a spherical potential (the isochrone, eq. 2.47) forms a rosette.

Example

T_r for the Keplerian case $\Phi(r) = -\frac{GM}{r}$
We have equation (1.59)

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-\Phi(r)) - \frac{h^2}{r^2}}}$$

Now r_1 & r_2 are determined from $\dot{r} = 0$, i.e.

$$2(E-\Phi(r)) - \frac{h^2}{r^2} = 0 \quad (1.64)$$

$$2E + \frac{2GM}{r} - \frac{h^2}{r^2} = 0 \quad (1.65)$$

$$r^2 + \frac{GM}{E}r - \frac{h^2}{2E} = 0 \quad (1.66)$$

$$\Leftrightarrow (r-r_1)(r-r_2) = 0 \quad (1.67)$$

$$\Rightarrow r_1 r_2 = -\frac{h^2}{2E}; \quad r_1 + r_2 = -\frac{GM}{E} \quad (1.68)$$

(remember $E < 0$ for a bound orbit).

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(remember $E < 0$ for a bound orbit).

Example

Rewrite (1.59) as

$$T_r = 2\int_{r_1}^{r_2} \frac{rdr}{\sqrt{2E(r-r_1)(r-r_2)}} = \frac{2}{\sqrt{2|E|}}\int_{r_1}^{r_2} \frac{rdr}{\sqrt{(r_2-r)(r-r_1)}} \tag{1.69}$$

if $r_1 < r < r_2$.
This is another of those integrals. If $R = a + bx + cx^2 = -r^2 + (r_1 + r_2)r - r_1 r_2$ and $\Delta = 4ac - b^2$ which becomes, using the variables here, $\Delta = -(r_1 - r_2)^2$ then

$$\int \frac{xdx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c}\frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{2cx+b}{\sqrt{-\Delta}}\right)$$

for $c < 0$ and $\Delta < 0$ (See G&R 2.261 and 2.264).

Example

The first term is 0 at r_1 and r_2 ($R = 0$ there), so

$$
\begin{aligned}
T_r &= \frac{2}{\sqrt{2|E|}}\left[\frac{r_1+r_2}{2}\right]\left[\sin^{-1}\left(\frac{-2r_2+r_1+r_2}{r_1-r_2}\right)\right. \\
&\left. -\sin^{-1}\left(\frac{-2r_1+r_1+r_2}{r_1-r_2}\right)\right] \\
&= \frac{2}{\sqrt{2|E|}}\left[\frac{r_1+r_2}{2}\right][\sin^{-1}(1)-\sin^{-1}(-1)] \\
&= \frac{2}{\sqrt{2|E|}}\frac{GM}{2(-E)}\left[\frac{\pi}{2}-(-\frac{\pi}{2})\right] \\
&= \frac{2\pi GM}{(-2E)^{\frac{3}{2}}}
\end{aligned}
\tag{1.70}
$$