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PART II ASTROPHYSICS

Stellar Dynamics and Structure of Galaxies

Examples Sheet 4

4.1 [02307Z(i)] A dwarf spheroidal galaxy has a velocity dispersion σ three times less than that of the centre of a globular cluster, while the core of the dwarf galaxy is 40 times larger. Use the virial theorem to estimate the mass of the dwarf galaxy relative to the mass of the globular cluster.

Assume that for any elliptical galaxy the luminosity is proportional to the square of the radius, and that its mass-to-ligth ratio is a constant. Use the Virial theorem to show that

$$L \propto \sigma^4$$

4.2 In spherical polar coordinates (r, θ, ϕ) the velocity is

$$\mathbf{v} = (\dot{r}, r\dot{\theta}, r\sin\theta\dot{\phi}) ,$$

and the acceleration is

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2,\ 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2,\ r\sin\theta\ddot{\phi} + 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi}).$$

Show that if both f and Φ are independent of θ and ϕ, the collisionless Boltzmann equation is

$$\frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{1}{r}\left(v_\theta^2 + v_\phi^2\right)\frac{\partial f}{\partial v_r} - \frac{1}{r}\left(v_r v_\theta - v_\phi^2\cot\theta\right)\frac{\partial f}{\partial v_\theta} - \frac{1}{r}v_\phi\left(v_r + v_\theta\cot\theta\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial r}\cdot\frac{\partial f}{\partial v_r} = 0.$$

4.3 Obtain the Jeans equations for a spherically symmetrical star cluster for which the velocity distribution is axisymmetrical about the radial direction:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho\overline{v_r}\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\rho \overline{v_r}\right) + \lambda \frac{\partial p}{\partial r} + 3\left(\lambda - 1\right)\frac{p}{r} + \rho \frac{\partial \Phi}{\partial r} = 0,$$

where $p = \frac{1}{3}\rho\left(\overline{v_r^2} + \overline{v_\theta^2} + \overline{v_\phi^2}\right)$ and $\lambda = 3\overline{v_r^2} / \left(\overline{v_r^2} + \overline{v_\theta^2} + \overline{v_\phi^2}\right)$

Explain why, in the steady state, the second equation differs from the hydrostatic equation for a fluid:

$$\frac{\partial p}{\partial r} + \rho \frac{\partial \Phi}{\partial r} = 0,$$

describing physically the significance of each term. [*You may assume that in spherical polar coordinates* (r, θ, ϕ) *the collisionless Boltzmann equation is as given in Q.4.3.*]

4.4 [97306(ii)] In a one-dimensional system the distribution function $f(x, v)$ is just a function of the energy $E = \frac{1}{2}v^2 + \Phi(x)$. Verify that this satisfies the collisionless Boltzmann equation.

If

$$f(E) = \frac{\rho_0}{\sqrt{2\pi}} \exp(-E)\ ,$$

where ρ_0 is a constant, deduce that $\Phi(x)$ satisfies

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} = 4\pi G \rho_0 \exp(-\Phi)\ .$$

Verify that $\Phi = 2\ln[\cosh\left(x/H\right)]$, where $H = (2\pi G\rho_0)^{-\frac{1}{2}}$, and hence obtain the density $\rho(x)$.

[*You may assume* $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.]

4.5 [96306(ii)] Consider a spherically-symmetric stellar-dynamical system with distribution function

$$f(E) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}}\ \exp[E/\sigma^2],$$

where $E = \Psi - \frac{1}{2}v^2$ is the reduced energy, where Ψ is the potential, v is velocity, σ is the (constant) dispersion, and ρ_1 is a constant. Find the density ρ of the system, and write down the Poisson equation for the system.

The equation of hydrostatic support for an isothermal gas of density $\rho(r)$ at temperature T is

$$\frac{kT}{m_0}\ \frac{d\rho}{dr} = -\rho \frac{Gm(r)}{r^2},$$

where k is Boltzmann's constant, m_0 is the mass per particle, and $m(r)$ is the total mass interior to radius r. Show that the stellar-dynamical system and the gaseous system have the same density structure when

$$\sigma^2 = \frac{kT}{m_0}.$$

[*For a spherically symmetrical function F,* $\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial F}{\partial r}\right)$.]

4.6 Show that it is possible to construct a spherically symmetric stellar distribution function for which all stars are on circular orbits ($v_r = 0$) and the density satisfies

$$\rho = \begin{cases} \rho_0 & 0 \le r < R \\ 0 & R \le r. \end{cases}$$

Do you think a similar distribution function might be constructed for any density distribution $\rho(r)$?

How might one test whether globular clusters have such distribution functions?

4.7 Consider a spherically symmetric stellar cluster with self-consistent isotropic stellar distribution function. Choose an arbitrary axis through the centre of the cluster. What is the net angular momentum about this axis?

Each star has an orbital angular momentum about the axis. At a particular instant in time, take each star with negative orbital angular momentum and reverse its velocity in space. What effect does this have on: (a) the angular momentum of the cluster; (b) the equilibrium of the cluster and (c) the density distribution?

Does a spherical star cluster necessarily have zero angular momentum?

Does an axially symmetric, but flattened, star cluster necessarily have non-zero angular momentum?

4.8 [96406(ii)] Give an account of the structure and evolution of a globular cluster. Topics you should cover include: the stellar population, the density profile and the velocity dispersion.

*4.9 A black hole is introduced into the centre of a cluster of stars. Ignore the dynamical effect of the black hole mass and assume that it swallows any star which passes close enough to it. How does the stellar distribution function evolve (a) on a dynamical timescale and (b) on a relaxation timescale? What physical effects determine the feeding rate of the black hole at late times? (See: Frank, J., Rees, M.J. MNRAS **176**, 633 (1976), Syer, D., Ulmer, A. MNRAS **306**, 35 (1999), Magorrian, J., Tremaine, S. MNRAS **309**, 447 (1999)).

4.10 [01407A(ii)]

Show that if the distribution function of a spherical system is of the form

$$f(\epsilon, L) = f_0(\epsilon) L^{-1}$$

where ϵ is the relative energy, L is the absolute value of the angular momentum, and $f_0(\epsilon)$ is an arbitrary function, then at any point in the system the velocity dispersions are related by

$$\overline{v_\theta^2} = \overline{v_\phi^2} = \frac{1}{2}\overline{v_r^2}$$

[Hint:You may use spherical polar coordinates in the velocity space and set $v_r = v\cos\alpha$, $v_\theta = v\sin\alpha\cos\beta$, $v_\phi = v\sin\alpha\sin\beta$ with $d^3\mathbf{v} = v^2\sin\alpha dv d\alpha d\beta$]