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PART II ASTROPHYSICS

Stellar Dynamics and Structure of Galaxies

Examples Sheet 2

2.1 Show that the potential within a homogeneous sphere of density ρ_0 is of the form

$$\Phi(r) = \frac{1}{2}\Omega^2 r^2 + \text{ const.},$$

where Ω is a constant which depends on ρ_0.

What can you say about the general nature of orbits in such a potential?

2.2 If $h(r)$ is the specific angular momentum of a circular orbit of radius r in a given spherically symmetric potential, show that the circular orbit at radius r is unstable if $\frac{d}{dr}\left(h^2(r)\right) < 0$.

2.3 For what spherically symmetric potential is a possible trajectory $r = \mathrm{a}\exp(\mathrm{b}\phi)$, where a and b are constants?

2.4 In a spherically symmetric system, write down the equation in terms of $u = 1/r$ whose roots are the inverses of the apocentre and pericentre distances. Consider the second derivative of $\Phi(u)$ with respect to u to show that if $E < 0$, and the potential $\Phi(r)$ is generated by a non-negative density distribution, the equation has either two or zero roots.

2.5 [95306(ii)] A spherical galaxy has a distribution of stars which has roughly constant density near its centre, and which falls to zero at large distance. The corresponding gravitational potential is the 'isochrone potential'

$$\Phi(r) = \frac{-GM}{b + \sqrt{b^2 + r^2}}$$

with b constant. By considering large radii, show that M is the total mass.

Show that the central density is

$$\rho(0) = \frac{3M}{16\pi b^3}$$

and that

$$\rho(r \gg b) \approx \frac{bM}{2\pi r^4}.$$

Derive the dependence of the circular speed on r.

[*For spherically symmetrical functions* F, ∇^2 *in spherical coordinates is*

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right).]$$

*2.6 Show that both for a Keplerian potential and for the isochrone potential, the radial period T_r is given by

$$T_r = \frac{2\pi GM}{(-2E)^{\frac{3}{2}}}$$

Hint: use the substitution

$$s = 1 + \sqrt{\frac{r^2}{b^2} + 1}\ , \text{ and the fact that } \int_{s_1}^{s_2} \frac{(s-1)ds}{\sqrt{(s_2 - s)(s - s_1)}} = \pi\left[\frac{1}{2}(s_1 + s_2) - 1\right]$$

2.7 [97405(ii)] A cylindrically symmetric potential $\Phi(R, z)$ has the property that $\Phi(R, z) = \Phi(R, -z)$. A particle is in a circular orbit $R = R_0,\ z = 0$ with angular velocity Ω_0. The orbit is perturbed slightly in such a way that the angular momentum about the z-axis remains unchanged. Show that the particle undergoes oscillations in the R-direction with frequency κ such that

$$\kappa^2 = 3\Omega_0^2 + \left.\frac{\partial^2\Phi}{\partial R^2}\right|_{R=R_0,\ z=0} .$$

Find the corresponding frequency ν with which it oscillates in the z-direction.

Show that if the matter giving rise to the potential Φ has zero density at $R = R_0$ and $z = 0$, then

$$\kappa^2 + \nu^2 = 2\Omega_0^2 .$$

[*In cylindrical polar coordinates*

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial R}, \frac{1}{R}\frac{\partial\Phi}{\partial\phi}, \frac{\partial\Phi}{\partial z}\right), \quad \nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}.]$$

2.8 Show that

$$\frac{dJ_0}{dx} = -J_1$$

$$\frac{dI_0}{dx} = I_1$$

and

$$\frac{dK_0}{dx} = -K_1,$$

where $J_n(x)$, $I_n(x)$ and $K_n(x)$ are the standard Bessel functions.

2.9 The gravitational potential of an exponential disk is

$$\Phi(R, z) = -2\pi G\Sigma_0 R_d^2 \int_0^\infty \frac{J_0(kR)\ e^{k|z|}dk}{\{1 + (kR_d)^2\}^{\frac{3}{2}}}$$

Derive and sketch the circular velocity in the plane z = 0.

*2.10 A uniform thin ring of mass m, radius a, is centred at the origin and is in the $\theta = \pi/2$ plane of a system of (r, θ, ϕ) coordinates.

For radii $r > a$ find an approximation to the gravitational potential of the form

$$\Phi(r, \theta) = -\frac{Gm}{r}\left\{1 - J_2\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \ldots\right\},$$

where J_2 is to be determined.

Write down the approximate form of the potential in (R, ϕ, z) coordinates close to the $z = 0$ plane, keeping terms to second order in z.

Show that the nodal precession of a slightly tilted orbit is retrograde.

Why would you expect all slightly tilted orbits close to the plane of a spiral galaxy to precess in the retrograde direction?