NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 08 June 2021 10:00am – 1:00pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.

Answers must be uploaded in separate pdf files, named with your candidate number, paper number, and the letter **X, Y, Z**, *according to the letter associated with the questions, each separated by an underscore. For example,* **1X, 3X, and 6X** *should be in the same file and* **5Z, 7Z, and 8Z** *in another, each file labelled* **1234A_paper1_X** *or* **1234A_paper1_Z**, *where '1234A' should be replaced with your candidate number.*

After the exam there will be **30 minutes** *for you to scan and upload your solutions to ProctorExam. During this time you* **may not do any writing**, *and* **must complete the online cover sheet** *linked to in the ProctorExam instructions, identifying all the questions you have attempted.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Paper (blank sheets, plain or lined)	*Astrophysics Formulae Booklet (as pdf)*
1 Rough Work Pad	*Approved Calculators Allowed*

You can begin the paper provided you have already received instructions to start the exam from the invigilator.
Stop writing when indicated to by the invigilator, or following your allotted time.

Question 1X - Relativity

(i) An accelerated, charged particle radiates energy in its instantaneous rest frame (IRF) at a rate

$$\mathcal{P}_0 = \frac{\mu_0 q^2 |\vec{a}_{\mathrm{IRF}}|^2}{6\pi c} ,$$

where q is the charge and $|\vec{a}_{\mathrm{IRF}}|$ is the magnitude of the 3D acceleration in the IRF. The radiation pattern in the IRF is symmetric under inversion through the position of the particle. By considering the total 4-momentum of the radiation emitted in the IRF in a short time interval, and Lorentz transforming, show that the rate at which energy is radiated is Lorentz invariant. [6]

Hence show that in a general inertial frame, the particle is instantaneously radiating at a rate

$$\mathcal{P} = -\frac{\mu_0 q^2}{6\pi c} a^\mu a_\mu ,$$

where a^μ is the particle's acceleration 4-vector. [4]

(ii) A photon of frequency ν propagating along the x-axis of an inertial reference frame scatters off an electron, with total energy E, moving along the negative x-direction. Show that the frequency of the scattered photon, $\bar{\nu}$, as a function of the angle that the photon scatters through, θ, is given by

$$\bar{\nu} = \frac{\nu(E + pc)}{E + pc\cos\theta + h\nu(1 - \cos\theta)} ,$$

where p is the magnitude of the initial 3-momentum of the electron and h is Planck's constant. [8]

If $h\nu < pc$, for what angle θ is $\bar{\nu}$ maximised? Determine this maximum frequency $\bar{\nu}_{\max}$. [2]

In an active galactic nucleus, thermal radiation emitted from the accretion disk at temperature $10^5\,\mathrm{K}$ scatters off free electrons in the corona, each with kinetic energy $0.2\,\mathrm{MeV}$. Estimate the maximum frequency of the radiation after scattering once. [4]

In the limit that $E \gg m_e c^2$, where m_e is the electron mass, and assuming $\gamma h\nu \ll m_e c^2$ (where γ is the Lorentz factor of the incident electron), show that

$$\bar{\nu}_{\max} \approx 4\nu\gamma^2 .$$ [6]

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a fluid with uniform density ρ_0 and uniform pressure p_0 in a static equilibrium (no gravitational field present). Let the fluid have a kinematic viscosity ν and zero bulk viscosity. Starting from the standard form of the Navier-Stokes equation, show that the dispersion relation for isothermal, compressible perturbations of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ is

$$\omega^2 + \frac{4}{3}i\omega\nu k^2 - c_s^2 k^2 = 0,$$

where k is the modulus of the vector $\mathbf{k}$. You should provide a suitable definition of c_s^2 as part of your answer. [7]

Compute the critical wavelength λ_{crit} in terms of ν and c_s that separates propagating from non-propagating modes. [3]

(ii) Consider an incompressible fluid with uniform density ρ_0 and uniform pressure p_0 that is in a static equilibrium *in a frame of reference that is rotating with constant angular velocity* $\boldsymbol{\Omega}$. Let the fluid have a kinematic viscosity ν and zero bulk viscosity. Show that perturbations satisfy

$$\frac{\partial\, \delta\mathbf{w}}{\partial t} = 2(\boldsymbol{\Omega}\cdot\nabla)\delta\mathbf{u} + \nu\nabla^2\delta\mathbf{w},$$

where $\delta\mathbf{w}$ is the perturbation in vorticity corresponding to the perturbation in fluid velocity $\delta\mathbf{u}$. [5]

In the case where $\nu = 0$ and the z-axis is aligned with the direction of $\boldsymbol{\Omega}$, show that plane wave perturbations obey the dispersion relation

$$\omega = \pm 2\Omega\frac{|k_z|}{k}.$$ [8]

Show that the modes of the plane wave perturbations are transverse. [2]

Without detailed calculation, write down the corresponding dispersion relation for the case $\nu \neq 0$ and interpret the behaviour. [5]

[*The Navier-Stokes equation for an isothermal fluid in a frame of reference rotating at angular velocity* $\boldsymbol{\Omega}$ *is*

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + 2\mathbf{u}\times\boldsymbol{\Omega} - \frac{1}{2}\nabla\left[(\boldsymbol{\Omega}\times\mathbf{r})^2\right] + \nu\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right].$$

]

TURN OVER...

Question 3X - Cosmology

(i) Assume that the Universe is spatially flat and described by the Friedmann–Robertson–Walker line element,

$$ds^2 = c^2dt^2 - R^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]\,,$$

where $R(t)$ is the scale factor. Show that the angular-diameter distance, $D_A(z)$, and luminosity distance, $D_L(z)$, to an object at redshift z are given in terms of the Hubble parameter $H(z)$ by

$$D_A(z) = \frac{c}{(1+z)}\int_0^z \frac{dz'}{H(z')}\,, \qquad D_L(z) = c(1+z)\int_0^z \frac{dz'}{H(z')}\,, \qquad (*)$$

independent of the matter content of the Universe. [10]

(ii) The deceleration parameter, $q(t)$, and jerk parameter, $j(t)$, in a Friedmann–Robertson-Walker (FRW) universe with scale factor $R(t)$ are defined as

$$q = -R\frac{d^2R/dt^2}{(dR/dt)^2}\,, \qquad j = R^2\frac{d^3R/dt^3}{(dR/dt)^3}\,.$$

Show that these quantities can be written as

$$q(z) = -1 + \frac{(1+z)}{2}\frac{(H^2)'}{H^2}\,, \qquad (\dagger)$$

$$j(z) = 1 - (1+z)\frac{(H^2)'}{H^2} + \frac{(1+z)^2}{2}\frac{(H^2)''}{H^2}\,, \qquad (\dagger\dagger)$$

where H is the Hubble parameter and primes denote differentiation with respect to redshift z. [10]

Show that in a spatially-flat FRW universe consisting of non-relativistic matter and a cosmological constant, the Hubble parameter is

$$H(z) = H_0\left[\Omega_m(1+z)^3 + \Omega_\Lambda\right]^{1/2}\,,$$

where Ω_m and $\Omega_\Lambda = 1-\Omega_m$ are the present-day densities contributed by matter and the cosmological constant, respectively, in units of the critical density and H_0 is the Hubble constant. [3]

Hence show that in such a universe the values of q and j at the present day are given by

$$q_0 = \frac{3}{2}\Omega_m - 1\,, \qquad j_0 = 1\,.$$

[2]

Outline, without detailed proof, how the relations (†) and (††) can be used to derive the following general approximation for the luminosity distance $D_L(z)$ in a spatially-flat universe, given by $(*)$ in Part (i):

$$D_L(z) = \frac{cz}{H_0}\left[1 + \frac{1}{2}(1 - q_0)z - \frac{1}{6}(1 - q_0 - 3q_0^2 + j_0)z^2 + \mathcal{O}(z^3)\right] .$$ [3]

A Type-1a supernova is observed at redshift $z = 0.5$. Will it appear fainter in a universe with: (a) $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$; or (b) $\Omega_m = 1$, $\Omega_\Lambda = 0$? The Hubble constant is the same in both cases. [2]

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) A star is composed of H (mass fraction $X = 0.7$), He (mass fraction $Y = 0.3$), and negligible amounts of heavier elements. Calculate the mean molecular weight immediately above and below the radius in the star where hydrogen and helium transition from fully neutral to fully ionized. [6]

Assume that the transition between ionized and neutral hydrogen and helium takes place over a very small radial distance, such that the pressure and temperature can be considered constant across the zone. What does this imply about the dynamical stability of the zone? [4]

(ii) An eclipsing-binary system has a parallax of 0.1 arcsec (with negligible error) and consists of two Solar-type stars with a semi-major axis of $500\,R_\odot$. The period is known very accurately. What is the angular size of each of the stars and of the semi-major axis? Give your answer in units of arcsec. [1]

If you can measure angles on the sky with a 1σ root mean square accuracy of 0.01 arcsec, what is the percentage accuracy of the measurement of the semi-major axis and of the radius of each star? [2]

If we now include an error in the measurement of the parallax of $\sigma_\Pi = 0.01\,\mathrm{arcsec}$, what is the percentage accuracy in the mass of the system? [7]

Assume that the stars emit as blackbodies with an effective temperature $T_{\rm eff} \simeq 5800\,\mathrm{K}$ and with a spectral radiance given by

$$F_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1},$$

where ν is the frequency, T is the temperature, k_B is Boltzmann's constant, h is Planck's constant, and c is the speed of light. If you measure the flux ratio between $\log_{10}(\nu/\mathrm{Hz}) = 14.0$ and $\log_{10}(\nu/\mathrm{Hz}) = 15.0$ with an accuracy of 10%, with what percentage precision can you determine the value of $T_{\rm eff}$? [10]

Question 5Z - Statistical Physics

(i) Write down the equation of state for an ideal gas, and an expression for its internal energy in terms of the heat capacity at constant volume, C_V. [3]

Starting from the first law of thermodynamics, find a relation between the heat capacity at constant pressure, C_p, and C_V. [4]

Hence give an expression for $\gamma = C_p/C_V$. [2]

Describe the meaning of an adiabatic process. [1]

(ii) Starting from the first law of thermodynamics, and using your results from Part (i) or otherwise, derive the equation for an adiabatic process in the pressure–volume (p, V) plane for an ideal gas. [4]

Consider a simplified Otto cycle (an idealised petrol engine) involving an ideal monatomic gas and consisting of the following four reversible steps:

$A \to B$: adiabatic compression from volume V_1 to volume $V_2 < V_1$;

$B \to C$: heat Q_1 *in* at constant volume;

$C \to D$: adiabatic expansion from volume V_2 to volume V_1;

$D \to A$: heat Q_2 *out* at constant volume.

Sketch the cycle in the (p, V) plane and in the temperature–entropy (T, S) plane. [6]

Derive an expression for the efficiency, $\eta = W/Q_1$, where W is the work done by the gas, in terms of the compression ratio $r = V_1/V_2$. [8]

How can the efficiency be maximized? [2]

TURN OVER...

Question 6X - Principles of Quantum Mechanics

(i) A group G of transformations acts on a quantum system with Hilbert space $\mathcal{H}$. Briefly explain why the Born rule implies these transformations may be represented by operators $U(g) : \mathcal{H} \to \mathcal{H}$ obeying

$$U^\dagger(g)\,U(g) = 1_{\mathcal{H}}\,,$$
$$U(g_1)\,U(g_2) = e^{i\phi(g_1,g_2)}\,U(g_1 \cdot g_2)\,,$$

for all $g_1, g_2 \in G$, where $\phi(g_1, g_2) \in \mathbb{R}$. Note that $g_1 \cdot g_2$ denotes the composition of g_1 and g_2. [4]

What additional property does $U(g)$ have when G is a *symmetry* of the Hamiltonian? [3]

Show that symmetries correspond to conserved quantities. [3]

(ii) The Coulomb Hamiltonian describing the gross structure of the hydrogen atom is invariant under reversal of the direction of time, $t \mapsto -t$. Suppose we try to represent time reversal by a unitary (linear) operator T obeying $U(t)T = TU(-t)$, where $U(t)$ is the time-evolution operator. Show that this would imply that hydrogen has no stable ground state. [8]

An operator A is *anti*-linear if

$$A(a|\alpha\rangle + b|\beta\rangle) = a^*\,A|\alpha\rangle + b^*\,A|\beta\rangle\,,$$

for all $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$ and all $a, b \in \mathbb{C}$. It is further *anti*-unitary if $|\alpha'\rangle = A|\alpha\rangle$ and $|\beta'\rangle = A|\beta\rangle$ satisfy

$$\langle\alpha'|\beta'\rangle = \langle\alpha|\beta\rangle^*$$

for all $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$. Show that the above instability is avoided if time reversal is instead represented by an anti-unitary operator satisfying $U(t)T = TU(-t)$. [6]

Given that the action of such an anti-unitary operator T on the position-basis states, $\{|\mathbf{x}\rangle\}$, is

$$T|\mathbf{x}\rangle = |\mathbf{x}\rangle\,,$$

show that the action on the position-space wavefunction is equivalent to complex conjugation. [3]

Show further that the action on the momentum-basis states, $\{|\mathbf{p}\rangle\}$, is

$$T|\mathbf{p}\rangle = |-\mathbf{p}\rangle\,.$$ [3]

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Assume that a test particle moves in the gravitational potential of a point mass M. Show that the specific angular momentum h of the test particle is conserved. [2]

Show further that the orbit can be described by the equation

$$\frac{1}{r} = \frac{GM}{h^2}\,(1 + e\cos\phi),$$

where r is the distance to the point mass, ϕ is an angle in the plane of the orbit and e is a constant. [5]

Deduce that there is no closed orbit that reaches $r < h^2/(2GM)$. [3]

(ii) The gravitational potential of a black hole of mass M can be approximated by $\Phi = -GM/(r - r_{\rm s})$, where r is the distance to the black hole and $r_{\rm s}$ is a constant characteristic radius. Show that a test particle in orbit around the black hole with specific angular momentum h satisfies

$$\frac{1}{2}\,\dot{r}^2 + \frac{h^2}{2\,r^2} - \frac{GM}{r - r_{\rm s}} = E,$$

where E is a constant and the overdot denotes a derivative with respect to time. [2]

Using this equation, or otherwise, determine the radius of the smallest stable circular orbit. [10]

Suppose a spacecraft to be in circular orbit at $2.5r_{\rm s}$ from a black hole. Assume the spacecraft fires its rockets to provide a small radial impulse either inwards or outwards. For each case describe the subsequent trajectory, and comment on the ultimate fate of the spacecraft. [8]

TURN OVER...

Question 8Z - Topics in Astrophysics

(i) Explain why the tidal interaction between a moon and a rapidly spinning host planet could result in the moon being 'evaporated' (i.e., becoming unbound from the planet). [3]

Consider a planet with the properties of the present day Earth orbiting a solar mass star, but with the actual Moon replaced by a small satellite that is formed well within the Hill sphere of the planet. Estimate the maximum mass of a satellite that could eventually be evaporated by this mechanism and comment on this answer in relation to the ultimate fate of the actual Moon. [You may assume if necessary that the mass of the actual Moon is $\sim 1/80$ that of the Earth.] [7]

(ii) A dust grain of radius a and internal density $\rho = 2000\,\mathrm{kg\,m^{-3}}$ is located at distance $R = 1\,\mathrm{au}$ from a star with solar mass and luminosity. Derive an expression for the net outward acceleration, f_o, of the dust grain resulting from the effects of both gravity and radiation pressure from the star. You may assume that the dust grain behaves like a perfect black body. [6]

Determine the grain size, a_b, for which $f_\mathrm{o} = 0$ and sketch f_o as a function of a. [4]

What is the terminal velocity of outflowing grains of size $0.5a_\mathrm{b}$ that originate at rest at $1\,\mathrm{au}$? [3]

A cloud of dust particles has a size distribution such that the fraction of particles with radii in the range a to $a + da$ is proportional to $a^{-3.5}da$ over a range such that the maximum dust size is much less than a_b. This cloud is released at $1\,\mathrm{au}$ from a star with solar properties following a planetesimal collision event. Show that if the dust particles are initially at rest, then at a long time after the event, the distance from the star, R, attained by grains of size a is related to a according to $R \propto a^{-0.5}$. [3]

How does the total mass of dust at a given time that is located between R and $R + dR$ scale with R? [4]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 10 June 2021 10:00am – 1:00pm

ASTROPHYSICS - PAPER 2

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.

Answers must be uploaded in separate pdf files, named with your candidate number, paper number, and the letter **X, Y, Z**, *according to the letter associated with the questions, each separated by an underscore. For example,* **1X, 3X, and 6X** *should be in the same file and* **5Z, 7Z, and 8Z** *in another, each file labelled* **1234A_paper2_X** *or* **1234A_paper2_Z**, *where '1234A' should be replaced with your candidate number.*

After the exam there will be **30 minutes** *for you to scan and upload your solutions to ProctorExam. During this time you* **may not do any writing**, *and* **must complete the online cover sheet** *linked to in the ProctorExam instructions, identifying all the questions you have attempted.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Paper (blank sheets, plain or lined)	*Astrophysics Formulae Booklet (as pdf)*
1 Rough Work Pad	*Approved Calculators Allowed*

You can begin the paper provided you have already received instructions to start the exam from the invigilator.
Stop writing when indicated to by the invigilator, or following your allotted time.

Question 1X - Relativity

(i) Let $x^\mu(\lambda)$ be a geodesic in spacetime with affine parameter λ. Starting from the geodesic equation in the form

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\,,$$

show that

$$\frac{d}{d\lambda}\left(g_{\tau\mu}\frac{dx^\mu}{d\lambda}\right) = \frac{1}{2}\frac{\partial g_{\nu\rho}}{\partial x^\tau}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\,. \qquad (*)$$ [7]

Comment on the implications of this result for spacetimes with symmetries such that the metric $g_{\mu\nu}$ is independent of some coordinate. [3]

(ii) Consider a static spacetime with line element of the form

$$ds^2 = g_{00}dt^2 + g_{ij}dx^i dx^j\,,$$

where $x^0 = t$, the spatial indices (e.g., i and j) run from 1 to 3 and g_{00} and g_{ij} depend only on the $\{x^k\}$. An implicit summation over repeated spatial indices should be understood in this part of the question. Starting from the geodesic equation in the form of $(*)$ from Part (i), show that

$$g_{00}\frac{dt}{d\lambda} = \text{const.}$$

and, for the case of a null geodesic,

$$\frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right) = \frac{g_{00}}{2}\frac{\partial}{\partial x^i}\left(\frac{g_{jk}}{g_{00}}\right)\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\,.$$ [9]

Hence show that

$$\frac{d}{dt}\left(\gamma_{ij}\frac{dx^j}{dt}\right) = \frac{1}{2}\frac{\partial\gamma_{jk}}{\partial x^i}\frac{dx^j}{dt}\frac{dx^k}{dt}\,,$$

where $\gamma_{ij} \equiv -g_{ij}/g_{00}$. [4]

Now consider any null curve $x^\mu(u)$ in this static spacetime, parameterised by a general parameter u and passing through points with spatial coordinates $x^i = a^i$ at $u = 0$ and $x^i = b^i$ at $u = 1$. Show that the elapsed coordinate time is

$$\Delta t = \int_0^1 \sqrt{\gamma_{ij}\frac{dx^i}{du}\frac{dx^j}{du}}\,du\,.$$ [2]

Show further that null curves for which Δt is extremal are geodesics in spacetime. [*Hint: apply the Euler–Lagrange equations to* $L \equiv \sqrt{\gamma_{ij} \frac{dx^i}{du} \frac{dx^j}{du}}$ *and note that* $dt/du = L$.] [5]

TURN OVER...

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider the steady-state, spherically-symmetric accretion of matter from an infinite, uniform medium onto a central gravitating object with mass M. Neglecting the self-gravity of the accreting matter, and adopting a barotropic equation of state, show that

$$\left(\frac{u^2}{c_s^2} - 1\right) \frac{\mathrm{d} \ln u}{\mathrm{d} r} = \frac{2}{r}\left(1 - \frac{r_s}{r}\right), \qquad (*)$$

where r is the radial distance from the object, $u(r)$ is the inward-directed radial velocity, $c_s(r)$ is the sound speed, and $r_s = GM/(2c_s^2)$. [7]

Clearly state the boundary condition for the accretion problem. [1]

Sketch the corresponding solution curves of $(*)$ on the $(r/r_s, u/c_s)$-plane. [2]

(ii) Consider a simple model for the solar wind consisting of a steady-state, spherically-symmetric, isothermal outflow of fully ionized hydrogen from the Sun. Let $u(r)$ be the outward-directed velocity of the flow, c_s the sound speed, and define $r_s = GM_\odot/c_s^2$. Assuming that the outflow is launched from close to the Sun with a velocity that is significantly subsonic, sketch the possible solution curves for this problem on the $(r/r_s, u/c_s)$-plane. [3]

The Parker Wind solution makes a transition from subsonic to supersonic flow. Show that in this case,

$$\left(\frac{u}{c_s}\right)^2 - 2\ln\left(\frac{u}{c_s}\right) = 4\ln\left(\frac{r}{r_s}\right) + \frac{4r_s}{r} - 3.$$

[4]

Adopting a temperature of $T = 10^6\,\mathrm{K}$, calculate the location of the sonic point and proceed to find an approximate value for the speed of the Parker Wind at the location of the Earth. [6]

Show that, at very large distances from the Sun, the pressure of the Parker Wind obeys $p \propto 1/(r^2\sqrt{\ln(r/r_s)})$. [4]

The 'solar breeze' solution remains subsonic at all radii. Given that the interstellar medium has an extremely small pressure, carefully explain why the Parker Wind is a viable description of the solar wind whereas the 'solar breeze' solution is not. [3]

Question 3X - Cosmology

(i) The contribution of relativistic particles of species i in thermal equilibrium at temperature T to the energy density of the Universe, $\rho_i c^2$, is given by

$$\rho_i c^2 = g_i \frac{4\pi c}{h^3} \int_0^\infty \frac{p^3 dp}{[\exp(pc/k_\mathrm{B}T) \pm 1]} \, ,$$

where p is the 3-momentum, g_i is the number of spin states and the $+$ sign corresponds to fermions and the $-$ sign to bosons. Show that the energy densities are given by

$$\rho_i c^2 = \left(\frac{g_i}{2}\right) aT^4 \qquad \text{for bosons}\, ,$$
$$\rho_i c^2 = \frac{7}{8}\left(\frac{g_i}{2}\right) aT^4 \quad \text{for fermions}\, ,$$

where [5]

$$a = \frac{8\pi^5 k_\mathrm{B}^4}{15 h^3 c^3} \, .$$

We can define an effective statistical weight, $g_\mathrm{eff}(T_\gamma)$, for relativistic particles as a function of the photon temperature T_γ:

$$g_\mathrm{eff}(T_\gamma) = \sum_\mathrm{bosons} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \sum_\mathrm{fermions} \frac{7}{8} g_i \left(\frac{T_i}{T_\gamma}\right)^4 \, , \qquad (*)$$

which allows for the possibility that particles are described by a thermal distribution function with a temperature T_i that may differ from the photon temperature. Sketch g_eff as a function of photon energy $k_\mathrm{B}T_\gamma$ over the range 0.1 MeV to 300 GeV and comment on notable features. [5]

[*You may assume that*

$$\int_0^\infty \frac{x^{\nu-1}}{\exp(x) - 1} dx = \Gamma(\nu)\zeta(\nu)\, , \qquad \int_0^\infty \frac{x^{\nu-1}}{\exp(x) + 1} dx = \left(1 - 2^{1-\nu}\right)\Gamma(\nu)\zeta(\nu)\, ,$$

where Γ *and* ζ *are the gamma and zeta functions and* $\Gamma(4)\zeta(4) = \pi^4/15$.]

(ii) The distribution function, f, of a homogenous species of collisionless particles in a Friedmann–Robertson–Walker (FRW) universe satisfies the collisionless Boltzmann equation

TURN OVER...

$$\frac{\partial f}{\partial t} - p\frac{\dot{R}}{R}\frac{\partial f}{\partial p} = 0\,, \qquad (**)$$

where R is the scale factor, overdots denote differentiation with respect to time t and p is the 3-momentum satisfying $E^2 = p^2c^2 + m^2c^4$, where m and E are the rest mass and energy of the particle. For a freely falling particle in a FRW background, $p \propto R^{-1}$. If neutrinos are highly relativistic at the time that they go out of thermal equilibrium, show that the neutrino distribution function of each flavour is

$$f = \frac{1}{h^3}\frac{1}{[\exp{(pc/k_{\rm B}T_\nu)} + 1]}\,,$$

with neutrino temperature $T_\nu \propto R^{-1}$, and that this satisfies $(**)$ irrespective of the neutrino mass m. [10]

Electrons and positrons annihilate at a temperature $k_{\rm B}T_\gamma \sim 1\,\text{MeV}$, boosting the photon temperature T_γ. Show that after e^+e^- annihiliation, the neutrino temperature is related to the photon temperature via

$$T_\nu \approx \left(\frac{4}{11}\right)^{1/3} T_\gamma\,.$$

[5]

In the Standard Model of particle physics, calculate the effective statistical weight $g_{\rm eff}$, given by $(*)$ of Part (i), before and after e^+e^- annihiliation. [5]

Question 4Y - Structure and Evolution of Stars

(i) The following three approximate relations apply to massive stars on the main sequence:

(a) the mass–luminosity relation

$$\log\left(\frac{L}{L_\odot}\right) \approx 0.78 + 2.76\,\log\left(\frac{M_\mathrm{i}}{M_\odot}\right),$$

where L is luminosity, and M_i is the initial mass;

(b) the mass-loss rate–luminosity relation

$$\log\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right) \approx -12.76 + 1.30\,\log\left(\frac{L}{L_\odot}\right),$$

where $\mathrm{d}M/\mathrm{d}t$ is in $M_\odot\ \mathrm{yr}^{-1}$; and

(c) the main-sequence lifetime–mass relation

$$\log \tau_\mathrm{MS} \approx 7.72 - 0.66\,\log\left(\frac{M_\mathrm{i}}{M_\odot}\right),$$

where τ_MS is the main sequence lifetime in years.

Use these relations to calculate the fraction of the initial mass that is lost by massive stars with $M_\mathrm{i} = 25$, 40, 60, 85, and $120\,M_\odot$ before they evolve off the main sequence. [6]

A star with $M_\mathrm{i} = 85\,M_\odot$ has a convective core that contains 83% of the stellar mass. Calculate the time after the star appears on the main sequence at which the products of nuclear burning will appear at the surface. [2]

How would such a star be classified at this time? [2]

(ii) If energy transport within a star is by radiative diffusion, the luminosity $L(r)$ at some radius r within the star can be written as

$$L(r) = -4\pi r^2\,\frac{16\sigma}{3}\,\frac{T(r)^3}{\rho(r)\kappa(r)}\,\frac{\mathrm{d}T(r)}{\mathrm{d}r},$$

where ρ is the density, T is the temperature, σ is the Stefan-Boltzmann constant, and the opacity κ is given by

TURN OVER...

$$\kappa(r) \propto \rho(r) T(r)^{-3.5} .$$

From these two equations show, using homology arguments, that as a pre-main sequence star contracts its luminosity changes with temperature according to the relation

$$L \propto T_{\rm eff}^{4/5}.$$ [14]

The path taken by a contracting star as it approaches the main sequence in the H-R diagram is called the Henyey track. Computer calculations show that $L \propto T_{\rm eff}^{4/5}$ is a satisfactory approximation of the Henyey tracks of massive stars, but becomes a progressively poorer fit to the tracks of stars with masses $M \lesssim 2M_\odot$. What conclusions can you draw from this statement? [6]

Question 5Z - Statistical Physics

(i) What systems are described by microcanonical, canonical and grand canonical ensembles? [5]

Under what conditions is the choice of ensemble irrelevant? [2]

Define the partition function and describe how it can be used to determine thermodynamic quantities of a system. [3]

(ii) In a simple model a meson consists of two quarks of mass m bound through a linear potential, $U(\mathbf{r}) = \alpha|\mathbf{r}|$, where $\mathbf{r}$ is the relative displacement of the two quarks and α is a positive constant. You are given that the classical (non-relativistic) Hamiltonian for the meson is

$$H(\mathbf{P}, \mathbf{R}, \mathbf{p}, \mathbf{r}) = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} + \alpha|\mathbf{r}|\,,$$

where $M = 2m$ is the total mass, $\mu = m/2$ is the reduced mass, $\mathbf{P}$ is the total momentum, $\mathbf{p} = \mu d\mathbf{r}/dt$ is the internal momentum, and $\mathbf{R}$ is the centre of mass position. Show that the partition function for a single meson in thermal equilibrium at temperature T in a three-dimensional volume V can be written as $Z_1 = Z_{\text{trans}} Z_{\text{int}}$, where

$$\begin{aligned} Z_{\text{trans}} &= \frac{V}{(2\pi\hbar)^3} \int d^3P\, e^{-\beta|\mathbf{P}|^2/(2M)}\,, \\ Z_{\text{int}} &= \frac{1}{(2\pi\hbar)^3} \int d^3r d^3p\, e^{-\beta|\mathbf{p}|^2/(2\mu)} e^{-\beta\alpha|\mathbf{r}|}\,, \end{aligned}$$

with $\beta = 1/(k_{\text{B}}T)$. [3]

Evaluate both Z_{trans} and Z_{int} in the large volume limit ($\beta\alpha V^{1/3} \gg 1$). [6]

What is the average separation of the quarks within the meson at temperature T? [3]

Now consider an ideal gas of N such mesons in a three-dimensional volume V. Calculate the total partition function of the gas. [3]

What is the heat capacity C_V of this system? [5]

[*You may assume that* $\int_{-\infty}^{+\infty} e^{-cx^2} dx = \sqrt{\pi/c}$.]

TURN OVER...

Question 6X - Principles of Quantum Mechanics

(i) Let $\{|n\rangle\}$ be a basis of eigenstates of a non-degenerate Hamiltonian H, with corresponding eigenvalues $\{E_n\}$. Show that the energy levels of the perturbed Hamiltonian $H+\lambda\,\Delta H$, correct to second order in the dimensionless constant λ, are

$$E_n(\lambda) = E_n + \lambda\langle n|\Delta H|n\rangle + \lambda^2 \sum_{m\neq n} \frac{|\langle m|\Delta H|n\rangle|^2}{E_n - E_m} + O(\lambda^3)\,.$$

[10]

(ii) A particle of mass m travels in one dimension under the influence of the perturbed harmonic-oscillator potential

$$V(X) = \frac{1}{2}m\omega^2 X^2 + \lambda\,\hbar\omega\,\frac{X^3}{L^3}\,,$$

where ω is a frequency and $L = \sqrt{\hbar/(2m\omega)}$ is a length scale. Show that to first order in λ, all energy levels coincide with those of the unperturbed harmonic oscillator. [5]

Calculate the energy of the ground state to second order in λ. [12]

Does perturbation theory in λ converge for this potential? Briefly explain your answer. [3]

[*You may wish to use the raising and lowering operators of the harmonic oscillator,*

$$A^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\,(m\omega X - iP) \quad \textit{and} \quad A = \frac{1}{\sqrt{2m\hbar\omega}}\,(m\omega X + iP)\;,$$

respectively.]

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Describe briefly the process of dynamical friction, and outline the physical arguments that lead to the following dynamical friction formula for the acceleration on an object of mass M moving through a sea of particles with mass density ρ:

$$\frac{\mathrm{d}v}{\mathrm{d}t} \propto \rho \frac{M}{v^2}. \qquad [5]$$

Describe briefly the effect of dynamical friction on globular clusters, and the observational evidence for tidal stripping of clusters by their host galaxies. [2]

Show that tidal stripping occurs outside the radius r_{t} within which the mean density of the cluster is approximately equal to that of the host galaxy inside the orbit of the cluster. [3]

(ii) A star in the disk of the Milky Way has a circular orbit with radius R and velocity V. The Galactic longitude of the star as seen from the Sun is l, and the distance from the Sun to the star is d. Show that the radial and transverse components of the star's motion with respect to the Local Standard of Rest at the Sun's Galactic radius R_0 are

$$V_r = \left(\frac{V}{R} - \frac{V_0}{R_0}\right) R_0 \sin l, \qquad V_t = \left(\frac{V}{R} - \frac{V_0}{R_0}\right) R_0 \cos l - \frac{V}{R}\, d,$$

where V_0 is the circular speed at the position of the Sun. [6]

If $d \ll R_0$ show that these expressions reduce to

$$V_r \approx A\, d \sin 2l, \qquad V_t \approx (A \cos\, 2l + B)\, d,$$

where A and B are the Oort constants. [7]

Suppose that the mass distribution of the Milky Way can be represented by the density distribution

$$\rho = \rho_{\mathrm{c}} \left(\frac{r_{\mathrm{c}}}{r}\right)^{\alpha}, \qquad \alpha < 3,$$

where r is the distance to the Galactic centre and ρ_{c}, r_{c} and α are constants. Calculate the Oort constants for this mass distribution. [5]

If observations of stars at $l = 45°$ indicate that $V_r/V_t = -1$, deduce the value of the index α. [2]

TURN OVER...

Question 8Z - Topics in Astrophysics

(i) The velocity dispersion of stars in the solar neighbourhood increases with stellar age, t, in proportion to $t^{0.33}$. Suggest, without calculation, a process that could increase the stellar velocity dispersion with age and suggest candidates for driving this process. [3]

The *Gaia* satellite has found that stars that host hot Jupiters have an average velocity dispersion of $36\,\mathrm{km\,s^{-1}}$, as compared to an average value of $43\,\mathrm{km\,s^{-1}}$ for stars that do not host hot Jupiters. Discuss possible explanations for this observation. [7]

(ii) An intermediate mass black hole of mass $10^3 M_\odot$ is located in the centre of a cluster and dominates the gravitating mass out to a distance of 0.5 pc. A solar-mass main-sequence star in the cluster follows an orbit with apocentre at 0.2 pc. Estimate how close the star must approach the black hole at pericentre in order that the total tidal energy deposited in the star over its main-sequence lifetime (4.5 Gyr) is comparable with its internal gravitational binding energy. [9]

Explain whether you expect the star's evolution to be affected by tidal effects in this case. [3]

Are relativistic effects important at pericentre? [2]

Do you expect the star to be able to support habitable planets? [2]

At the end of the star's main-sequence lifetime, it becomes a red giant, expanding to a radius of around 1 au. Explain how tidal effects from the black hole would affect the star during this evolutionary stage. [2]

Would there be any observable consequences of this interaction? [2]

[*You may assume that the tidal energy injected into a star of mass M_2, radius R_2 when it approaches within a distance a_{peri} of a mass M_1 is given by $E_{\mathrm{tidal}} \sim GM_1^2 R_2^5 / a_{\mathrm{peri}}^6$.*]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Saturday 12 June 2021 10:00am – 1:00pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.

Answers must be uploaded in separate pdf files, named with your candidate number, paper number, and the letter **X, Y, Z**, *according to the letter associated with the questions, each separated by an underscore. For example,* **1X, 3X, and 6X** *should be in the same file and* **5Z, 7Z, and 8Z** *in another, each file labelled* **1234A_paper3_X** *or* **1234A_paper3_Z**, *where '1234A' should be replaced with your candidate number.*

After the exam there will be **30 minutes** *for you to scan and upload your solutions to ProctorExam. During this time you* **may not do any writing**, *and* **must complete the online cover sheet** *linked to in the ProctorExam instructions, identifying all the questions you have attempted.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Paper (blank sheets, plain or lined)	*Astrophysics Formulae Booklet (as pdf)*
1 Rough Work Pad	*Approved Calculators Allowed*

You can begin the paper provided you have already received instructions to start the exam from the invigilator.
Stop writing when indicated to by the invigilator, or following your allotted time.

Question 1X - Relativity

(i) A massive particle moves freely in the equatorial plane ($\theta = \pi/2$) outside a non-rotating black hole of mass M, with line element

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) ,$$

where $\mu \equiv GM/c^2$. Show that

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k \quad \text{and} \quad r^2 \dot{\phi} = h \,,$$

where k and h are constants and overdots denote differentiation with respect to proper time. [2]

In the case $k > 1$, the particle can reach $r = \infty$ asymptotically. Find the speed there, v_∞, as measured by an observer in the equatorial plane at fixed r and ϕ. Express your answer in terms of k. [4]

Show further that the impact parameter b for such motion is

$$b = \frac{h}{c\sqrt{k^2 - 1}} \,.$$ [4]

(ii) For the massive particle in Part (i), show that the radial motion is determined by

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}c^2 \left(k^2 - 1\right) ,$$

where the effective potential

$$V_{\text{eff}}(r) = -\frac{\mu c^2}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) .$$ [4]

Sketch the effective potential for several representative values of $h/(\mu c)$ and explain why, no matter how large h, a particle incident from infinity with sufficiently large k will be captured by the black hole. [5]

By considering the maximum value of $V_{\text{eff}}(r)$, or otherwise, show that the condition for a particle with speed $v_\infty \ll c$ as $r \to \infty$ to be captured by the black hole is that the impact parameter

$$b < \frac{4\mu c}{v_\infty} \,.$$ [7]

Determine the equivalent bound on the impact parameter for capture in the limit $v_\infty \to c$. [4]

Question 2Y - Astrophysical Fluid Dynamics

(i) The expressions of the conservation of mass and angular momentum for a geometrically-thin accretion disk are

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0,$$
$$\frac{\partial}{\partial t}(R\Sigma u_\phi) + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) - \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{d\Omega}{dR}\right) = 0,$$

where Σ is the surface density, (u_R, u_ϕ) describes the radial and azimuthal components of the velocity field, respectively, R is radial distance, t is time, ν is kinematic viscosity and Ω is angular velocity. Show that, for accretion onto a point mass M,

$$u_R = -\frac{3}{R^{1/2}\Sigma}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2}). \qquad [7]$$

Show that the local mass-accretion rate is

$$\dot{M}(R) = 6\pi R^{1/2}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2}). \qquad [3]$$

(ii) Using expressions from Part (i), show that the surface density of a geometrically-thin accretion disk around a point mass evolves according to

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2})\right]. \qquad (\dagger) \qquad [3]$$

Consider a simple model in which viscosity is given by $\nu = \nu_0 R$, where ν_0 is a constant. Define $x = R^{1/2}$. Show that ($\dagger$) can be recast in a form

$$\frac{\partial \Psi}{\partial t} = \frac{3\nu_0}{4}\frac{\partial^2 \Psi}{\partial x^2},$$

where you should appropriately define the variable Ψ. [6]

Show that the local mass-accretion rate $\dot{M} = 3\pi\, \partial\Psi/\partial x$. [3]

At time $t = 0$, a mass m is dumped into a very narrow orbiting ring around the central object at radius $R = R_0$. Assuming that it evolves according to the simple viscous model considered above, derive an expression for the mass-accretion rate onto the central object (at $R = 0$) at subsequent times. [6]

TURN OVER...

Sketch this mass-accretion rate and comment on the behaviour at early and late times. [2]

[*You may assume without proof that the Green's function for the equation* $\partial y/\partial t = D\partial^2 y/\partial x^2$ *is* $G(x_0, x; t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x_0-x)^2/(4Dt)}$.]

Question 3X - Cosmology

(i) Consider a spatially-flat Friedmann–Robertson–Walker universe described by the line element

$$ds^2 = c^2 dt^2 - R^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)]\,,$$

where $R(t)$ is the scale factor. Show that the proper size of the particle horizon is given by

$$d_{\rm ph}(t) = cR(t)\int_0^t \frac{dt'}{R(t')}\,.$$ [2]

If $R(t) \propto t^p$, with $p < 1$, show that

$$d_{\rm ph}(t) = \frac{ct}{1-p}\,. \qquad (*)$$ [1]

Assume that the universe is dominated by matter with an equation of state relating pressure, P, and density, ρ, by

$$P = w\rho c^2\,,$$

where w is a constant. Find the value of w that corresponds to the critical index $p = 1$ in $(*)$. [5]

Briefly discuss the significance of this result in the context of inflationary cosmology. [2]

(ii) Consider a spatially-flat universe that goes through an inflationary phase of exponential expansion with constant Hubble parameter H_I, i.e., $R(t) \propto e^{H_I t}$, over the time interval $t_{\rm i} < t < t_{\rm f}$. Using results from Part (i), show that the particle horizon during the inflationary phase grows as

$$d_{\rm ph}(t) = \frac{c}{H_I}\left(e^{H_I(t-t_{\rm i})} - 1\right), \qquad t_{\rm i} < t < t_{\rm f}\,.$$ [5]

Discuss how this result might alleviate the *horizon problem* of standard hot big bang cosmology. [5]

In an alternative universe, the energy density is dominated by a scalar field ϕ that obeys the equations of motion (in Planck units)

TURN OVER...

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}\,,$$
$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\,,$$

where overdots denote differentiation with respect to time t and $V(\phi)$ is the potential. Show that these equations are satisfied exactly if

$$V(\phi) = V_0\ \exp\left(-\sqrt{\frac{2}{p}}\phi\right)\,,$$
$$\phi = \sqrt{2p}\ln\left(t\sqrt{\frac{V_0}{p(3p-1)}}\right)\,,$$

and that the scale factor varies as a power law $R(t) \propto t^p$. [8]

For what values of p does inflation occur in this case? [2]

Question 4Y - Structure and Evolution of Stars

(i) A star of mass M and radius R is in hydrostatic equilibrium. Derive an integral expression for the gravitational potential energy of the star, U. [4]

If the density is uniform, derive U in terms of the mass, M, and radius, R, of the star and show that for a given M, U is proportional to $1/R$. [3]

Show that the total thermal energy of the star, K, is given by

$$K = 4\pi \int_0^R \frac{3}{2} P\, r^2\, dr\,,$$

where P is the pressure, and r is the radial distance from the centre of the star. [3]

(ii) In a $10\, M_\odot$ star the $1\, M_\odot$ core collapses to produce a Type II supernova. Assume that 100% of the energy released by the collapsing core is converted to neutrinos and that 1% of the neutrinos are absorbed by the overlying envelope to power the ejection of the supernova remnant. Estimate the final radius of the stellar remnant, assuming that the energy liberated is just enough to eject the remaining $9\, M_\odot$ to infinity. [*Hint: You will need to assume an appropriate density structure in the star's envelope.*] [12]

Do you consider the assumption made above to be consistent with observations of supernova remnants? [2]

What is the typical velocity of the ejecta, if the energy absorbed by the envelope is 10^{51} erg? [3]

The detection of a Type II supernova in a globular cluster is announced. Why might one be skeptical of this claimed discovery? [3]

TURN OVER...

Question 5Z - Statistical Physics

(i) A gas of non-interacting particles has energy–momentum relationship

$$E = A(\hbar k)^{\alpha}, \qquad (*)$$

for some constants $A > 0$ and $\alpha > 0$, and spin degeneracy g_s. Show that the density of states in a d-dimensional volume V with $d \geqslant 2$ is given by

$$g(E)dE = CVE^{d/\alpha - 1},$$

where C is a constant that you should determine. [You may denote the surface area of a unit $(d-1)$-dimensional sphere by S_{d-1}.] [8]

Write down the Bose–Einstein distribution for the average number of identical Bosons in a state with energy $E_r \geqslant 0$ in terms of $\beta = 1/(k_B T)$ and chemical potential μ. [2]

(ii) Explain why $\mu < 0$ for the Bose–Einstein distribution of Part (i). [2]

Using your results from Part (i), or otherwise, show that an ideal quantum Bose gas with the energy–momentum relationship $(*)$ has

$$p\,V = D\,E\,,$$

where p is the pressure and D is a constant that should be determined. [9]

For such a Bose gas, write down an expression for the number of particles that do not occupy the ground state, and use this to determine the values of α for which there exists a Bose–Einstein condensate at sufficiently low temperatures. [9]

Question 6X - Principles of Quantum Mechanics

(i) A quantum system with total angular momentum j_1 is combined with another of total angular momentum j_2. What are the possible values of the total angular momentum j of the combined system? [2]

For a given j, what are the possible values of angular momentum along any axis? [1]

Consider two systems with $j_1 = j_2$. Explain why all the states with $j = 2j_1 - 1$ are antisymmetric under exchange of the angular momenta of the two subsystems, while all the states with $j = 2j_1 - 2$ are symmetric. [7]

(ii) Consider the systems in Part (i) for the case $j_1 = j_2 = 1$. Construct the state with zero total angular momentum in terms of the angular momentum states $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ of the individual systems. [6]

An exotic particle X of spin 0 and negative intrinsic parity decays into a pair of indistinguishable particles Y. Each Y particle has spin 1 and the decay process conserves parity. Find the probability that the total spin of the two Ys along some given axis is observed to be $\hbar$ *and* their directions of travel with respect to this axis are within the angular range $\pi/4$ and $3\pi/4$. [You may assume the spherical harmonic function $Y_1^{-1}(\theta, \phi) \propto \sin\theta e^{-i\phi}$.] [14]

[*You may wish to make use of the following action of the angular momentum raising and lowering operators:*

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle \,. \qquad]$$

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) The phase space density function $f(\mathbf{x}, \mathbf{v})$ for a distribution of particles with equal mass m is a function of E alone, where $E = \frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{x})$ and $\Phi(\mathbf{x})$ is the gravitational potential. Show that $f(E)$ is a solution of the collisionless Boltzmann equation. [4]

Suppose $\mathcal{E} = -E + \Phi_0$ and $\Psi = -\Phi + \Phi_0$, where Φ_0 is a constant chosen such that $f(\mathcal{E}) > 0$ for $\mathcal{E} > 0$ and $f(\mathcal{E}) = 0$ for $\mathcal{E} \leq 0$. Show that the mass density of a spherical system can be written as

$$\rho(\Psi) = 2^{5/2}\,\pi\, m \int_0^{\Psi} f(\mathcal{E})\sqrt{\Psi - \mathcal{E}}\,\mathrm{d}\mathcal{E}.$$ [6]

(ii) Using the notation in Part (i), the distribution function $f(\mathcal{E})$ of the spherical system has the form

$$\begin{aligned} f(\mathcal{E}) &= F\,\mathcal{E}^{n-3/2} \qquad &&\text{for} \qquad \mathcal{E} > 0, \\ &= 0 &&\text{for} \qquad \mathcal{E} \leq 0, \end{aligned}$$

where F is a constant and n is a positive integer. Using the results of Part (i), or otherwise, show that the potential Ψ satisfies

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) + 4\pi\, G\, C\, \Psi^n = 0, \qquad (*)$$

where C is a constant that depends on F and n, an expression for which should be given. [9]

If $n = 5$ and $s = r\sqrt{4\pi G C \Psi_0^{n-1}}$, where $\Psi_0 = \Psi(0)$, show that

$$\Psi = \frac{\Psi_0}{\sqrt{1 + s^2/3}}$$

is a solution to $(*)$. [4]

Hence show that the density is non-zero everywhere and that the total mass is finite. [2]

Calculate the total mass. [5]

[*For a spherically symmetric function* $F(r)$, $\nabla^2 F = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}F}{\mathrm{d}r}\right)$.]

Question 8Z - Topics in Astrophysics

(i) Consider planetesimals that form in a protoplanetary disk at a time t_{f} after the formation of the star and its disk. Derive a constraint on t_{f} for heating due to short-lived radionuclides to lead to the melting of planetesimals with radius 50 km. You may assume that the mass fraction of ^{26}Al in planetesimal-forming solids at $t = 0$ is $X_0 = 10^{-7}$, the energy per decay of ^{26}Al is $\epsilon = 10^{-12}$ J, the half-life of ^{26}Al is $t_{1/2} = 10^5$ yr, and the enthalpy of fusion of rock is $\Delta H_{\mathrm{f}} = 10^6\ \mathrm{J\,kg^{-1}}$. You can neglect the energy cost of heating the planetesimals to the point of melting. [9]

Justify any assumptions you have made for heat transport in the planetesimals. You may assume that the thermal diffusivity of rock is $D = 10^{-6}\ \mathrm{m^2\,s^{-1}}$. [1]

(ii) Using the same assumptions about physical properties as in Part (i) where necessary, calculate the minimum radius, R_{min}, for a planet to melt due to the gravitational potential energy released during its accretion. You may assume a planet mass–radius scaling of $M \propto R^2$ in a range that includes the Earth, where radius is defined by the extent of the rock-dominated part of the planet excluding any atmosphere. [5]

Determine whether a planet of size R_{min} is likely to be able to retain a steam atmosphere. You may assume the atmosphere to be isothermal with $T = 1000$ K. [4]

Consider an Earth-like planet immediately after its formation. This planet is composed of 1% water by mass that entirely partitions into an isothermal 1000 K steam atmosphere. Determine the height in the atmosphere at which the pressure is 1 mbar. [9]

Assuming this planet could be observed in transit immediately following its formation, what is the increase in transit depth due to the steam atmosphere? You may assume that an optical depth of unity is reached at a pressure of 1 mbar. [2]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 14 June 2021 10:00am – 1:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may not attempt more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts. The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Each sheet should contain the number, letter, and part of the question being attempted, and a page number count for this question.

Answers must be uploaded in separate pdf files, named with your candidate number, paper number, and the letter **X, Y, Z**, *according to the letter associated with the questions, each separated by an underscore. For example,* **1X, 3X, and 6X** *should be in the same file and* **5Z, 7Z, and 8Z** *in another, each file labelled* **1234A_paper4_X** *or* **1234A_paper4_Z**, *where '1234A' should be replaced with your candidate number.*

After the exam there will be **30 minutes** *for you to scan and upload your solutions to ProctorExam. During this time you* **may not do any writing**, *and* **must complete the online cover sheet** *linked to in the ProctorExam instructions, identifying all the questions you have attempted.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Paper (blank sheets, plain or lined)	*Astrophysics Formulae Booklet (as pdf)*
1 Rough Work Pad	*Approved Calculators Allowed*

You can begin the paper provided you have already received instructions to start the exam from the invigilator.
Stop writing when indicated to by the invigilator, or following your allotted time.

Question 1X - Relativity

(i) A beam of photons, each with 4-momentum p^α, has a photon number density n in some inertial frame in Minkowski spacetime. By considering the energy density, momentum density and the flux of momentum, show that the energy–momentum tensor is

$$T^{\alpha\beta} = \frac{nc^2}{E} p^\alpha p^\beta \, ,$$

where E is the energy of each photon. [5]

Two photons in the beam are separated by distance ℓ along the direction of the beam. Determine their separation and energy in an inertial frame moving at relative velocity βc in the direction of the beam, and hence show that n/E is Lorentz invariant. [5]

(ii) The Schwarzschild solution for a mass M expressed in outgoing Eddington–Finkelstein coordinates is

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^{*2} + 4\frac{\mu c}{r} dt^* dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \, ,$$

where $\mu = GM/c^2$. By introducing the alternative time coordinate $u \equiv ct^* - r$, show that the transformed line element is

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) du^2 + 2du dr - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \, . \qquad (*)$$

[3]

Now consider an extension of the line element $(*)$ in which the mass M depends on u. Find the connection coefficients Γ^u_{uu}, Γ^r_{uu} and Γ^r_{ur} in this case. *You may assume for the rest of the question that these are the only non-zero coefficients of the form* $\Gamma^\alpha_{u\beta}$. [7]

The Ricci tensor can be expressed in terms of the connection coefficients as

$$R_{\alpha\beta} = -\partial_\rho \Gamma^\rho_{\alpha\beta} + \partial_\alpha \Gamma^\rho_{\rho\beta} + \Gamma^\rho_{\sigma\beta}\Gamma^\sigma_{\alpha\rho} - \Gamma^\rho_{\alpha\beta}\Gamma^\sigma_{\sigma\rho} \, .$$

Given that $\Gamma^\alpha_{\alpha\beta} = (2/r)\delta^r_\beta + \cot\theta \delta^\theta_\beta$, show that

$$R_{uu} = \frac{2}{r^2}\frac{d\mu}{du} \, .$$

[5]

Given that all other components of the Ricci tensor are zero, use the Einstein field equations to show that the energy–momentum tensor must take the form

$$T_{\alpha\beta} = -\frac{c^2}{4\pi r^2}\frac{dM}{du} l_\alpha l_\beta \, ,$$

where l_α are the components of a null dual-vector that you should specify. [3]

Using results from Part (i), or otherwise, give a physical interpretation of this solution in the case where $dM/du < 0$. [2]

TURN OVER...

Question 2Y - Astrophysical Fluid Dynamics

(i) Starting from the ideal fluid equations in conservative form, show that the Rankine-Hugoniot jump conditions can be written in a form that only involves velocity components perpendicular to the shock. [10]

(ii) Suppose that a perpendicular shock forms in an ionized gas that possesses a uniform magnetic field $\mathbf{B}_0$. Explain carefully why the Rankine-Hugoniot conditions are unaffected by the presence of the magnetic field if the magnetic field is aligned with the flow direction and the shock front is perpendicular to the flow direction. [5]

A magnetized white dwarf with mass M and radius R accretes cold (with negligble pressure), monatomic ($\gamma = 5/3$), gas down its magnetic poles. Close to the surface of the white dwarf in the vicinity of one of the magnetic poles, the accreting gas forms a column with cross-sectional area A and density ρ_1 that falls with the free-fall velocity (i.e., zero total energy) vertically towards the stellar surface before encountering a perpendicular adiabatic shock. The magnetic field in the accretion column has strength B_0 and is also aligned vertically. Use this information to calculate the density, pressure, and temperature of the post-shock gas in terms of M, R and ρ_1. [8]

The accretion column will undergo a structural change (i.e., "collapse") if the sound speed exceeds the Alfvén speed ($v_{\mathrm{A}} = \sqrt{B^2/(\rho\mu_0)}$, where μ_0 is the vacuum magnetic permeability) in the post-shock region. Estimate the critical mass accretion rate $\dot{M}_{\mathrm{crit}}$ above which the accretion column will collapse, giving your answer in terms of A, B_0, M and R. [4]

Estimate $\dot{M}_{\mathrm{crit}}$ and the corresponding luminosity in the case of a white dwarf with $M = 1\,M_{\odot}$, $R = 7000\,\mathrm{km}$, magnetic field $B_0 = 10^3\,\mathrm{T}$, and an accretion column that has a circular cross-section with radius $r = 300\,\mathrm{km}$. [3]

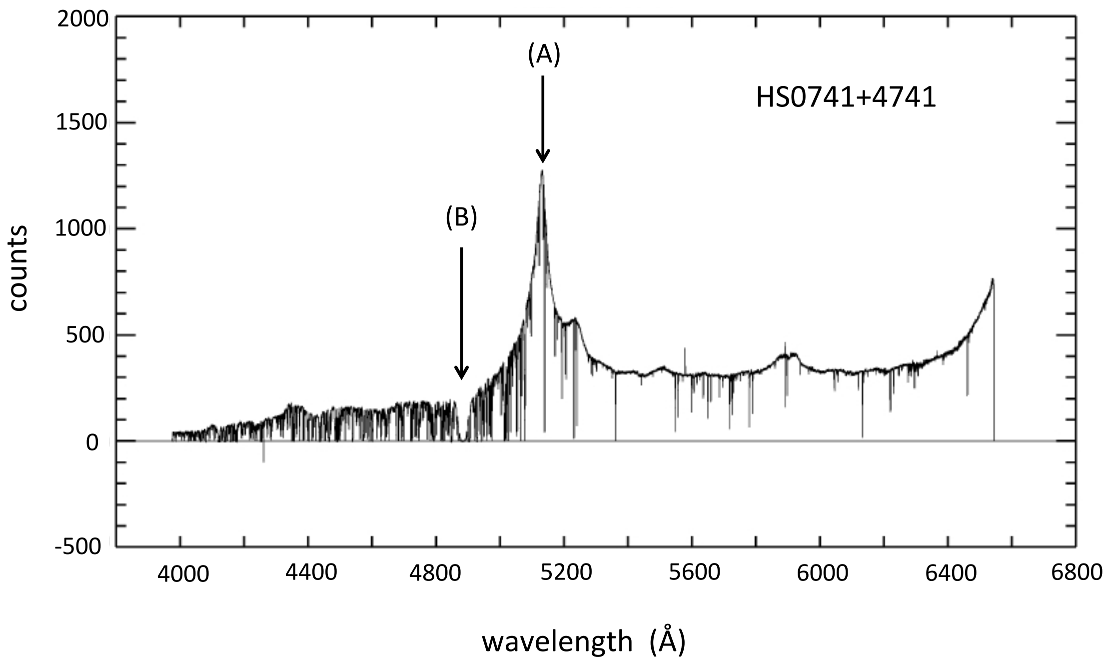
[*If M_1 is the Mach number of the incoming flow, you may use without proof the following relations between pre- and post-shock quantities:*

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2},$$
$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}.$$

]

Question 3X - Cosmology

(i) The figure below shows a spectrum of the quasar HS0741+4741.



Explain the main features in this spectrum in as much detail as you can, in particular:

(a) identify the emission line (A) and estimate the redshift of the quasar;

(b) explain why there are many more absorption lines on the blue side of the emission line (A) compared to the red side; and

(c) explain why the absorption line (B) is so much broader and deeper than neighbouring lines. [8]

Estimate the velocity width in $\mathrm{km\,s^{-1}}$ of an optically-thin Lyα absorption line if the intergalactic medium has a temperature of $T = 10^4\,\mathrm{K}$. [2]

[*The rest-frame wavelength of the Lyα transition is* 1215.7 Å.]

TURN OVER...

(ii) It has been proposed that by observing large numbers of quasar absorption lines with a stable high-resolution spectrograph it might be possible to measure a change in the source redshift over a time interval from t_0 to $t_0 + \Delta t_0$ at the observer. If the source redshift is z_s when observed at t_0, and the light observed in Δt_0 is emitted over a time interval t_s to $t_s + \Delta t_s$, show that the change in the source redshift is

$$\begin{aligned}\Delta z &= \frac{R(t_0 + \Delta t_0)}{R(t_s + \Delta t_s)} - \frac{R(t_0)}{R(t_s)} \\ &\approx \left[-H(z_s) + (1 + z_s)H_0\right] \Delta t_0 \,, \qquad (*)\end{aligned}$$

where overdots denote differentiation with respect to time, $R(t)$ is the scale factor and $H = \dot{R}/R$ is the Hubble parameter with present-day value H_0. [8]

Changes in redshift, Δz, are often expressed in terms of an equivalent *spectroscopic velocity shift*, $\Delta v \equiv c\Delta z/(1 + z_s)$. Assuming the Hubble parameter is given by

$$H(z) = H_0 \Omega_m^{1/2} (1 + z)^{3/2} \,,$$

where $\Omega_m = 0.3$ is the present-day matter density parameter, show from $(*)$ that the spectroscopic velocity shift is

$$\Delta v = -cH_0 \Delta t_0 \left[\Omega_m^{1/2} (1 + z_s)^{1/2} - 1\right] \,. \qquad (**)$$ [4]

Estimate Δv if $\Delta t_0 = 100\,$years and $z_s = 3$. [3]

Using your result from Part (i), compare this Δv to the typical velocity width of a Lyα absorption line and comment on what you find. [2]

The acceleration of the Sun towards the Galactic centre is estimated to be $a \approx 2.3 \times 10^{-10}\,\mathrm{m\,s^{-2}}$. Is the velocity shift associated with the Solar motion comparable to the velocity shift computed from $(**)$? [3]

[*You may adopt* $1/H_0 = 1.45 \times 10^{10}$ *years.*]

Question 4Y - Structure and Evolution of Stars

(i) In a non-relativistic white dwarf, the energy density of the degenerate gas can be written as

$$U_{\mathrm{e}} \propto n_{\mathrm{e}}^{5/3} \, ,$$

where n_{e} is the number density of electrons. By considering the sum of gravitational and kinetic energy, where the latter is $E_{\mathrm{K}} \propto U_{\mathrm{e}} V$ (where V is the volume), show that a white dwarf radius is inversely proportional to the cube root of its mass. [7]

Hence, show that the white dwarf density is proportional to the square of its mass and give a qualitative physical explanation for this proportionality. [3]

(ii) At the end of the AGB phase, a solar mass star ejects its remaining envelope to expose its degenerate He core as a $0.5\,M_{\odot}$ white dwarf of radius $R \simeq 7000\,\mathrm{km}$. Using the virial theorem (or otherwise), estimate the temperature of the white dwarf at this early stage. [8]

Suggest an observational test that uses light in the visible range to verify qualitatively your previous answer. [6]

How does your answer compare with the measured values of effective temperature, T_{eff}, in most white dwarfs? Give reasons for your answer. [4]

In the Milky Way galaxy, why are no white dwarfs known with $T_{\mathrm{eff}} < 3000\,\mathrm{K}$. [2]

TURN OVER...

Question 5Z - Statistical Physics

(i) The (Helmholtz) free energy F is defined in terms of the energy E, temperature T and entropy S by $F = E - TS$. Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V ,$$

where p and V denote the pressure and volume, respectively, of the system. [5]

Explain why the free energy is the appropriate thermodynamic potential to consider at fixed T, V and particle number N. [5]

(ii) Explain what is meant by a *first-order* phase transition and a *second-order* phase transition. [3]

Consider a ferromagnet with free energy

$$F(T, m) = F_0(T) + \frac{a}{2}(T - T_{\mathrm{c}})m^2 + \frac{b}{4}m^4 + \frac{c}{6}m^6 ,$$

where T is the temperature, m is the magnetization, and $a, c, T_{\mathrm{c}} > 0$, $b \leqslant 0$ are constants. Find the equilibrium values of m at different temperatures, as well as any metastable states, quantifying the transition between different regimes. [8]

Explaining your reasoning, sketch F as a function of m in the regimes identified and give the order of the phase transition. [4]

For $b = 0$ compute the entropy and heat capacity at high and low temperatures. [5]

Question 6X - Principles of Quantum Mechanics

(i) A quantum system has Hamiltonian $H = H_0 + V(t)$. Let $\{|n\rangle\}_{n\in\mathbb{N}_0}$ be an orthonormal basis of H_0 eigenstates, with corresponding energies $\{E_n\}$. When $t < 0$, $V(t) = 0$ and the system was in state $|0\rangle$. Calculate the probability it is found to be in state $|1\rangle$ at time $t > 0$, correct to leading order in V. [10]

(ii) Now suppose $\{|0\rangle, |1\rangle\}$ form a basis of the Hilbert space for the system in Part (i), with respect to which

$$\begin{pmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{pmatrix} = \begin{pmatrix} \hbar\omega_0 & \Theta(t)\,\hbar v\, e^{i\omega t} \\ \Theta(t)\,\hbar v\, e^{-i\omega t} & \hbar\omega_1 \end{pmatrix},$$

where $\Theta(t)$ is the Heaviside step function and v is a constant. Calculate the exact probability that the system is in state $|1\rangle$ at time t. [12]

Show that your approximate result in Part (i) is consistent with the exact probability. [4]

Let $P_{\max}(\omega)$ be the maximum exact probability attained for any $t > 0$. For which frequency ω is $P_{\max}$ maximized? [4]

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) In cylindrical polar coordinates (R, ϕ, z),

$$(\dot{v}_R, \dot{v}_\phi, \dot{v}_z) = \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}, -\frac{v_R\, v_\phi}{R} - \frac{1}{R}\frac{\partial \Phi}{\partial \phi}, -\frac{\partial \Phi}{\partial z} \right),$$

where v_R, v_ϕ and v_z are the velocity components, overdots denote differentiation with respect to time t, and Φ is the gravitational potential. Use this to derive the collisionless Boltzmann equation for an axisymmetric system in terms of the phase space coordinates $(R, \phi, z, v_R, v_\phi, v_z)$ and the gravitational potential. [4]

By taking a first moment of the collisionless Boltzmann equation show that the corresponding Jeans equation for an axisymmetric system in steady state can be written as

$$\frac{\partial}{\partial R}\left(\nu\, \overline{v_R v_z}\right) + \frac{\partial}{\partial z}\left(\nu\, \overline{v_z^2}\right) + \nu\, \frac{\overline{v_R v_z}}{R} + \nu\, \frac{\partial \Phi}{\partial z} = 0,$$

where

$$\nu = \int f\, \mathrm{d}^3\mathbf{v}, \qquad \overline{v_i} = \frac{1}{\nu}\int f\, v_i\, \mathrm{d}^3\mathbf{v}, \qquad \overline{v_i v_j} = \frac{1}{\nu}\int f\, v_i\, v_j\, \mathrm{d}^3\mathbf{v},$$

and f is the distribution function. [6]

(ii) Use the Jeans equation given in Part (i) to show that, for motion satisfying $\overline{v_R v_z} \ll \overline{v}_z^2$, near the plane of a highly flattened axisymmetric system in steady state

$$\frac{\partial}{\partial z}\left[\frac{1}{\nu}\frac{\partial}{\partial z}\left(\nu\, \overline{v_z^2}\right)\right] = -4\pi\, G\, \rho,$$

where ρ is the mass density. [11]

Consider a disc galaxy where all the mass is contained at a central point and the mass density of the disc is negligible. The root-mean-square of the z-component of the velocity of the stars in the disc, which may be considered as having zero mass, is σ_z, which is independent of z. At radius R the number density of stars as a function of z is $\nu(z) = \nu_0\, e^{-z^2/(2z_0^2)}$, where ν_0 and $z_0 \ll R$ are constants.

What is the relation between σ_z and z_0? [5]

If the stellar motions are interpreted (incorrectly) as being due to self-gravity of mass in the disc, what is the inferred mean mass density in the plane within a scale length z_0 of $z = 0$? [4]

Question 8Z - Topics in Astrophysics

(i) A planet of mass M_{p} is on a circular orbit at a separation a from a star of mass $M_\star \gg M_{\mathrm{p}}$. Derive an expression for the reflex motion, $v_\star$, of the star in response to the planet. [4]

Spectroscopic observations of a solar mass star show an absorption line in its optical spectrum to shift by 0.001 nm with a period of 10 days. What is the minimum planet mass in orbit around the star to explain this observation? [5]

Can this line-shift, of a single line, be observed in a spectrum taken with a resolution $R = 10^5$? [1]

(ii) A protoplanetary disk is locally isothermal and has a radial pressure gradient $p \propto r^{-n}$, where r is the radius and n is a constant power law index. Show that the azimuthal gas velocity in the disk can be written as

$$v_{\phi,\mathrm{g}} = v_{\mathrm{K}}(1-\eta)^{1/2},$$

where v_{K} is the Keplerian velocity in the disk, $\eta = nc_{\mathrm{s}}^2/v_{\mathrm{K}}^2$, and c_{s} is the sound speed of the gas. [7]

In a disk with a minimum-mass solar-nebula surface density, $\Sigma \propto r^{-3/2}$, and with a sound speed that scales as $c_{\mathrm{s}} \propto r^{-\alpha}$, show that $\frac{dp}{dr} \propto r^{-4-\alpha}$. [6]

If a pressure bump now develops in the disk of radial extent Δr that is capable of trapping solids, show that the timescale for accumulation of particles in this pressure bump is a factor $(\Delta r/r)^2$ of the particle infall time. You may assume that the radial velocity of particles is given by $v_r = \eta v_{\mathrm{K}}/2$. [5]

How would the formation of such a pressure bump affect planet formation in its vicinity? [2]

[*You may assume* $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.]

END OF PAPER