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**PART II ASTROPHYSICS**

**Stellar Dynamics and Structure of Galaxies**

Examples Sheet 3

3.1 Show that the surface density $\Sigma(R)$ and volume density $\rho(r)$ of a spherical mass distribution are related by

$$\Sigma(R) = 2\int_R^\infty \frac{\rho(r) r dr}{\sqrt{r^2 - R^2}} \ .$$

By using the substitution $x = 1/R^2$ and $y = 1/r^2$, rewrite this in the form of Abel's integral equation

$$f(x) = \int_0^x \frac{g(y)dy}{\sqrt{x-y}} \ ,$$

where $f(x)$ is given and $g(y)$ is unknown. The solution of the equation is

$$g(y) = \frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{f(x)dx}{\sqrt{y-x}} \ .$$

Deduce that $\rho(r)$ is obtained uniquely from $\Sigma(R)$.

If $\Sigma(R) \propto R^{-1}$, show that $\rho(r) \propto r^{-2}$.

3.2 Assume the sun is 10 kpc from the Galactic Centre and Oorts constants are A = −B = 15 km/s/kpc.

(a) Calculate the circular velocity at the sun and estimate the mass interior to 10 kpc.

(b) What is the age of the sun in Galactic years?

(c) How many radial oscillations does the sun undergo in one Galactic orbit?

3.3 The local standard of rest (LSR) at radius $R$ in the Galaxy is the origin of a coordinate system which is in the plane of the Galaxy and has radial, azimuthal and vertical velocity $(u, v, w) = (0, V_c(R), 0)$ where $V_c$ is the circular velocity at radius $R = 8.5$ kpc. At the solar radius, $V_c \approx 200$ km s$^{-1}$ and the mean velocity dispersion of stars is 20 km s$^{-1}$. Estimate approximately the vertical and radial excursions of a typical stellar orbit, explaining your assumptions carefully.

An observer moving with the LSR measures relative velocities of stars within a radius of about 50 pc. It is found that $\langle u \rangle = \langle w \rangle = 0$, but that $\langle v \rangle \neq 0$. where the $\langle\,\rangle$ denote averages. How might this result be interpreted? What sign do you expect $\langle v \rangle$ to have?

*How can one measure the solar motion relative to the LSR?

3.4 The core of a globular cluster has density $\rho_c = 8 \times 10^3 M_\odot \mathrm{pc}^{-3}$ and radius $r_c = 1.5$ pc. The stars have masses $M = 0.7M_\odot$, radii $R = 0.7R_\odot$ and line of sight velocity dispersion $\sigma_r = 7$ km s$^{-1}$. Calculate the number of physical collisions which have occurred in the cluster core (a) with and (b) without taking account of gravitational focussing. How might one test this estimate observationally?

What is the corresponding result for an open cluster with $\rho_c = 100 M_\odot$ pc$^{-3}$, $r_c = 1$ pc, $\sigma_r = 1$ km s$^{-1}$, $M = 2M_\odot$, $R = 2R_\odot$?

3.5 A typical cluster of galaxies has core radius $r_c = 250$ kpc and velocity dispersion $\sigma_r = 800$ km s$^{-1}$. The fractional volume of the cluster that is occupied by galaxies is $\approx 10^{-3}$. How does this value compare with (a) stars in the core of a globular cluster; (b) field stars in the Galaxy? How often do galaxy collisions occur? What observational evidence is there?

3.6 [96407(ii)] The solar system has a size of 30 a.u. and an age of $4.5 \times 10^9$ y. The velocity dispersion of stars in the solar neighbourhood is approximately 20 km s$^{-1}$. Use these facts to derive an approximate upper limit to the stellar density $n_*$ in units of stars per cubic parsec in the solar neighbourhood, stating your assumptions carefully.

An outdated model of the formation of the solar system is based on the idea that the Sun underwent a grazing collision with another star like the Sun. During the collision, material was torn off and later condensed to form planets. Given $n_*$, estimate the timescale between such events. Assuming this model to be correct, give an approximate upper limit to the fraction of stars that might be expected to have planetary systems.

If, in fact, $n_* \sim 0.1$ pc$^{-3}$, and there are $10^{11}$ stars in the Galaxy, how many planetary systems in the Galaxy would this model produce in the age ($\sim 10^{10}$ y) of the Universe?

3.7 For a Maxwellian distribution of velocities with one-dimensional dispersion $\sigma$, show that (a) the mean speed is $\bar{v} = \sigma\sqrt{8/\pi}$; (b) the mean square speed is $\bar{v}^2 = 3\sigma^2$; (c) the mean square of one component of velocity is $\bar{v}_x^2 = \sigma^2$; (d) the mean square relative speed of any two particles is $\overline{v_{rel}^2} = 6\sigma^2$; and (e) the fraction of particles with $v^2 > 4\bar{v}^2$ is 0.00738.

3.8 [97406(ii)] Explain what is meant by the relaxation timescale $t_\mathrm{e}$ and the crossing time $t_\mathrm{cr}$ for a self-gravitating system of $N$ stars (with $N \gg 1$). Comment briefly on why $t_\mathrm{e}/t_\mathrm{cr}$ is an increasing function of $N$.

If the stars have exactly a Maxwellian distribution of velocity, the fraction of stars that have speeds exceeding the escape speed is $\epsilon = 7.4 \times 10^{-3}$. Explain why it is reasonable to estimate the evaporation timescale, $t_{\rm evap}$, on which stars escape from the system to be $t_{\rm evap} \sim \epsilon^{-1} t_{\rm e}$.

The relaxation timescale is given approximately by

$$t_{\rm e} \sim \frac{N}{10 \log N} \, t_{\rm cr} \ .$$

Estimate $t_{\rm cr}, t_{\rm e}$ and $t_{\rm evap}$ for a globular cluster of mass $10^5 \ M_\odot$, radius 5 pc and typical stellar mass $0.7 \ M_\odot$. Comment briefly on your results.

3.9 [97406(i)] A heavy particle of mass $m$ passes at speed $V$ through a stationary sea of light particles. The mean density of the light particles is $\rho$. Explain briefly why there is a significant enhancement in the density of light particles over a region of size $\sim Gm/V^2$ directly behind the heavy particle.

Deduce that the motion of the heavy particle is subject to a gravitational drag force $F$ where

$$F \sim G^2 m^2 \rho / V^2 \ .$$

3.10 A galaxy has density profile $\rho(r) = v_c^2/4\pi G r^2$ where $v_c$ is a constant. A globular cluster mass $M$ is on a circular orbit at radius $r_0$. Show that its angular momentum per unit mass is $L = r_0 v_c$. Applying the gravitational drag formula, estimate the time taken for the cluster to plunge into the galactic centre. Show that for M31 for which $v_c \simeq 250$ km s$^{-1}$ and $M \simeq 5 \times 10^6 M_\odot$, there is a radius within which one does not expect to find many globular clusters.