NST2AS NATURAL SCIENCES TRIPOS Part II

Mon 5 June 2023 1:30pm – 4:30pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than 6 questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer for each Part on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **5Z and 6Z** *should be in one bundle and* **1X, 3X and 4X** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X – Relativity

(i) (a) Explain what is meant by a *geodesic curve* in a manifold and an *affine parameter*. [2]

(b) Write down the geodesic equation in an affine parameterisation and show that it can be obtained by applying the Euler–Lagrange equations,

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = \frac{\partial \mathcal{L}}{\partial x^a}\,,$$

to $\mathcal{L} = g_{ab}\dot{x}^a\dot{x}^b$, where dots denote differentiation with respect to an affine parameter, $\dot{x}^a = dx^a/d\lambda$. [5]

(c) The line element on the surface of a cylinder embedded in 3D Euclidean space is given by $ds^2 = dz^2 + a^2 d\phi^2$, where a is the radius of the cylinder, ϕ is the polar angle and z is the Cartesian coordinate along the axis of the cylinder. Write down the geodesic equations for the cylinder and solve them to determine $\phi(\lambda)$ and $z(\lambda)$ for a general geodesic curve. [3]

(ii) (a) Consider the spacetime line element

$$ds^2 = c^2dt^2 - \left(\frac{R^2}{r^2+\alpha^2}\right)dr^2 - R^2d\theta^2 - (r^2+\alpha^2)\sin^2\theta d\phi^2\,, \qquad (*)$$

with $R^2 = r^2 + \alpha^2\cos^2\theta$, where α is a constant. By considering the t and ϕ components of the geodesic equation, or otherwise, show that

$$J = \frac{d\phi}{dt}(r^2+a^2)\sin^2\theta$$

is an integral of motion for a free particle. [6]

(b) Give a physical interpretation of J. [2]

(c) By considering the coordinate transformation

$$\begin{aligned} x &= \sqrt{r^2+\alpha^2}\sin\theta\cos\phi\,,\\ y &= \sqrt{r^2+\alpha^2}\sin\theta\sin\phi\,,\\ z &= r\cos\theta\,, \end{aligned}$$

show that $(*)$ is Minkowski spacetime. [7]

(d) Sketch the surface $r =$ const. and the surface $\theta =$ const. in terms of the x, y, z coordinates. [5]

Question 2Y – Astrophysical Fluid Dynamics

(i) (a) The entropy form of the energy equation for an ideal gas is

$$\frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\left(\frac{\rho}{p}\right)\dot{Q},$$

where p is the pressure, ρ is the density, γ is the standard ratio of specific heat capacities, $K = p/\rho^\gamma$, and $\dot{Q}$ is the net cooling rate per unit mass. Describe the processes through which the gas might be locally heated or cooled. [4]

(b) Consider an ideal gas that is not experiencing any explicit heating or cooling processes, but that is subject to the action of thermal conduction, such that there is a thermal energy flux $\mathbf{q} = -\chi\,\nabla T$, where T is the gas temperature and $\chi(T)$ is the coefficient of thermal diffusivity. Show that the entropy form of the energy equation can be written as

$$\frac{1}{K}\left[\frac{\partial K}{\partial t} + (\mathbf{u}\cdot\nabla)K\right] = \frac{(\gamma-1)}{p}\nabla\cdot(\chi\,\nabla T)\,,$$

where $\mathbf{u}$ is the fluid velocity. [6]

(ii) (a) Consider an equilibrium system in which the gas from Part (i) is distributed uniformly through space with density ρ_0, pressure p_0, temperature T_0, and is static with $\mathbf{u} = 0$. We now introduce small perturbations (δp, $\delta\rho$, δT, $\delta\mathbf{u}$) into the system. Using any information from Part (i), show that the perturbed entropy equation to linear order is

$$\frac{1}{p_0}\frac{\partial\,\delta p}{\partial t} - \frac{\gamma}{\rho_0}\frac{\partial\,\delta\rho}{\partial t} = \frac{(\gamma-1)\chi_0 T_0}{p_0}\nabla^2\left(\frac{\delta p}{p_0} - \frac{\delta\rho}{\rho_0}\right),$$

where $\chi_0 \equiv \chi(T_0)$. [4]

(b) By decomposing the perturbations into plane waves, i.e., perturbations in quantity X have the form $\delta X \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, show that the dispersion relation governing the perturbations is

$$\omega(\omega^2 - k^2c_\mathrm{s}^2) = -i\omega_\mathrm{cond}(\omega^2 - k^2c_\mathrm{s}^2/\gamma),$$

where c_s is the adiabatic sound speed and you should define ω_cond in terms of quantities already specified. [10]

(c) The weak conduction limit is defined by $\omega_\mathrm{cond} \ll \omega$. By writing $\omega = kc_\mathrm{s} + \epsilon$, where $\epsilon \sim O(\omega_\mathrm{cond}/\omega) \ll 1$, show that the system contains damped sound waves and determine the damping rate Γ in terms of ω_cond and γ. [4]

(d) In addition to the sound waves, the cubic dispersion relation contains one other mode. Determine the frequency ω and provide a physical interpretation for this mode. [2]

Question 3X – Cosmology

(i) (a) Consider a Friedmann-Robertson-Walker (FRW) universe with scale-factor $R(t)$. Show that light with wavelength λ_e emitted at time t_e is redshifted to a wavelength λ_0 at the present day t_0, where

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{R(t_0)}{R(t_e)}.$$ [5]

(b) Show that in an FRW universe the Hubble parameter $H(t) = R^{-1}dR/dt$ can be written as

$$H(t) = H_0 \left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]^{1/2},$$

where H_0 is the Hubble parameter at the present day, and Ω_r, Ω_m, Ω_K and Ω_Λ are the density parameters at the present day contributed by radiation, non-relativistic matter, curvature and a cosmological constant Λ. [5]

(ii) (a) Assuming a spatially flat FRW universe composed of non-relativistic matter with present day density parameter Ω_m and a cosmological constant, show that the age of the universe, at the time light is emitted from an object with an observed redshift z, is

$$t(z) = \frac{2}{3H_0(1-\Omega_m)^{1/2}} \sinh^{-1}\left[\left(\frac{1-\Omega_m}{\Omega_m}\right)^{1/2}(1+z)^{-3/2}\right], \qquad (*)$$

where H_0 is the present day value of the Hubble parameter. [6]

[*You may assume that* :

$$\int \frac{dx}{(a+bx^2)^{1/2}} = \frac{1}{\sqrt{b}} \sinh^{-1}\left[\left(\frac{b}{a}\right)^{1/2} x\right].$$]

(b) In the high redshift limit, $(1+z) \gg [(1-\Omega_m)/\Omega_m]^{1/3}$. Show that (*) reduces to

$$t(z) \approx \frac{2}{3H_0\Omega_m^{1/2}} \frac{1}{(1+z)^{3/2}}.$$ [2]

QUESTION CONTINUED ON NEXT PAGE

(c) Consider a black hole of mass M_{bh} accreting at the Eddington luminosity

$$L_{Ed} = \frac{4\pi G M_{bh} m_p c}{\sigma_T},$$

where σ_T is the Thomson cross section. If the efficiency of conversion of rest mass to radiation, ϵ, is assumed to be constant, show that the black hole grows according to

$$M_{bh}(t) = M_{bh}(t_i) \exp\left(\frac{(1-\epsilon)}{\epsilon}\frac{(t-t_i)}{t_{\rm Ed}}\right) \qquad (**)$$

and evaluate the characteristic timescale $t_{\rm Ed}$. [6]

(d) If black holes are formed with $M_{bh}(t_i) = 200\ M_\odot$ at some very early time $t_i \ll t$ and subsequently grow according to (**) with radiative efficiency $\epsilon = 0.1$, calculate the maximum redshift z_* at which you would expect to find a supermassive black hole with mass $M_{bh}(z_*) = 2 \times 10^9\ M_\odot$ in a universe with $\Omega_m = 0.3$ and $H_0 = 67.5$ km s^{-1}Mpc^{-1}. [3]

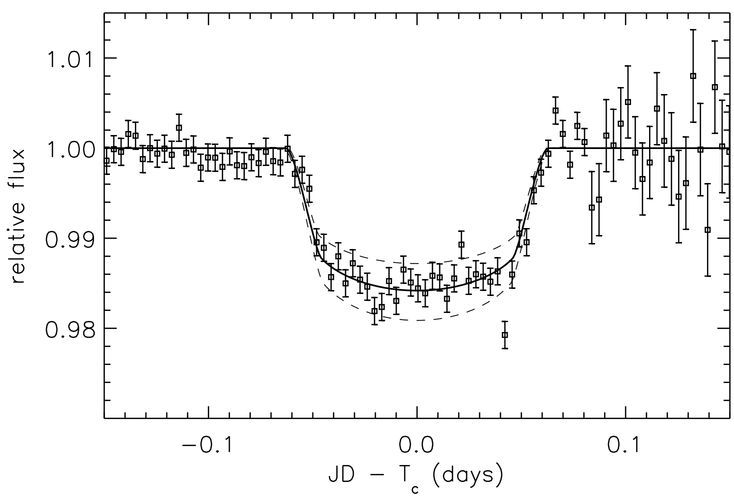
(e) How would you interpret the existence of black holes with masses $M_{bh} > 2 \times 10^9 M_\odot$ at redshifts $z > z_*$? [3]

Question 4X – Structure and Evolution of Stars

(i) (a) Sketch the Hertzsprung-Russell diagram of a Globular Cluster, *labelling the axes carefully.* Include in your sketch the approximate positions of: the main sequence, the horizontal branch, red giants, white dwarfs, and a solar mass star. [5]

(b) The James Webb Space Telescope has recently released observations of the young cluster NGC 346 in the Small Magellanic Cloud galaxy. This cluster is thought to be only 3 Myr old. Sketch what you imagine that the Hertzsprung-Russell diagram of NGC 346 may look like once the relevant data have been assembled, and comment on the most obvious differences from your plot in (a). [5]

(ii) A planet orbits a $1M_\odot$ solar-type star and transits in front of the star once every 3.5 days. The light curve of the transit is shown below:



(a) From the depth of the eclipse, calculate the radius of the planet, assuming that it is completely dark. [4]

(b) What other information might be inferred from the shape of the light curve? [4]

(c) If the mass of the planet is $0.001M_\odot$ (approximately one Jupiter mass), calculate the radial velocity amplitude of the star due to the orbiting planet. [4]

QUESTION CONTINUED ON NEXT PAGE

(d) Another similar planet orbits a white dwarf which is a remnant of a $1M_\odot$ solar-type star. The orbit is sufficiently wide that the planet survived during the late stages of stellar evolution. With a very sensitive instrument the orbit of the planet around the white dwarf can be followed. The planet's period is 244 yr, and the mass of the white dwarf is $0.6\,M_\odot$. What is the length of the semi-major axis of the white dwarf–planet system? [4]

(e) As projected onto the sky, the orbit of the planet around the white dwarf appears to be a perfect circle of radius $1''$, but the white dwarf, instead of being in the centre of the circle, is 60% of the way to the edge. Show that the true orbit is an ellipse, with eccentricity $e = 3/5$, and calculate the size of the semi-minor axis and the distance to the system. [4]

Question 5Z – Statistical Physics

(i) (a) What types of systems are described by *microcanonical*, *canonical* and *grand canonical ensembles*? [5]

(b) Under what conditions is the choice of ensemble irrelevant? [5]

(ii) (a) Consider a classical particle of mass m moving non-relativistically in two-dimensional space, with coordinates x and y, enclosed inside a circle of radius R and attached by a spring to the centre. The particle therefore moves in a potential

$$V(r) = \begin{cases} \frac{1}{2}\kappa r^2 & \text{for } r < R\,, \\ \infty & \text{for } r \geqslant R\,, \end{cases}$$

where κ is the spring constant and $r^2 = x^2 + y^2$. The particle is coupled to a heat reservoir at temperature T. Calculate the partition function for the particle. [5]

(b) Calculate the average energy $\langle E \rangle$ and the average potential energy $\langle V \rangle$ of the particle. [6]

(c) Compute $\langle E \rangle$ in the two limits $\frac{1}{2}\kappa R^2 \gg k_{\mathrm{B}}T$ and $\frac{1}{2}\kappa R^2 \ll k_{\mathrm{B}}T$, where k_{B} is the Boltzmann constant. How do these two results compare with what is expected from equipartition of energy? [6]

(d) Compute the partition function for a collection of N identical non-interacting such particles. [3]

Question 6Z – Principles of Quantum Mechanics

(i) (a) Write down the Hamiltonian for a quantum harmonic oscillator of frequency ω in terms of the creation and annihilation operators A and $A^\dagger$. You may work in units where $\hbar = 1$. Define the number operator N and state all commutation relations among A, $A^\dagger$ and N. [3]

(b) Show that the eigenvalues of N are real, non-negative integers. [7]

(ii) (a) Consider a system of two independent harmonic oscillators of frequency $\omega_{\mathrm{A}} = 1$ and $\omega_{\mathrm{B}} = 2$, respectively, and let A, $A^\dagger$, B and $B^\dagger$ be their corresponding creation and annihilation operators. Find the five lowest eigenvalues of the Hamiltonian H_0 of the combined system and determine their degeneracy. Work in units where $\hbar = 1$. [7]

(b) The system is perturbed so that it is now described by the new Hamiltonian $H = H_0 + \lambda H'$, where $H' = A^\dagger A^\dagger B + AAB^\dagger$. Using degenerate perturbation theory, calculate to $\mathcal{O}(\lambda)$ the energies of the eigenstates associated with the level $E_0 = \frac{9}{2}$. Write down the eigenstates, to $\mathcal{O}(\lambda)$, associated with these perturbed energies. [10]

(c) By explicit evaluation show that these are in fact exact eigenstates of H with these energies as eigenvalues. [3]

[*Hint: you may use without proof that for the harmonic oscillator* $A\,|n\rangle = \sqrt{n}\,|n-1\rangle$ *and* $A^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$]

Question 7Y – Stellar Dynamics and the Structure of Galaxies

(i) (a) The equation of an ellipse with semi-major axis, a, and eccentricity, e, can be written

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y^2}{a^2(1-e^2)}\right) = 1,$$

where (x, y) is a Cartesian coordinate system with origin at the centre of the ellipse and the focus of the ellipse is located at $x = ae$, $y = 0$. The line XY connects the two points on the ellipse where $x = ae$. Calculate the fraction of the area of the ellipse for which $x > ae$. [7]

(b) The ellipse describes the path of a test particle in the gravitational potential of a point mass M which is located at the focus of the ellipse. Use Kepler's second law to show that the fraction of time that the particle spends between XY and pericentre is given by

$$f = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}(e) - \frac{e}{\pi}(1-e^2)^{1/2}. \qquad (*)$$

[3]

(ii) (a) A satellite executes a circular orbit around the Sun with an orbital radius of 1 au. A radial impulse is applied so that it attains apocentre at the orbital radius of Jupiter, which can be assumed to be on a circular orbit at 5 au from the Sun, and to be coplanar with the satellite. The angle between the satellite-Sun vector and the Jupiter-Sun vector is θ_{imp} at the moment that the impulse is applied to the satellite. What value of θ_{imp} will ensure that the satellite intercepts Jupiter at apocentre? You may use the result in $(*)$ without proof. [6]

(b) Calculate the relative velocity v_{rel} between the satellite and Jupiter at the point that they meet. [3]

QUESTION CONTINUED ON NEXT PAGE

(c) In practise it is decided to adjust the impulse so that the apocentre of the satellite's orbit is slightly beyond 5 au, so that the satellite and Jupiter have a close encounter at a point when the satellite still has a small outward radial velocity. In this configuration, as it approaches Jupiter (but before its motion is affected by the gravitational attraction of Jupiter), the velocity vector of the satellite in the frame of Jupiter is inclined at an angle β to the tangential direction (i.e., the direction of Jupiter's motion). Show that if, following the dynamical encounter between the satellite and Jupiter, the tangential component of the satellite's velocity relative to Jupiter is reversed, then

$$\cos\beta = \left(1 + \frac{v_{\rm rel}^2 r_{\rm peri}}{G M_{\rm J}}\right)^{-1},$$

where $M_{\rm J}$ is the mass of Jupiter and $r_{\rm peri}$ is the distance of closest approach between the satellite and Jupiter. [5]

(d) If the mass and radius of Jupiter are $10^{-3} M_\odot$ and 7×10^4 km, use the value of $v_{\rm rel}$ from (b) to evaluate $\beta_{\rm min}$, the minimum value of β for which a collision is avoided. [3]

(e) Discuss whether the satellite can avoid a collision with Jupiter and still have enough energy to be ejected from the Solar system. [3]

Question 8Y – Topics in Astrophysics

(i) (a) Write the expression for Kepler's third law and define the terms. [1]

(b) Provide a physical description of the Hill radius of an object. [2]

(c) Show that the Hill radius, $r_{\rm H}$, for a planet of mass $M_{\rm p}$ in orbit around a star of mass $M_\star$ at a semi-major axis a is given by

$$r_{\rm H} = a\left(\frac{M_{\rm p}}{3M_\star}\right)^{1/3}. \qquad (*)$$

[7]

(ii) (a) A planet is moving on a circular orbit through a protoplanetary disk that contains pebbles. The radial drift velocity of the pebbles is given by

$$v_{r,{\rm d}} = \frac{-\eta\, v_{\rm K}}{\tau_{\rm f} + \tau_{\rm f}^{-1}},$$

where η is a dimensionless parameter, $v_{\rm K}$ is the Keplerian velocity and $\tau_{\rm f}$ is the pebble's dimensionless frictional timescale. The azimuthal velocity of the pebbles is given by

$$v_{\phi,{\rm d}} = v_{\rm K} - \frac{\tau_{\rm f}^{-1}\eta\, v_{\rm K}}{2(\tau_{\rm f} + \tau_{\rm f}^{-1})}.$$

Giving your answer in terms of η and $v_{\rm K}$, derive an approximate expression for the magnitude of the relative velocity Δv between pebbles with $\tau_{\rm f} \le 1$ and the planet as they cross its orbit. [4]

(b) Provide a physical interpretation of your result in (a). [2]

(c) Show that the velocity dispersion of pebbles, σ_v, due to Keplerian shear at the Hill radius of the planet is given by

$$\sigma_v = \frac{3}{2}\Omega_{\rm p} r_{\rm H},$$

where $\Omega_{\rm p}$ is the Keplerian angular velocity of the planet and $r_{\rm H}$ is its Hill radius. [6]

(d) Derive an expression for the transition mass for a growing planet between drift- and Hill-limited accretion of pebbles. If necessary you may assume the result in equation $(*)$ from Part (i). Give your answer in terms of Δv, $\Omega_{\rm p}$, and known constants. [6]

(e) Comment on how the transition mass for a planet's growth being drift- or Hill-limited depends on the disk's physical properties. [2]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Tues 6 June 2023 1:30pm – 4:30pm

ASTROPHYSICS - PAPER 2

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Tags	

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Question 1X – Relativity

(i) Consider the spacetime with line element

$$ds^2 = c^2dt^2 - A^2(t)dx^2 - B^2(t)\left(dy^2 + dz^2\right) .$$

Show that the non-vanishing Christoffel symbols (other than those related by the symmetry $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$) are

$$\Gamma^t_{xx} = \frac{AA'}{c^2}\,, \quad \Gamma^t_{yy} = \Gamma^t_{zz} = \frac{BB'}{c^2}\,, \quad \Gamma^x_{tx} = \frac{A'}{A}\,, \quad \Gamma^y_{ty} = \Gamma^z_{tz} = \frac{B'}{B}\,,$$

where $A' = dA/dt$ and $B' = dB/dt$. [10]

(ii) (a) For the spacetime in Part **(i)**, show that component R_{tt} of the Ricci tensor is

$$R_{tt} = \frac{A''}{A} + 2\frac{B''}{B}\,.$$ [5]

[*You may use* $R_{\mu\nu} = -\partial_\rho\Gamma^\rho_{\mu\nu} + \partial_\mu\Gamma^\rho_{\rho\nu} + \Gamma^\tau_{\rho\nu}\Gamma^\rho_{\mu\tau} - \Gamma^\tau_{\mu\nu}\Gamma^\rho_{\rho\tau}$.]

(b) Given that the other non-zero components of the Ricci tensor are

$$R_{xx} = -\frac{A^2}{c^2}\left(\frac{A''}{A} + 2\frac{A'}{A}\frac{B'}{B}\right)\,, \quad R_{yy} = R_{zz} = -\frac{B^2}{c^2}\left[\frac{B''}{B} + \frac{A'}{A}\frac{B'}{B} + \left(\frac{B'}{B}\right)^2\right]\,,$$

determine the Ricci scalar. [3]

(c) A dust-like fluid with mass density ρ is at rest in the x, y, z coordinates of this spacetime, so that the only non-zero component of the energy–momentum tensor is $T^{tt} = \rho$. Construct the Einstein field equations in terms of A and B and their derivatives. [5]

(d) Determine the possible values of the constants m and n such that

$$A(t) = A_0t^m + A_1t^n \quad \text{and} \quad B(t) = B_0t^m$$

are solutions of the Einstein field equations, where A_0, A_1 and B_0 are constants. [5]

(e) How does the density ρ evolve with time for $A_1 = 0$? [2]

Question 2Y – Astrophysical Fluid Dynamics

(i) (a) The expressions for the conservation of mass and angular momentum for an axisymmetric geometrically thin accretion disk are

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) &= 0, \\ \frac{\partial}{\partial t}(R\Sigma u_\phi) + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) - \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right) &= 0, \end{aligned}$$

where $\Sigma(R,t)$ is the surface density of the disk, $u_R(R)$ is the radial velocity, $u_\phi(R)$ is the azimuthal velocity, $\Omega(R)$ is the angular velocity of the flow, R is the radial distance, and ν is the viscosity of the flow. Show that, for accretion onto a point mass M,

$$u_R = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2}).$$

[7]

(b) By making any approximations necessary, show that the radial velocity is of order $-u_R \sim \nu/R$. [1]

(c) Hence show that the mass accretion rate is of order $\dot{M} \sim \nu\Sigma$. [2]

(ii) (a) Using any results from Part (i) necessary, show that

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right].$$

[3]

(b) Assuming hydrostatic equilibrium and an isothermal temperature profile in the vertical direction, show that the vertical thickness of the disk (in a sense that you should define more rigorously) is $H = c_\mathrm{s}/\Omega$, where c_s is the isothermal sound speed of the gas and $\Omega(R)$ is the angular velocity of the flow. [4]

(c) Use dimensional analysis to argue that $\nu = \alpha c_\mathrm{s} H$, where α is a dimensionless parameter of order or less than unity. [4]

QUESTION CONTINUED ON NEXT PAGE

(d) In some settings, accretion disks can have regions where a thermal instability (driven by the ionization of hydrogen) causes the gas to quickly transition from $T_{\rm c} \sim 10^3\,{\rm K}$ to $T_{\rm h} \sim 10^4\,{\rm K}$ over a small range of mass accretion rates $\dot{M}_1 \to \dot{M}_2$. Explain the effect of this transition on the viscosity, and why this results in an inverse relationship between Σ and ν, so that $\partial \ln \Sigma / \partial \ln \nu < 0$. [4]

(e) With reference to a sketch of $\dot{M}$ against Σ, or otherwise, explain without detailed calculation why a disk close to the ionization threshold may be subject to an instability. [5]

Question 3X – Cosmology

(i) (a) The number density of particles of species i with momentum in the range p to $p+dp$ in thermal equilibrium at temperature T is given by

$$dn_i(p) = \frac{4\pi}{h^3}\frac{g_i p^2 dp}{\left[\exp\left(\frac{E-\mu_i}{k_B T}\right) \pm 1\right]} \quad \begin{cases} +1 & \text{for fermions} \\ -1 & \text{for bosons} \end{cases},$$

where $E^2 = p^2c^2 + m_i^2c^4$, g_i is the number of spin states, k_B is the Boltzmann constant and μ_i is the chemical potential. Show that in the non-relativistic limit, the particle number density is

$$n_i = \frac{\pi^{3/2}}{h^3} g_i (2m_i k_B T)^{3/2} \exp\left[\frac{(\mu_i - m_i c^2)}{k_B T}\right]. \qquad (*)$$ [5]

[*You may assume that* $\int_0^\infty x^2 \exp(-\alpha x^2)dx = \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}$.]

(b) Discuss the implications of (*) for the relic density of massive particles created in thermal equilibrium in the early Universe. [3]

(c) Protons and neutrons, of number density n_p and n_n respectively, remain in thermal equilibrium via weak interactions until the temperature drops to $T_{\text{freeze}} \sim 0.8$ MeV. How does the baryon density $n_B = n_p + n_n$ vary with temperature during nucleosynthesis? [1]

(d) Explain your answer. [1]

(ii) (a) Discuss the evolution of the neutron-proton ratio n_n/n_p during the epoch of nucleosynthesis in the standard Big Bang Cosmology. [6]

(b) Using the result (*) of Part (i), show that during nucleosynthesis the proton chemical potential μ_p behaves as

$$\mu_p = m_p c^2 + k_B T \ln\left[\left(\frac{k_B T}{m_p c^2}\right)^{3/2} 2\zeta(3)\sqrt{\frac{2}{\pi}}\frac{\eta}{(1+n_n/n_p)}\right],$$

where η is the baryon-to-photon ratio, n_B/n_γ, the photon number density is $n_\gamma = 16\pi\zeta(3)(k_B T/hc)^3$, and $\zeta(3) = 1.202$ is a Riemann zeta function. [6]

(c) Assuming $\eta = 5 \times 10^{-10}$, estimate the values of μ_p in units of MeV at $T = 100$ MeV and $T = 1$ MeV. [5]

(d) During this period, the contribution of protons to the total entropy of the universe increases logarithmically as the temperature decreases. Give a physical explanation for this behaviour. [3]

Question 4X – Structure and Evolution of Stars

(i) (a) Two early-type stars in the same cluster start their lives on the H-burning main sequence with the same mass: $M_{\mathrm{A}} = M_{\mathrm{B}} = 5\,M_{\odot}$. Star A is single. Star B is a member of a binary system and throughout its life on the main sequence loses mass to a compact companion at an average rate $\dot{M} = 1 \times 10^{-8} M_{\odot}\ \mathrm{yr}^{-1}$. Which of the two stars will leave the main sequence first and why? [3]

(b) An astronomer equipped with a photometer and two broad-band filters, V and B, measures the following magnitudes for two stars in the constellation of Pegasus: $m_V(\alpha\,\mathrm{Peg}) = 2.45$, $m_B(\alpha\,\mathrm{Peg}) = 2.45$; and $m_V(\beta\,\mathrm{Peg}) = 2.40$, $m_B(\beta\,\mathrm{Peg}) = 4.04$. On the basis of this information alone, which of the two stars would you consider more likely to be the closer one to the Sun? What other information would you require to definitely establish which is closer? [7]

(ii) (a) The gas density within a star decreases from the centre to the surface as a function of radial distance r according to

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R}\right)^2\right]$$

where ρ_c is the core density and R is the star's radius. Find a function, $m(r)$, describing how the mass contained within radius r varies with r. [5]

(b) Derive the relation between the total mass of the star M and R. [3]

(c) What is the average density of the star in units of the core density ρ_c? [3]

(d) The gravitational potential energy of a star of mass M and radius R is given by:

$$U_g = -\alpha \frac{GM^2}{R},$$

where α is a constant of order unity determined by the distribution of matter within the star. Find the value of α for the density profile given above. [9]

Question 5Z – Statistical Physics

(i) (a) A simple one-dimensional model of a rubber molecule consists of a chain of n links, where n is fixed. Each link has a fixed length a and can be oriented in either the positive or negative direction. A unique state i of the molecule is specified by giving the orientation of each link and the molecule's length in this state is l_i. If n_+ links are oriented in the positive direction and n_- in the negative direction, then $n = n_+ + n_-$ and the length of the molecule is $l = (n_+ - n_-)a$. All links have the same energy. What is the range of possible values of l? What is the number of states of the molecule for fixed n_+ and n_-? [3]

(b) Now consider an ensemble with $M \gg 1$ copies of the molecule in which m_i members are in state i. Write down an expression for the mean length L. By introducing Lagrange multipliers τ and α show that the most probable configuration for the $\{m_i\}$ with given L is found by maximising

$$\ln\left(\frac{M!}{\prod_i m_i!}\right) + \tau \sum_i m_i\, l_i - \alpha \sum_i m_i\,.$$

Hence show that the most probable configuration has

$$p_i = e^{\tau l_i}/Z\,,$$

where p_i is the probability for finding an ensemble member in state i and $Z = \sum_i e^{\tau l_i}$ is the partition function. [7]

(ii) (a) Consider the ensemble described in Part (i). Show that Z can be expressed as

$$Z = \sum_l g(l)\, e^{\tau l}$$

where the meaning of $g(l)$ should be explained. Hence show that

$$Z = \sum_{n_+=0}^{n} \frac{n!}{n_+!n_-!} \left(e^{\tau a}\right)^{n_+} \left(e^{-\tau a}\right)^{n_-}\,,$$

where $n = n_+ + n_-$. [3]

QUESTION CONTINUED ON NEXT PAGE

(b) Show that the free energy G for the system at temperature T is

$$G = -nk_{\mathrm{B}}T \ln\left(2\cosh \tau a\right) ,$$

where k_{B} is the Boltzmann constant. G is the Gibbs free energy because the setup corresponds to a system with fixed tension rather than with a fixed length. Hence, show that

$$L = -\frac{1}{k_{\mathrm{B}}T}\left(\frac{\partial G}{\partial \tau}\right) \quad \text{and} \quad \tanh \tau a = \frac{L}{na} ,$$

where $L = \langle\, l \,\rangle$ is the average length. [7]

(c) Why is the tension f in the rubber molecule equal to $k_{\mathrm{B}}T\tau$? [6]

(d) Now assume that $na \gg L$. Show that the chain satisfies Hooke's law $f \propto L$. What happens if f is held constant and T is increased? [4]

Question 6Z – Principles of Quantum Mechanics

(i) A two-state quantum system has Hamiltonian H_0 with eigenvectors $|-\rangle$ and $|+\rangle$, and corresponding eigenvalues $E_- < E_+$. The system is perturbed by

$$\Delta H = \lambda \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} ,$$

where λ is a real constant. To first order in λ, the perturbed Hamiltonian, eigenvalues, and eigenstates are:

$$\begin{aligned} H &= H_0 + \Delta H , \\ E_\pm(\lambda) &= E_\pm + \lambda \delta E_\pm \\ |-, \lambda\rangle &= |-\rangle + \lambda \left(a_- |-\rangle + a_+ |+\rangle\right) \\ |+, \lambda\rangle &= |+\rangle + \lambda \left(b_- |-\rangle + b_+ |+\rangle\right) . \end{aligned}$$

Starting from the full Schrödinger equation, show that the first-order corrections to the energy eigenstates and eigenvalues obey $\delta E_\pm = a_- = b_+ = 0$ and $a_+ = b_- = i/(E_+ - E_-)$. Explicitly derive any necessary results of first-order perturbation theory in λ. [10]

(ii) (a) Consider the system described in Part (i). Find the exact eigenstates and eigenvalues and show that they agree with the results of perturbation theory to first order in λ. [14]

(b) Determine the radius of convergence in λ of the solution under first-order perturbation theory. [6]

Question 7Y – Stellar Dynamics and the Structure of Galaxies

(i) (a) Write down the effective potential of a particle with specific angular momentum h orbiting in a spherical potential $\Phi(r)$, explaining the physical meaning of each term. [2]

(b) Show that if the angular velocity for particles on circular orbits at radius r is $\Omega_{\rm c}(r)$, the frequency of radial oscillations $\kappa(r)$ is given by

$$\kappa(r)^2 = \frac{1}{r^3}\frac{d}{dr}\left(r^4\Omega_{\rm c}(r)^2\right). \qquad (*)$$

[6]

(c) What does this result suggest about the expected radial profile of the specific angular momentum of circular orbits in real systems? [2]

[You may assume without proof that for a particle moving in a potential $V(x)$ with an equilibrium at $x = x_0$, the frequency ω of small oscillations about $x = x_0$ is given by $\omega^2 = \frac{{\rm d}^2V}{{\rm d}x^2}|_{x=x_0}$.]

(ii) (a) A disc of stars orbits in the $z = 0$ plane of a spherical potential $\Phi = \Phi_0 \ln r$. Define the local standard of rest (LSR) at radius $r = R$ in the $z = 0$ plane and derive both the angular velocity of the LSR, and the frequency of radial oscillations about the LSR, for which you may assume equation $(*)$ without proof. Is the orbital precession prograde or retrograde? [5]

(b) Determine the frequency of azimuthal excursions about the LSR. In what direction does the orbit precess in the frame of the LSR? Provide a labelled sketch of the motion of a test particle undergoing epicyclic excursions in the frame of the LSR. [5]

(c) Consider the case that the disc of stars has a surface density profile

$$\Sigma = \Sigma_0 \exp(-R/R_0).$$

Explain why the average azimuthal velocity of particles at any location is *not* equal to the velocity of the LSR. What is the sign of the deviation? [5]

(d) Explain why the orbital eccentricity of stars increases with their age and why the gas clouds from which they form have low orbital eccentricity. Hence explain which types of stars should be used to obtain the best estimates of the LSR velocity. [5]

Question 8Y – Topics in Astrophysics

(i) (a) The number of asteroids with absolute magnitudes in the range H to $H + \mathrm{d}H$ is given by $N(H)\mathrm{d}H \propto 10^{\alpha H}\mathrm{d}H$, where α is a constant. The asteroid size distribution is such that the number with radii in the range r to $r + \mathrm{d}r$ is $N(r)\mathrm{d}r \propto r^{-q}\mathrm{d}r$, where q is a constant. Find an expression to link the constants α and q. [5]

(b) What constraints do observations of the asteroid belt place on planet formation processes and dynamics in the early Solar system? [5]

(ii) (a) A transit survey is conducted to detect habitable exoplanets, defined as those with a radius of $R_{\mathrm{p}} = 1$ Earth radius and an equilibrium temperature of $T_{\mathrm{p}} = 300\,\mathrm{K}$. Derive an expression for the transit probability of a habitable zone planet in terms of R_{p}, T_{p}, the mass of the host star $M_\star$, and known constants. You may assume a stellar mass–radius relation of $M_\star \propto R_\star$ and a stellar luminosity–mass relation of $L_\star \propto M_\star^3$, and that habitable exoplanets behave like black bodies. [7]

(b) Stars in the vicinity of the Sun have a number density of $\bar{\rho} = 1\,\mathrm{pc}^{-3}$, and the fraction of stars with masses in the range $M_\star$ to $M_\star + \mathrm{d}M_\star$ is $\propto M_\star^{-2}$ for $M_\star > 0.1 M_\odot$. Calculate the distance to which stars must be surveyed before, on average, one star with a transiting habitable exoplanet has been found. You may assume that all stars have a planet in their habitable zone and that there are no limits on the sensitivity of the transit detection. [9]

(c) What is the most probable spectral type and mass of the star the planet will be found around, and what is the depth of the transit signal that would be observed for such a system? [4]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 8 June 2023 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt all 6 questions.

Each question is divided into Part (i) *and Part* (ii), *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i). *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer for each Part on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **5Z and 6Z** *should be in one bundle and* **1X, 3X and 4X** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X – Relativity

(i) (a) An atom in an excited state has (rest) mass m^*. This atom (labelled "1") makes a transition to the ground state of mass m (labelled "2") by emitting a photon γ,

$$1 \to 2 + \gamma\,.$$

Show that the energy of the photon in the rest-frame S of the original excited atom is

$$E_\gamma = \frac{m^{*2} - m^2}{2m^*} c^2\,. \qquad [6]$$

(b) Determine the recoil speed v_2 of the atom in the ground state ("2") in the frame S. [4]

(ii) (a) For the set-up in Part **(i)**, the emitted photon subsequently collides with another atom of the same type, which is in the ground state (labelled "3"). The photon is absorbed and the atom transitions to the excited state with mass m^* (labelled "4"), i.e.,

$$3 + \gamma \to 4\,.$$

Using conservation of 4-momentum, express $p_{3\parallel} \equiv \hat{e} \cdot \vec{p}_3$ in terms of the energy E_3 of the atom "3". Here, $\hat{e}$ is the unit vector in the direction of propagation of the photon and $\vec{p}_3$ is the relativistic 3-momentum of atom "3", all quantities being defined in the frame S. [6]

(b) By considering the energy–momentum invariant for atom "3", $E_3^2 - |\vec{p}_3|^2 c^2 = m^2 c^4$, show that the condition on E_3 that is required for the atom to absorb the photon and transition to the excited state "4" is

$$E_3 \geq \frac{m^2 + m^{*2}}{2m^*} c^2\,. \qquad [6]$$

(c) What is this condition in terms of the speed of atom "3"? [6]

(d) How does the minimum possible speed of atom "3" compare to the recoil speed v_2 of atom "2" determined in Part **(i)**? [2]

Question 2Y – Astrophysical Fluid Dynamics

(i) (a) Consider an equilibrium consisting of a uniform plasma of density ρ_0 threaded by a uniform field $\mathbf{B}_0 = B_0\hat{\mathbf{x}}$. Show that, for perturbations with $\mathbf{k} = k_y\hat{\mathbf{y}}$, the magnetic force term in the MHD equations becomes $-(1/\mu_0)\nabla(B^2/2)$. [4]

(b) By considering the direct analogy to sound waves, show that these waves are dispersion-free and propagate with speed $(c_\mathrm{s}^2 + B_0^2/\mu_0)^{1/2}$. [6]

[You may assume without proof that the linear perturbations of the quantity $\nabla[p + (1/\mu_0)B^2/2]$ are $(c_\mathrm{s}^2 + B_0^2/\mu_0)\nabla(\delta\rho)$, where p is the gas pressure, c_s is the sound speed, and $\delta\rho$ is the perturbation in density.]

(ii) (a) Consider a small patch of a plasma of density ρ_0 that, in its equilibrium state, is rotating about the z-axis with angular velocity $\Omega(R)$, where R is the distance from the axis. The plasma is threaded by a weak magnetic field $\mathbf{B}_0$ which is aligned with the axis of rotation. With some simplifying assumptions the dispersion relation governing the evolution of axisymmetric perturbations is

$$\omega^4 - \omega^2\left[4\Omega^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} + 2(\mathbf{k}\cdot\mathbf{v}_\mathrm{A})^2\right] + (\mathbf{k}\cdot\mathbf{v}_\mathrm{A})^2\left[(\mathbf{k}\cdot\mathbf{v}_\mathrm{A})^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)}\right] = 0,$$

where the perturbation in quantity X is assumed to have form $\delta X \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ and $\mathbf{v}_\mathrm{A} \equiv \mathbf{B}_0/\sqrt{\rho_0\mu_0}$. With the aid of a diagram, explain why

$$(\mathbf{k}\cdot\mathbf{v}_\mathrm{A})^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}(\ln R)} < 0$$

is a sufficient but not necessarily required condition for instability. [5]

(b) We now assume that the rotation profile is Keplerian, $\Omega \propto R^{-3/2}$. Show that the fastest growing unstable mode has growth rate $\Gamma = 3\Omega/4$, and occurs when $(\mathbf{k}\cdot\mathbf{v}_\mathrm{A})^2 = 15\Omega^2/16$. [10]

(c) Sketch the growth rate Γ as a function of $|(\mathbf{k}\cdot\mathbf{v}_\mathrm{A})|$. [5]

Question 3X – Cosmology

(i) (a) Consider a Friedmann-Robertson-Walker universe with negligible spatial curvature and scale factor $R(t)$ that is dominated by a uniform scalar field with equation of state

$$P = -\rho c^2. \qquad (*)$$

Show that the Einstein field equations are solved if

$$R(t) \propto \exp(Ht),$$

where H is a constant. [3]

(b) Show that the scale factor can be written in terms of the conformal time $\tau = \int dt/R(t)$ as

$$R(\tau) = -\frac{1}{H\tau},$$

and that the singularity, $R = 0$, is pushed back to $\tau = -\infty$. [3]

(c) Explain how an inflationary phase in the early Universe driven by a scalar field with equation of state (*) solves the horizon problem of conventional Big Bang cosmology. [4]

(ii) (a) Observations of distant Type Ia supernovae have shown that the Universe at late times is experiencing accelerated expansion $d^2R/dt^2 > 0$. If the Universe is dominated by dark energy with equation of state,

$$P_{\rm DE} = -w\rho_{\rm DE}c^2, \qquad (**)$$

where w is a constant, show that accelerated expansion requires $w < -1/3$. [2]

(b) Show that the density of dark energy with equation of state (**) evolves as

$$\rho_{\rm DE} \propto R^{-3(1+w)}.$$

[3]

QUESTION CONTINUED ON NEXT PAGE

(c) If the universe is spatially flat and is composed of non-relativistic matter and dark energy with density parameters at time t_0 of Ω_m and $\Omega_{\rm DE} = (1-\Omega_m)$ respectively, show that if the dark energy equation of state parameter w is less than -1, the scale factor goes to infinity in a finite time,

$$t_{\rm rip} = \frac{1}{H_0 \Omega_m^{1/2}} \frac{1}{(-3w)} \left(\frac{1-\Omega_m}{\Omega_m} \right)^{\frac{1}{2w}} \frac{\Gamma\left(-\frac{1}{2w}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2w}\right)}{\Gamma(\frac{1}{2})},$$

where H_0 is the Hubble parameter at time t_0 and $\Gamma(x)$ is the Gamma function. [11]

[*You may assume that* :

$$\int_0^\infty \frac{x^{1/2}dx}{(1+\beta x^\gamma)^{1/2}} = \frac{1}{\gamma}\beta^{-\frac{3}{2\gamma}} \frac{\Gamma\left(\frac{3}{2\gamma}\right)\Gamma\left(\frac{1}{2}-\frac{3}{2\gamma}\right)}{\Gamma(\frac{1}{2})}.$$

]

(d) Give a physical interpretation of the result in Part (ii)(c). [4]

Question 4X – Structure and Evolution of Stars

(i) (a) Sketch the behaviour of the radius of the Sun as a function of age. Set the origin as the time the Sun first appears on the Hertzsprung-Russell diagram. Annotate the plot to indicate the evolutionary phases corresponding to significant changes in the Solar radius. [10]

(ii) (a) Show that the equations of mass continuity and hydrostatic equilibrium can be combined into the second-order differential equation:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right] = -4\pi G\rho.$$ [4]

(b) For a gas with an equation of state, $P = K\rho^\gamma$, where K is a constant, use the above equation to derive a second-order differential equation involving only density and radius. Using the dimensionless variables $r' \equiv r/R_*$ and $\rho' \equiv \rho/\rho_0$, show that the term $K\rho_0^{\gamma-2}/R_*^2$ is a dimensionless constant, and hence that $R_* \propto \rho_0^{\gamma/2-1}$. [10]

(c) White dwarfs obey the equation of state $P = K\rho^\gamma$, with $\gamma = 5/3$ for non-relativistic conditions and $\gamma = 4/3$ in the relativistic regime. Using the above result, $R \propto \rho_0^{\gamma/2-1}$, show that for non-relativistic white dwarfs $R \propto M^{-1/3}$, while for relativistic white dwarfs R is independent of M. [4]

(d) What is the significance of the mass-radius relations of Part (ii)(c)? [2]

Question 5Z – Statistical Physics

(i) (a) State the formula for the Bose-Einstein distribution for the mean occupation numbers n_r of discrete single-particle states r with energies $E_r \geqslant 0$ in a gas of identical ideal bosons in terms of the chemical potential μ and $\beta = 1/(k_{\mathrm{B}}T)$, where k_{B} is the Boltzmann constant, and T is the temperature. Write down expressions for the total particle number N and the total energy E when the single-particle states can be treated as continuous with energies $E \geqslant 0$ and density of states $g(E)$. [3]

(b) Consider the bosonic vibrational modes (phonons) in a two-dimensional crystal with dispersion relation $\omega = C|\mathbf{k}|^{\alpha}$, where ω is the frequency, $\mathbf{k}$ is the wavevector, and $C > 0$ and $0 < \alpha < 2$ are constants. The crystal is square with area A. Show that the density of states is

$$g(\omega) = B\omega^{b}\,, \qquad (*)$$

where B and b are constants that you should determine. You may assume that the phonons have two polarizations. [7]

(ii) (a) Consider the crystal described in Part (i)(b). Calculate the Debye frequency ω_{D} by identifying the number of single-phonon states with the total number of degrees of freedom, $2n$, where n is the number of atoms in the crystal. What is the Debye temperature T_{D}? You may leave your answers in terms of B and b, as defined in $(*)$, and other constants. [6]

(b) Derive an expression for the total energy, leaving your answer in integral form with the integral over $x = \beta\hbar\omega$. You may also leave your answer in terms of B and b, as defined in $(*)$, and other constants. [3]

(c) Now consider the case $b = 3$. Calculate the heat capacity at constant volume C_V in the limit that the temperature $T \gg T_{\mathrm{D}}$. Show that $C_V \sim T^d$ in the limit $T \ll T_{\mathrm{D}}$, where d is a real number that you should determine. Comment on these two results. [11]

Question 6Z – Principles of Quantum Mechanics

(i) (a) Consider a composite system of several distinguishable particles. Describe how the multiparticle state is constructed from single-particle states. [3]

(b) For the case of two identical particles, describe how considering the interchange symmetry leads to the definition of bosons and fermions. [7]

(ii) (a) Consider two non-interacting, identical particles, each with spin 1. The single-particle, spin-independent Hamiltonian $H(\mathbf{X}_i, \mathbf{P}_i)$ has non-degenerate eigenvalues E_n, labelled by $n \in \{0, 1, 2, 3, \dots\}$, and corresponding wavefunctions $\psi_n(\mathbf{x}_i)$, where $i \in \{a, b\}$ labels each particle. In terms of these single-particle wavefunctions and single-particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the two-particle states and energies for the ground state and the first excited state. [10]

(b) Assume now that E_n is a linear function of n. Find the degeneracy of the N^{th} energy level of the two-particle system for N even and N odd. [10]

Question 7Y – Stellar Dynamics and the Structure of Galaxies

(i) (a) The distribution function for a spherical system is described by

$$f = A\epsilon^{n-3/2}$$

for $\epsilon > 0$, and $f = 0$ otherwise, where $\epsilon = \Phi_0 - E$, E is the particle energy per unit mass, Φ_0 is the gravitational potential at the outer edge of the system, and A and n are constants. Explain why this distribution function implies that the particle velocities are isotropic. [1]

(b) The potential Ψ is defined by $\Psi = \Phi_0 - \Phi$, where Φ is the gravitational potential. Derive the form of the power law relationship between ρ and Ψ for this distribution function. [4]

(c) The first moment Jeans equation for a steady state spherically symmetric system with isotropic velocity dispersion σ is given by

$$-\rho\nabla\Phi = \nabla P, \qquad (*)$$

where $P = \rho\,\sigma^2$. Use this to derive the form of the power law relationship between P and ρ for this distribution function and comment on your results. [4]

(d) How is P to be understood in the context of a stellar system? [1]

(ii) (a) The potential generated by a spherically symmetric star cluster is given by

$$\Phi = \Phi_0 \left(1 + (r/b)^2\right)^{-1/2},$$

where Φ_0 and b are constants. Evaluate the density distribution $\rho(r)$ that produces this potential. [4]

(b) If $\rho(0) = 10^4\, M_\odot\,\mathrm{pc}^{-3}$ and $b = 1\,\mathrm{pc}$, determine the value of Φ_0. Show that a black hole falling radially inwards with velocity at infinity of $v_\infty = 200\,\mathrm{km\,s^{-1}}$ is accelerated by less than 1% when it passes through the centre of the cluster. [4]

(c) Under the assumption that the stellar velocity distribution is isotropic, use the Jeans equation $(*)$ from Part (i) to determine the pressure at the cluster centre, and hence evaluate the velocity dispersion of the stars at $r = 0$. Hence show that the relative velocity v_{rel} between the black hole and the stars in the cluster is dominated by the motion of the black hole. [6]

QUESTION CONTINUED ON NEXT PAGE

(d) Stars are swallowed by the black hole if they pass within its Schwarzschild radius $R_{\rm s} = 2GM/c^2$, where M is the black hole's mass. Assuming that $M = 10^3 M_\odot$ and that the typical stellar mass is $1 M_\odot$, estimate the number of stars that the black hole will swallow as it crosses the cluster. [6]

[You may assume without proof that the black hole's cross-section for swallowing stars is given by $\pi R_{\rm s}^2 \left(1 + \frac{2GM}{R_{\rm s} v_{\rm rel}^2}\right)$*.]*

Question 8Y – Topics in Astrophysics

(i) (a) Consider a planet of mass M that is orbited at a distance a by a moon of mass m. The planet is spinning with angular frequency Ω and has a moment of inertia I. The moon's spin can be neglected. Show that the total angular momentum in the system is given by

$$J_{\text{tot}} = I\Omega + mM\sqrt{\frac{Ga}{M+m}}. \qquad [3]$$

(b) Hence show that

$$\dot{a} = -2I\dot{\Omega}/(\mu a\omega),$$

where the dot denotes a differentiation with respect to time, $\mu = mM/(M+m)$ is the reduced mass, and ω is the angular velocity of the planet-moon system about its centre of mass. [3]

(c) It has been measured that the Moon is receding from the Earth at a rate of 3.8 cm per year. Use this to estimate the rate of change of the length of the day in ms per year. You may assume that the Earth has a uniform density and so has a moment of inertia of $(2/5)M_{\oplus}R_{\oplus}^2$, and that the Moon is 80 times less massive than the Earth. [4]

(ii) (a) Consider the general planet-moon system described in Part (i) and use any equations therein. Give an expression for the total energy E in the system and hence show that the rate of change of this energy is given by

$$\dot{E} = I\dot{\Omega}(\Omega - \omega). \qquad [3]$$

(b) Henceforth consider the Earth-Moon system. By considering its angular momentum, or otherwise, show that

$$\Omega/\omega \approx A(a/a_{\text{now}})^{3/2} - B(a/a_{\text{now}})^2,$$

where a is the Earth-Moon distance at any given time, with a_{now} as its present day value, and A and B are constants whose numerical values you should estimate from the information provided in Part (i)(c). [4]

(c) Explain the physical meaning of Ω/ω and sketch its dependence on orbital separation, quantifying wherever possible, and commenting on its past and future evolution. [5]

QUESTION CONTINUED ON NEXT PAGE

(d) Estimate the current rate of energy loss due to tides and comment on how this energy is lost from the system. [3]

(e) Compare this with the energy input from solar radiation to estimate the resulting increase in the Earth's equilibrium temperature. [3]

(f) The energy in the tide raised on the Earth by the Moon is $\sim Gm^2\, R_\oplus^5\, a^{-6}$, where $R_\oplus$ is the Earth's radius and m is the mass of the Moon. Without detailed calculation, comment on the rate of energy loss due to tides shortly after the Moon formed. [2]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 9 June 2023 9:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

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Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X – Relativity

(i) (a) A particle moves around a circle of radius r with constant angular speed ω in some inertial frame S. The circle lies in the plane $z = 0$ and is centred on the origin. Write down the worldline of the particle, expressing time ct and the Cartesian coordinates x, y and z as functions of the particle's proper time τ. [3]

(b) Define the *acceleration 4-vector* and determine its components in S for the particle above. [3]

(c) Find the magnitude of the 3D acceleration of the particle in its instantaneous rest frame. [4]

(ii) (a) An observer $\mathcal{O}$ and a distant star are at rest in an inertial frame in Minkowski spacetime. The star subtends a small solid angle $d\Omega$ at the observer. The specific intensity of the radiation from the star is $I(\nu)$, as measured by $\mathcal{O}$, such that the power arriving per area dA (perpendicular to the line of sight to the star), per frequency interval $d\nu$ around ν and per solid angle $d\Omega$ is $I(\nu)d\nu dA d\Omega$. At some instant, a second observer $\mathcal{O}'$ is coincident with $\mathcal{O}$ but moving at speed βc along the line of sight towards the star. Determine the ratio of frequencies ν'/ν and of solid angles subtended by the star, $d\Omega'/d\Omega$, where primes denote quantities measured by $\mathcal{O}'$, and verify that $\nu d\nu d\Omega$ is the same for both observers. [8]

(b) By considering the rate of arrival of photons per area for both observers, show that the specific intensity measured by $\mathcal{O}'$ satisfies

$$\frac{I'(\nu')}{\nu'^3} = \frac{I(\nu)}{\nu^3} \,. \qquad [6]$$

(c) Describe the appearance of the Sun to an observer passing Earth with speed $0.5c$ towards the Sun. Include a discussion of the apparent surface temperature and its angular size. You may assume that the Sun radiates as a blackbody at temperature 5800 K in its rest frame and has an angular diameter of $0.5°$ as seen from the Earth. [6]

Question 2Y – Astrophysical Fluid Dynamics

(i) (a) A small spherical asteroid of density $\rho_{\rm a}$ and radius R enters the Earth's atmosphere at an angle α relative to the vertical and a highly supersonic speed v. It disintegrates once it has passed through a column of air with mass camparable to that of the asteroid. By treating the Earth's atmosphere to be isothermal and plane-parallel, show that the asteroid disintegrates at a height above the surface of

$$z_{\rm dis} \approx H \ln\left[\frac{3}{4\cos\alpha}\left(\frac{H}{R}\right)\left(\frac{\rho_0}{\rho_{\rm a}}\right)\right],$$

where ρ_0 is the density at the base of the Earth's atmosphere, $H = c_{\rm s}^2/g$ is the scale-height of the Earth's atmosphere, $c_{\rm s}$ is the isothermal sound speed of the atmosphere, and g is the acceleration due to gravity close to the Earth's surface. [8]

(b) Derive an approximate condition on the size that the asteroid needs to be to impact the surface. [2]

(ii) (a) The asteroid of Part (i) disintegrates at a height $z_{\rm dis}$ above the ground, dumping its entire kinetic energy into a small volume of air. This results in a fireball, a hot and low density expanding bubble driving a shock front into the surrounding air. By using dimensional analysis, or otherwise, and any results from Part (i) necessary, and assuming that radiative cooling of the heated air is negligible, show that the radius of the fireball evolves at early time according to

$$r = \xi_0 \left(\frac{\rho_{\rm a} R^3 v^2 t^2}{\rho_0}\right)^{1/5} \exp\left(\frac{z_{\rm dis} g}{5c_{\rm s}^2}\right),$$

where t is the time elapsed since the disintegration event, and ξ_0 is a dimensionless constant that you do not need to determine. [6]

(b) The fireball "stalls" when its expansion velocity drops to the speed of sound $c_{\rm s}$ in the ambient atmosphere. Assuming $\xi_0 = 1$, determine the maximum size reached by the fireball when a 5 m asteroid of density $3500\,{\rm kg\,m^{-3}}$ traveling with a Mach number of 50 disintegrates at a height where the density of air is $1.0\,{\rm kg\,m^{-3}}$. [8]

(c) Qualitatively describe the subsequent evolution of both the shock and the hot gas making up the fireball. [6]

Question 3X – Cosmology

(i) (a) Consider a spherical cloud of characteristic density ρ and sound speed c_s. Use dimensional arguments to show that gravity will dominate over pressure forces if the size of the gas cloud exceeds the Jeans length

$$\lambda_J \sim c_s \left(\frac{1}{G\rho}\right)^{1/2}.$$

[3]

(b) Prior to recombination, baryons and radiation are tightly coupled by Thomson scattering and behave like a single fluid with sound speed

$$c_s = \frac{c}{\sqrt{3}}\left(\frac{3}{4}\frac{\overline{\rho}_b}{\overline{\rho}_\gamma} + 1\right)^{-1/2},$$

where $\overline{\rho}_b$ is the mean baryon density and $\overline{\rho}_\gamma$ is the mean radiation density. Assuming $\overline{\rho}_b/\overline{\rho}_\gamma = 920/(1+z)$, estimate the comoving size of the Jeans length just prior to recombination which occurs at redshift $z \approx 1000$ when the Universe is about 380,000 years old. [4]

(c) Discuss briefly the signatures of acoustic oscillations in the cosmic microwave background and in the matter distribution. [3]

(ii) (a) In the absence of pressure, linear perturbations in the matter distribution in a Friedmann-Robertson-Walker universe satisfy the following equation:

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} - 4\pi G\overline{\rho}_m\delta = 0,$$

where δ is the fractional matter overdensity $\delta = (\rho_m - \overline{\rho}_m)/\overline{\rho}_m$, $\overline{\rho}_m$ is the mean matter density, $H = R^{-1}dR/dt$ is the Hubble parameter and R is the scale factor. Show that the growth function

$$f = \frac{d\ln\delta}{d\ln R},$$

satistfies the linear perturbation equation

$$\frac{df}{d\ln R} + f^2 + \left[2 + \frac{1}{2}\frac{d\ln H^2}{d\ln R}\right]f - \frac{3}{2}\Omega_m(z) = 0, \qquad (*)$$

where $\Omega_m(z)$ is the matter density parameter at redshift z. [6]

QUESTION CONTINUED ON NEXT PAGE

(b) Assume that the universe is spatially flat and dominated by non-relativistic matter and a cosmological constant with present day density parameters $\Omega_m(0)$ and $\Omega_\Lambda(0)$, where $\Omega_\Lambda(0) = 1 - \Omega_m(0)$. As a trial solution write $f = \Omega_m(z)^\gamma$, where γ is a constant. Show that (*) requires

$$3\gamma\Omega_m(z)^{\gamma-1}[\Omega_m(z)^2 - \Omega_m(z)] \qquad +$$

$$\Omega_m(z)^{2\gamma} + \left[2 - \frac{3}{2}\Omega_m(z)\right]\Omega_m(z)^\gamma - \frac{3}{2}\Omega_m(z) = 0. \qquad (**)$$

[9]

(c) Equation (**) is satisfied if:

1. $\gamma \approx 0.550$ for $\Omega_m(z) = 0.70$;
2. $\gamma \approx 0.560$ for $\Omega_m(z) = 0.30$;
3. $\gamma \approx 0.575$ for $\Omega_m(z) = 0.10$.

Verify the solution for case 2. [2]

(d) Discuss how these results can be used to approximate the true linear growth function $f(z)$ for this cosmology. [3]

Question 4X – Structure and Evolution of Stars

(i) (a) Explain what astronomers mean by the term *'proper motion'*. [2]

(b) The Gaia satellite has measured the following parallaxes for two stars: $\pi(\alpha\,\mathrm{Cam}) = 0.1\,\mathrm{arcsec}$ and $\pi(\beta\,\mathrm{Cam})$=0.001 arcsec. Which of the two stars would you expect to show the higher proper motion and why? [2]

(c) Two solar-type stars are both at a distance of 50 pc from the Sun. One star is a member of the halo population, while the other is a disk star. Which of the two stars would you expect to show the higher proper motion? Which other physical properties would you expect to be different between the two stars? [3]

(d) A star at a distance of 10 pc is travelling at $5\,\mathrm{km\ s^{-1}}$ along a path perpendicular to our line of sight. What is its proper motion, in seconds of arc per century? [3]

(ii) (a) The gravitational binding energy of a star of mass M and radius R is given by:

$$U = -\frac{\alpha G M^2}{R},$$

where α is a constant. Such a star contracts homologously at constant effective temperature, radiating at a rate $L(t)$.

Derive expressions for $L(t)$ and $R(t)$ for a star of fixed mass which at time $t = 0$ had $L = L_0$ and $R = R_0$. [10]

(b) Show that at late times L and R display power law dependences on time. [5]

(c) Where would such a star be found in the Hertzsprung-Russell diagram? [5]

Question 5Z – Statistical Physics

(i) Give *Clausius' statement* of the second law of thermodynamics and *Kelvin's statement* of the second law of thermodynamics. Show that these two statements are equivalent. [10]

(ii) (a) Consider a classical ideal gas and assume that the number of particles N is fixed. Write down the equation of state for the ideal gas. Write down an expression for its internal energy in terms of the heat capacity C_V at constant volume. [2]

(b) Describe the meaning of an *adiabatic process*. Using the first law of thermodynamics, derive the relationship between pressure p and volume V for an adiabatic process occurring in the ideal gas. [5]

(c) Consider a Diesel cycle involving the ideal gas and consisting of the following four reversible steps:

$A \to B$: Adiabatic compression;

$B \to C$: Expansion at constant pressure with heat in Q_1;

$C \to D$: Adiabatic expansion;

$D \to A$: Cooling at constant volume with heat out Q_2.

Sketch this cycle in the (p, V)-plane and in the (T, S)-plane. Derive equations for the curves DA and BC in the temperature-entropy (T, S)-plane. [8]

(d) For this Diesel cycle, derive an expression for the efficiency, $\eta = W/Q_1$, where W is the work out, in terms of the temperatures at points A (T_A), B (T_B), C (T_C), and D (T_D). [5]

Question 6Z – Principles of Quantum Mechanics

(i) (a) A composite system is made of two sub-systems with total angular momentum j_1 and j_2, respectively. Let $\mathbf{J} = \{J_x, J_y, J_z\}$ be the angular momentum operator of the composite system and $|j, m\rangle$ a basis of eigenstates of $\mathbf{J}^2$ and J_z. Write $\mathbf{J}$ and the associated ladder operators $J_\pm$ in terms of the angular momentum operators $\mathbf{J}_{1,2}$ of each sub-system. [*Use units in which $\hbar = 1$ throughout this Question.*] [2]

(b) State the possible values of j in terms of j_1 and j_2 and show that $j = 0$ only if $j_1 = j_2$. [3]

(c) Write down all the states with $m \geq j_1 + j_2 - 1$ in terms of the states of the sub-systems. [5]

[*The states $|j, m\rangle$ obey $J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$.*]

(ii) (a) Consider the system described in Part (i). Explain why, if it exists, the state with $j = 0$ must be of the form

$$|0, 0\rangle = \sum_{m_1 = -j_1}^{j_1} \alpha_{m_1} |j_1, m_1\rangle_1 |j_1, -m_1\rangle_2 \ .$$

[3]

[*Use units in which $\hbar = 1$ throughout this Question.*]

(b) By considering $J_+ |0, 0\rangle$, determine a relation between α_{m_1+1} and α_{m_1}, and hence find $|\alpha_{m_1}|$. [12]

(c) If the system is in the state $|j_1, j_1\rangle_1 |j_1, -j_1\rangle_2$, compute the probability of measuring zero for the combined total angular momentum. [5]

[*The states $|j, m\rangle$ obey $J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$.*]

Question 7Y – Stellar Dynamics and the Structure of Galaxies

(i) (a) The first moment Jeans equation describing the steady state structure of an axisymmetric stellar distribution is given by

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\rho R\overline{v_R v_z}\right) + \frac{\partial}{\partial z}\left(\rho\overline{v_z^2}\right) + \rho\frac{\partial\Phi}{\partial z} = 0.$$

Explain how each of these terms describes the change of the z-momentum per unit volume and identify the physical source of each term. Why is the first term sub-dominant in the Galactic disc compared with the other two terms? [4]

(b) Explain how the second term can be evaluated by an astronomer located at the mid-plane of the Galactic disc using a population of stars located at different distances in a direction perpendicular to the disc plane. Your answer should mention the observations that should be made and how they can be used to construct this term. [4]

(c) Describe an astronomical issue that can be addressed once this term has been evaluated. [2]

(ii) (a) A binary star, with components of masses $0.7M_\odot$ and $0.5M_\odot$ in a circular orbit with separation 2 au, travels at velocity $10\,\mathrm{km\,s^{-1}}$ in the core of a globular cluster. The binary undergoes an exchange interaction with a star of mass $0.7M_\odot$ travelling at $5\,\mathrm{km\,s^{-1}}$. After the interaction, the $0.5M_\odot$ star is released from the original binary and a new binary traveling at $6\,\mathrm{km\,s^{-1}}$ is formed consisting of the two stars each of mass $0.7M_\odot$, with an orbital period of 1 year. If the escape velocity from the cluster core is $15\,\mathrm{km\,s^{-1}}$, describe the fate of the $0.5M_\odot$ star. [10]

(b) Discuss the role of interactions between binary and single stars in driving the evolution of globular clusters. [5]

(c) The binary undergoes further interactions with stars in the cluster core which raise the eccentricity of the binary while conserving energy. What is the eccentricity threshold which results in a stellar collision at pericentre if the stars have radii of $0.75R_\odot$? [2]

(d) How would the effect of the collision be observable to astronomers long after the event? [3]

Question 8Y – Topics in Astrophysics

(i) (a) A cluster of N stars has a mass function $f(m)$, defined such that $f(m)\,dm$ is the number of stars with masses in the range m to $m+dm$. Assume that the mass function is

$$f(m) = \begin{cases} 0, & \text{for } m < m_{\rm min}, \\ k(m/m_{\rm min})^{-1.3}, & \text{for } m_{\rm min} < m < m_{\rm t}, \\ k(m_{\rm t}/m_{\rm min})^{-1.3}(m/m_{\rm t})^{-2.3} & \text{for } m > m_{\rm t}, \end{cases}$$

where k is a constant, $m_{\rm min}$ is the minimum mass, and $m_{\rm t}$ is the mass at the transition between the two power laws. Derive an expression for the total number of stars in the distribution in the range $m_{\rm min}$ to $m_{\rm max} > m_{\rm t}$, and hence show for $m_{\rm min} \ll m_{\rm t}$ that $N \approx k\, m_{\rm min}/0.3$. [5]

(b) By considering the total number of stars above a given mass $m' > m_{\rm t}$ that would be expected from this mass function, determine the mass $m_{\rm max}$ of the most massive star expected to be found in the cluster. [3]

(c) Hence, determine the number of stars in the cluster required for it to be likely to contain at least one star of mass sufficient to become a core collapse supernova, which requires a mass of $> 20\, M_\odot$. You may assume that $m_{\rm min} = 0.1\, M_\odot$ and $m_{\rm t} = 0.5\, M_\odot$. [2]

(ii) (a) A young star, named X3a, has been observed at 2.6 arcsec from the $4 \times 10^6 M_\odot$ supermassive black hole (SMBH) at the centre of the Milky Way Galaxy, which is at a distance of 8 kpc from the Sun. Its projected separation from the SMBH was found to change by 0.065 arcsec over 10 years, and spectroscopic measurements found its radial velocity to be 49 km/s relative to the SMBH. Assuming its orbit to be circular, determine its distance from the SMBH in parsecs. [4]

(b) Without detailed calculation, comment on the orientation of the orbit with respect to our line of sight and comment on whether these measurements are likely to be compatible with a circular orbit. [3]

(c) The mass of X3a is $15\, M_\odot$. An additional much fainter source, X3b, is detected near X3a at a projected separation of 0.12 arcsec. Do you expect the two objects to be gravitationally bound to each other? [2]

QUESTION CONTINUED ON NEXT PAGE

(d) It is suggested that X3a formed in the nearby cluster IRS13, subsequently becoming gravitationally unbound from it. That cluster is currently at 1 arcsec projected separation from X3a and at a similar physical distance from the SMBH. Observations show IRS13 to be made up of 7 massive stars with total mass $\sim 100\,M_\odot$ within a region of angular diameter 0.5 arcsec. Explain why there is likely a black hole at the centre of the cluster and provide a constraint on its mass. [4]

(e) The age of X3a is determined to be 0.04 Myr. Estimate the ejection velocity from the cluster and so comment on the plausibility of X3a's formation in IRS13. [3]

(f) The cluster is assumed to have formed at ~ 10 parsec from the SMBH, having migrated inwards due to interactions with nearby objects also orbiting the SMBH. Describe the dynamical processes at work within the cluster that might have contributed to the scenario outlined in parts (d) and (e) above. [4]

[You may assume that the Roche lobe of an object of mass m, orbiting at a distance of R from another object of mass $M \gg m$, has a radius $R\left(\frac{m}{3M}\right)^{1/3}$.]

END OF PAPER