PAPER IV

Course

RELATIVITY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

1

Examiner

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

CHALLINOR

Section

[✓ I] / II

Draft Mark Scheme

Comments

(i) Let $p^\mu = \frac{E}{c}(1, \vec{e})$ where $\vec{e}$ is a unit 3-vector giving the direction of propagation. The energy density is $T^{00} = nE$; the momentum density is $\frac{T^{i0}}{c} = n\frac{E}{c}\vec{e}^{\,i}$; as each particle is moving at $c\vec{e}$ 3-velocity, the flux of momentum $T^{ij} = \begin{pmatrix}\text{momentum}\\ \text{density}\end{pmatrix}^i c\vec{e}^{\,j}$

$$= \frac{nE}{c}\vec{e}^{\,i} c \vec{e}^{\,j} = nE\,\vec{e}^{\,i}\vec{e}^{\,j}.$$

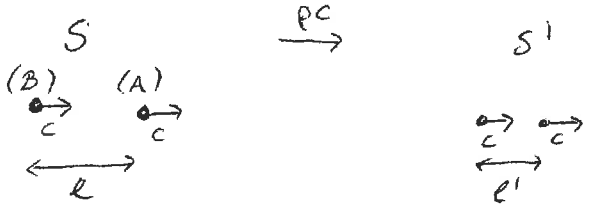
Comparing with the expression $T^{\mu\nu} = \frac{nc^2}{E}p^\mu p^\nu$, we have

[1] $$T^{00} = \frac{nc^2}{E}\left(\frac{E}{c}\right)^2 = nE \checkmark$$

SIMILAR TO "DUST" CASE IN NOTES

[2] $$T^{i0} = \frac{nc^2}{E}p^i p^0 = \frac{nc^2}{E}\frac{E}{c}\vec{e}^{\,i}\frac{E}{c} = nE\vec{e}^{\,i} \checkmark$$

[2] $$T^{ij} = \frac{nc^2}{E}\frac{E}{c}\vec{e}^{\,i}\frac{E}{c}\vec{e}^{\,j} = nE\vec{e}^{\,i}\vec{e}^{\,j} \checkmark$$



l' = distance between the photons in S' with $\Delta t' = 0$.

Let $\Delta x' = l'$.

SIMILAR TO EXAMPLES SHEET

Then L.T. $\Rightarrow$ $\Delta x = \gamma(\Delta x' + \beta c\Delta t') = \gamma l'$

$c\Delta t = \gamma(c\Delta t' + \beta\Delta x') = \gamma\beta l'$.

These events are not simultaneous in S; the measurement of photon A is later than B by Δt, and B moves $c\Delta t$ in this time.

Please do not write below this line

Page

1

Course

RELATIVITY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

1

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Examiner

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

CHALLINOR

Section

[✓ I] / [✓ II]

Draft Mark Scheme

Comments

(i) CONTINUED

Follows that the separation of the photons in S at equal times (i.e., ℓ) is

[3]
$$\ell = \Delta x - c\Delta t = \gamma \ell'(1-\beta) \Rightarrow \underline{\underline{\ell' = \frac{\ell}{\gamma(1-\beta)}}}$$

Since $n \propto \frac{1}{\ell}$, $\frac{n'}{n} = \frac{\ell}{\ell'} = \gamma(1-\beta)$.

[1]
The energy in S' is E' where $\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta\frac{E}{c}\right)$

$$\Rightarrow \underline{\underline{E' = \gamma E(1-\beta)}}.$$

[1]
Follows that $\underline{\underline{\frac{n'}{E'} = \frac{n}{E}}} \to$ L.I. (as it must be for the expression for $T^{\mu\nu}$ to make sense).

UNSEEN INTERPRETATION

(ii) $ct^* = u + r \Rightarrow d(ct^*) = du + dr$.

[3]
Follows that
$$ds^2 = \left(1-\frac{2\mu}{r}\right)(du+dr)^2 + \frac{4\mu}{r}dr(du+dr) - \left(1+\frac{2\mu}{r}\right)dr^2 - r^2 d\Omega^2$$

$$= \left(1-\frac{2\mu}{r}\right)du^2 + \left(1-\frac{2\mu}{r} + \frac{4\mu}{r} - 1 - \frac{2\mu}{r}\right)dr^2 + \left(2\left(1-\frac{2\mu}{r}\right) + \frac{4\mu}{r}\right)du\,dr - r^2 d\Omega^2$$

$$= \underline{\underline{\left(1-\frac{2\mu}{r}\right)du^2 + 2du\,dr - r^2 d\Omega^2}}$$

UNSEEN CALCULATION

Page

2

Please do not write below this line

Course

RELATIVITY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

1

Examiner

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

CHALLINOR

Section

I / [✓ II]

Draft Mark Scheme

Comments

(ii) CONTINUED

[1]

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2\mu}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 0 & 1 & & \\ 1 & -(1-\frac{2\mu}{r}) & & \\ & & -\frac{1}{r^2} & \\ & & & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

[2]

$$\Gamma^u_{uu} = \frac{1}{2} g^{ua}(2\partial_u g_{au} - \partial_a g_{uu})$$
$$= \frac{1}{2} g^{ur}(2\partial_u g_{ru} - \partial_r g_{uu})$$
$$= -\frac{1}{2}\partial_r(1-2\mu/r) = \underline{\underline{-\mu/r^2}}$$

UNSEEN FOR THIS METRIC

[2]

$$\Gamma^r_{uu} = \frac{1}{2} g^{ra}(2\partial_u g_{au} - \partial_a g_{uu})$$
$$= \frac{1}{2} g^{ru}(2\partial_u g_{uu} - \partial_u g_{uu}) + \frac{1}{2} g^{rr}(2\partial_u g_{ru} - \partial_r g_{uu})$$
$$= \frac{1}{2}\left(-\frac{2\mu'}{r}\right) + \frac{1}{2}\left(1-\frac{2\mu}{r}\right)\partial_r\left(1-\frac{2\mu}{r}\right) \qquad \left({}' \equiv \frac{d}{du}\right)$$
$$= \underline{\underline{-\frac{\mu'}{r} + \left(1-\frac{2\mu}{r}\right)\frac{\mu}{r^2}}}$$

[2]

$$\Gamma^r_{ur} = \frac{1}{2} g^{ra}(\partial_u g_{ar} + \partial_r g_{au} - \partial_a g_{ur})$$
$$= \frac{1}{2} g^{ru}(\partial_u g_{ur} + \partial_r g_{uu}) + \frac{1}{2} g^{rr}(\partial_u g_{rr} + \partial_r g_{ru})$$
$$= \frac{1}{2}\partial_r\left(1-\frac{2\mu}{r}\right) \cancel{+ \frac{1}{2}\left(1-\frac{2\mu}{r}\right)\frac{d}{du}\left(1-\frac{2\mu}{r}\right)}$$
$$= \underline{\underline{\frac{\mu}{r^2}}} \cancel{+ \left(1-\frac{2\mu}{r}\right)\frac{\mu'}{r}}$$

Please do not write below this line

Page

Course

RELATIVITY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

1

Examiner

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

CHALLINOR

Section

I / [✓ II]

Draft Mark Scheme

Comments

(ii) CONTINUED

$$R_{uu} = -\partial_\rho \Gamma^\rho_{uu} + \partial_u \Gamma^\rho_{\rho u} + \Gamma^\rho_{\sigma u}\Gamma^\sigma_{u\rho} - \Gamma^\rho_{uu}\Gamma^\sigma_{\sigma\rho}$$

$$= -\partial_u \Gamma^u_{uu} - \partial_r \Gamma^r_{uu} + \Gamma^\rho_{ru}\Gamma^r_{u\rho} + \Gamma^\rho_{uu}\Gamma^u_{u\rho}$$

↘ requires $\rho = r$ ↘ requires $\rho = u$

$$-\Gamma^u_{uu}\Gamma^\sigma_{\sigma u} - \Gamma^r_{uu}\Gamma^\sigma_{\sigma r}$$

$$= -\partial_u\left(\frac{-\mu}{r^2}\right) - \partial_r\left[\frac{\mu}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{\mu'}{r}\right] + \left(\frac{\mu}{r^2}\right)^2 + \left(\frac{-\mu}{r^2}\right)^2$$

$$+\frac{\mu}{r^2}\times 0 - \left[\frac{\mu}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{\mu'}{r}\right]\frac{2}{r},$$

[5]

UNSEEN FOR THIS METRIC

using $\Gamma^\sigma_{\sigma r} = \frac{2}{r}$

$$\text{Hence } R_{uu} = \frac{\mu'}{r^2} + \frac{2\mu}{r^3}\left(1-\frac{2\mu}{r}\right) - \frac{\mu}{r^2}\frac{2\mu}{r^2} - \frac{\mu'}{r^2} + 2\frac{\mu^2}{r^4}$$

$$-\frac{2\mu}{r^3}\left(1-\frac{2\mu}{r}\right) + \frac{2\mu'}{r^2}$$

$$= \underline{\underline{\frac{2\mu'}{r^2}}}.$$

Since $g^{uu} = 0$, $R = 0$, so EFEs $\Rightarrow R_{\mu\nu} = \frac{-8\pi G}{c^4} T_{\mu\nu}$.

Follows that $T_{\mu\nu} = \frac{-c^4}{8\pi G}\frac{2}{r^2}\frac{d}{du}\left(\frac{GM}{c^2}\right)\delta^u_\mu \delta^u_\nu$

$$= \underline{\underline{\frac{-c^2}{4\pi r^2}\frac{dM}{du}}}\,\delta^u_\mu\delta^u_\nu$$

[3]

UNSEEN CALCULATION

This is of the form given with $\underline{\underline{l_\mu = \delta^u_\mu}}$. This is null since $g^{\mu\nu} l_\mu l_\nu = g^{uu} = 0$

[Note $l_\mu = \partial_\mu u$]

Page

4

Please do not write below this line

Course

RELATIVITY

NST Astrophysics Part II: 2020-21

Model Solution

Question Number

1

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Examiner

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

CHALLINOR

Section

I / [✓ II]

Draft Mark Scheme

Comments

(ii) CONTINUED

For $\frac{dM}{du} < 0$, have $T_{\mu\nu} = (+ve)\, l_\mu l_\nu$ where l_μ is null $(= \partial_\mu u)$. Have a non-vacuum solution of the EFEs (and not stationary). The form of $T_{\mu\nu}$ is appropriate for radiation flowing radially outwards → describes a star losing mass to radiation. [2]

UNSEEN

—o—

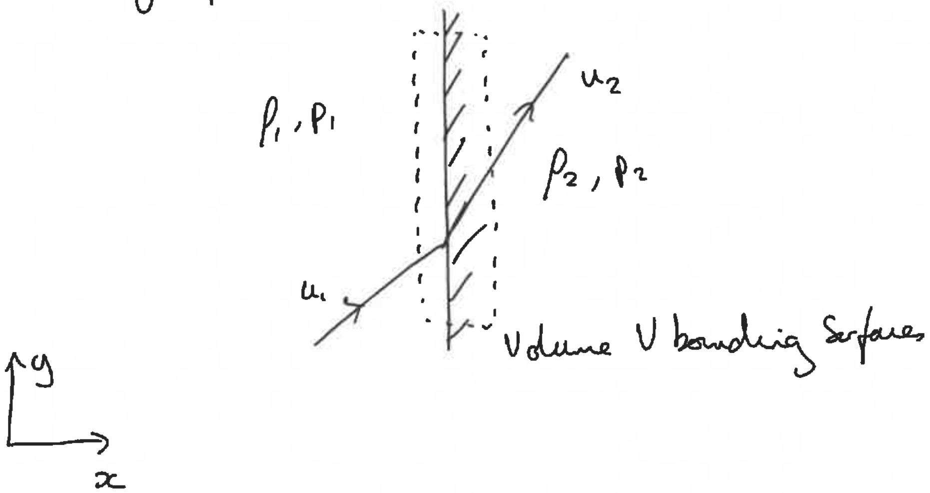
Please do not write below this line

Paper IV

(AFD)

2 i) Rankine-Hugoniot jump conditions

Shock jump:



Mass equation: $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0$

$$\Rightarrow \frac{\partial}{\partial t}\int_V \rho \, dV + \int_V \nabla\cdot(\rho\underline{u})\, dV = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}\int_V \rho \, dV}_{0\text{, as no mass build up at shock}} + \underbrace{\int_S \rho \underline{u}\cdot d\underline{s}}_{= \rho_2 u_{2x} - \rho_1 u_{1x}} = 0$$

$$\Rightarrow \rho_1 u_{1x} = \rho_2 u_{2x} \quad // \qquad \text{(RH1)}$$

2 //

Momentum eq. $\cancelto{0}{\frac{\partial}{\partial t}(\rho\underline{u})} + \nabla\cdot(\rho\underline{u}\underline{u} + p\underline{\underline{I}}) = 0$

$$\Rightarrow \quad \int_S (\rho\underline{u}\underline{u} + p\underline{\underline{I}})\cdot d\underline{S} = 0$$

x- component of vector expression is

$$[\rho u_x u_x + p]_1^2 = 0$$

$\Rightarrow \rho_1 u_{1x}^2 + p_1 = \rho_2 u_{2x}^2 + p_2$ // $\quad$ RH2a $\quad$ 3

y- component of vector expression is

$$[\rho u_x u_y]_1^2 = 0$$

$$\Rightarrow \rho_1 u_{1x} u_{1y} = \rho_2 u_{2x} u_{2y}$$

As $\dfrac{\rho_1 u_{1x}}{\rho_2 u_{2x}} = 1$ from RH1

then $u_{1y} = u_{2y}$ // $\quad$ RH2b $\quad$ 2

Energy equation: $\frac{\partial E}{\partial t} + \nabla \cdot \left[(E+p)\underline{u} \right] = 0$

$$\Rightarrow \int_S (E+p)\underline{u} \cdot d\underline{S} = 0$$

$$\Rightarrow \left[(E+p) u_x \right]_1^2 = 0$$

Since $E = \rho \left(\frac{1}{2} u^2 + \mathcal{E} + \Psi \right)$

$\mathcal{E}$ = Specific internal energy

Ψ = gravitational potential energy

Then $(E_1 + p_1) u_{1x} = (E_2 + p_2) u_{2x}$

$$\Rightarrow \rho_1 u_{1x} \left(\frac{1}{2} u_1^2 + \mathcal{E}_1 + \Psi_1 + p_1/\rho_1 \right) = \rho_2 u_{2x} \left(\frac{1}{2} u_2^2 + \mathcal{E}_2 + \Psi_2 + p_2/\rho_2 \right)$$

$$\frac{1}{2}\left(u_{1x}^2 + u_{1y}^2 \right) + \mathcal{E}_1 + \Psi_1 + p_1/\rho_1 = \frac{1}{2}\left(u_{2x}^2 + u_{2y}^2 \right) + \mathcal{E}_2 + \Psi_2 + p_2/\rho_2$$

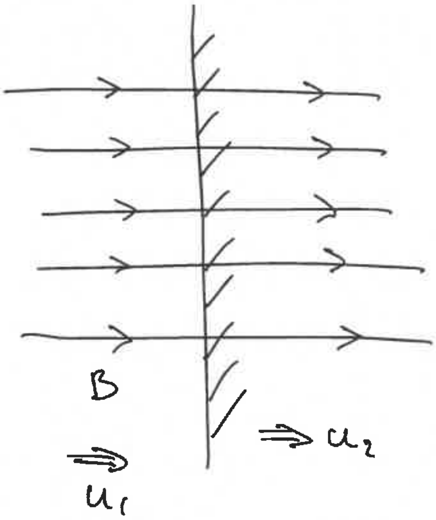
Now, $\Psi_1 = \Psi_2$, $u_{1y} = u_{2y}$

$$\therefore \frac{1}{2} u_{1x}^2 + \mathcal{E}_1 + p_1/\rho_1 = \frac{1}{2} u_{2x}^2 + \mathcal{E}_2 + p_2/\rho_2$$ //

RHS

3//

As required, all only include x component of velocity ii)



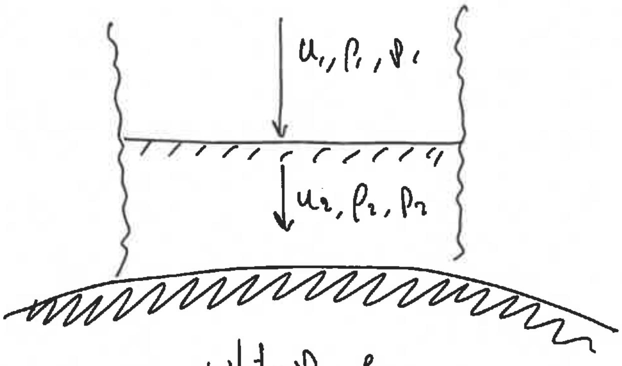
Magnetic field lines are frozen into the plasma. Since the velocity $\underline{u}_1 // \underline{B}$, and the shock is $\perp$ so that $\underline{u}_2 // \underline{u}_1$, the magnetic field will be carried through the shock unchanged.

Thus, there will be no change in the momentum flux or energy flux associated with magnetic field.

$\Rightarrow$ magnetic terms will not change the the R-H conditions. //

$\underline{S}$//

$M, R \quad \gamma = 5/3, \ \lambda, \ \rho_1, \ B_0$



Free-fall $\quad u_1 = \sqrt{\frac{2GM}{R}}$

$$c_s^2 = \gamma P/\rho$$

Cold incoming flow, so can assume $c_{s,1} \ll u_1$

$$\Rightarrow M_1 \gg 1$$

2//

Density jump: $\dfrac{\rho_2}{\rho_1} = \dfrac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}$

$$\rightarrow \frac{\gamma+1}{\gamma-1} \quad \text{as } M_1 \to \infty$$

$$= 4 \qquad \text{for } \gamma = 5/3$$

$$\therefore \rho_2 = 4\rho_1$$ 2

Pressure jump
$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}$$

$$\rightarrow \frac{2\gamma M_1^2}{\gamma+1}$$

$$= \frac{5}{4} M_1^2$$

$$\Rightarrow p_2 = \frac{5}{4}\frac{u_1^2}{c_s^2} p_1$$

$$= \frac{5}{4}\frac{3}{5} u_1^2 \rho_1$$

$$= \frac{3}{4}\cdot\frac{2GM}{R}\rho_1$$

$$= \frac{3}{2}\frac{GM}{R}\rho_1$$ 2

Post shock temperature from ideal gas law:

$$p_2 = \frac{\rho_2 k_B T_2}{\mu m_p}$$

$$\Rightarrow T_2 = \frac{p_2}{\rho_2}\frac{\mu m_p}{k_B}$$

$$= \frac{3}{2} \cdot \frac{GM}{R} \frac{\rho_1}{\rho_2} \frac{\mu m_p}{k_B}$$

$$= \frac{3}{8} \frac{GM}{R} \frac{\mu m_p}{k_B} \quad // \qquad \underline{\underline{2}}$$

Collapse if $C_s > v_A = (B^2/\rho\mu_0)^{1/2}$ $\qquad A, B_0, M, R$

i.e @ $C_s^2 = V_A^2$

$$\frac{\gamma p_2}{\rho_2} = \frac{B_0^2}{\mu_0 \rho_2}$$ (B₀ constant through column) $\underline{\underline{2}}$

$$p_2 = \frac{3}{5} \frac{B_0^2}{\mu_0}$$

$$\frac{3}{2} \frac{GM}{R} \rho_1 = \frac{3}{5} \frac{B_0^2}{\mu_0}$$

Now, $\dot{M} = \underset{\substack{\sim \\ A}}{\pi r^2} \rho_1 u_1 \Rightarrow \rho_1 = \frac{\dot{M}}{\pi r^2} \left(\frac{R}{2GM}\right)^{1/2}$

$$\Rightarrow \frac{3}{2\sqrt{2}} \left(\frac{GM}{R}\right)^{1/2} \cdot \frac{\dot{M}}{\pi r^2} = \frac{3}{5} \frac{B_0^2}{\mu_0}$$

$$\dot{M} = \frac{2\sqrt{2}}{5} \pi r^2 \frac{B_0^2}{\mu_0} \left(\frac{R}{GM}\right)^{1/2}$$

$$= \frac{2\sqrt{2}}{5} \Lambda \frac{B_0^2}{\mu_0} \left(\frac{R}{GM}\right)^{1/2} //$$

2/

$\dot{M}_{crit}$, $M = 1 M_\odot$ $R = 7000$ km $B_0 = 10^3$ T $r = 300$ km

$\dot{M} \approx 2.9 \times 10^{16}$ kg s^{-1} //

1/

$$L \sim \frac{GM}{R} \dot{M} \approx 5.5 \times 10^{29} \text{ W}$$

2/

Specific energy
per unit mass

Paper IV Cosmology

Course
COSMOLOGY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number
3X

Examiner
CHALLINOR

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper
1 / 2 / 3 / [✓ 4]

Lecturer
EFSTATHIOU

Section
[✓ I] / II

Draft Mark Scheme

Comments

The prominent emission line (A) is the Lyα line, with rest-frame wavelength 1215.7 Å. In the observed spectrum, this is redshifted to 1215.7 Å $\times (1+z) \simeq 5150$ Å,

[2] hence $1+z \simeq \frac{5150}{1215.7} \Rightarrow \underline{z \simeq 3.24}$

[1] The lines at longer wavelength are mostly "metal" lines (e.g., CIV, SiII, MgII)

[2] At shorter wavelengths, most of the absorption lines are Lyα absorption by neutral hydrogen in low density (~ mean density) clouds. Since a resonant line, only need low column density to produce a significant absorption line.

Since there is transmission, we can deduce that the IGM is highly ionised at $z \simeq 3$

[3] Most of the lines are optically thin, with column densities $N_{HI} \sim 10^{13}\ \text{cm}^{-2}$, but (B) is a damped Lyman-α system with $N_{HI} \gtrsim 10^{20}\ \text{cm}^{-2}$. This line is optically thick ($\tau \gtrsim 10^4$ at line centre), so no transmission ($e^{-\tau}$) there. The line shape is determined by Doppler and collisional broadening, with damping wings of the Lorentzian profile from collisional broadening.

Please do not write below this line

Page
1

Course

COSMOLOGY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

3X

Examiner

CHALLINOR

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

EFSTATHIOU

Section

[✓ I] / [✓ II]

Draft Mark Scheme

Comments

For a Maxwellian distribution,

$$p(v_{\parallel}) \propto e^{-\frac{1}{2} m v_{\parallel}^2 / k_B T},$$

which is Gaussian with $\sigma_v^2 = \frac{k_B T}{m_p}$ since hydrogen

[2]

The velocity width is usually quoted as $\sqrt{2}\sigma_v$ so

$$\sqrt{2}\sigma_v = \frac{\sqrt{2 k_B T}}{\sqrt{m_p}} = \underline{12 \cdot 8 \, \mathrm{km\,s^{-1}}}$$

/10

(ii) $1 + z_s = \frac{R(t_0)}{R(t_s)}$. At time $t_0 + \Delta t_0$, the new redshift is $1 + z_s' = \frac{R(t_0 + \Delta t_0)}{R(t_s + \Delta t_s)}$, so

$$\Delta z = z_s' - z_s = \frac{R(t_0 + \Delta t_0)}{R(t_s + \Delta t_s)} - \frac{R(t_0)}{R(t_s)}$$

$$= \frac{R(t_0)(1 + H_0 \Delta t_0 + \quad)}{R(t_s)(1 + H(t_s)\Delta t_s + \cdots)} - \frac{R(t_0)}{R(t_s)}$$

$$\simeq \left[H_0 \Delta t_0 - H(t_s)\Delta t_s\right] \frac{R(t_0)}{R(t_s)}$$

$H(t_s)$

[8] However, $\Delta t_s = \frac{R(t_s)}{R(t_0)} \Delta t_0$, so $\Delta z = \underline{\left[H_0(1+z_s) - H(t_s)\right] \Delta t_0}$

Please do not write below this line

Course

COSMOLOGY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

3X

Examiner

CHALLINOR

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

EFSTATHIOU

Section

I / [✓ II]

Draft Mark Scheme

Comments

$$\Delta z = \left[H_0(1+z_s) - H_0\Omega_m^{1/2}(1+z_s)^{3/2}\right]\Delta t_0 ,$$

so
$$\Delta V = \frac{c}{(1+z_s)} H_0(1+z_s)\left[1 - \Omega_m^{1/2}(1+z_s)^{1/2}\right]\Delta t_0 .$$

(4)

$$= -cH_0\Delta t_0\left[\Omega_m^{1/2}(1+z_s)^{1/2} - 1\right]$$

$$\frac{\Delta V}{c} = \frac{10^2\,\mathrm{yr}}{1.45\times10^{10}\,\mathrm{yr}}\left[\sqrt{0.3}\times 2 - 1\right] = 6.6\times10^{-10}$$

[3]

$$\Rightarrow \Delta V = 0.20\ \mathrm{m\,s^{-1}}$$

The typical velocity width of a line in the IGM is ~10 km s^{-1}, so need to measure ΔV to around 10^{-5} of the Doppler width. Very challenging! This would require monitoring of many lines.

[2]

The change in the velocity of the sun in 100yr along the line of sight to a quasar that is parallel to the acceleration is $a\Delta t_0 = 0.7\ \mathrm{m\,s^{-1}}$.

This is comparable to the redshift drift and so would have to be corrected for.

[3]

/20

Please do not write below this line

Page

Stars Q4

Paper IV

4i) $u_e \propto n_e^{5/3}$, n_e is volume density of electrons

$E_e \propto u_e V$, V is volume

For kinetic energy $E_k \propto u_e V \propto n_e^{5/3} V$

$$\text{Now} \quad n_e \propto \frac{M}{R^3}$$

$$\Rightarrow n_e^{5/3} \propto \frac{M^{5/3}}{R^5}$$

$$\Rightarrow n_e^{5/3} V \propto \frac{M^{5/3}}{R^2}$$

GPE $E_g \propto -\frac{M^2}{R}$

1 ∴ total energy $E_{tot} = B\frac{M^{5/3}}{R^2} - C\frac{M^2}{R}$, where B & C are constants

Equilibrium radius found when $\frac{dE_{tot}}{dR} = 0$

So, $\frac{dE_{tot}}{dr} = -2B\frac{M^{5/3}}{R^3} + C\frac{M^2}{R^2} = 0$

$$\therefore \quad R = \frac{2BM^{5/3}}{CM^2}$$

$$= \frac{2B}{C} M^{-1/3}$$

⑦

Stars Q4

$$\rho = \frac{M}{\frac{4}{3}\pi r^3}$$

from above $\ell \propto M^{-1/3}$

$$\therefore \rho \propto M \cdot M$$
$$\propto M^2 \quad //$$

①

this result is a consequence of the star deriving its support from electron degeneracy pressure. The electrons must be more closely confined to generate the larger pressure to support a more massive star //

②

ii) Obtain T by recalling WD formed by the contraction of thermally unsupported stellar core -> radius at which degeneracy stops contraction.

Virial theorem says thermal energy of this core ~ $\frac{1}{2}$ potential energy

i.e.,

1/ $$\langle K \rangle = \frac{1}{2} U_g = \frac{1}{2}\frac{3}{5}\frac{GM^2}{R}$$

Now need to relate $\langle T \rangle$ to $\langle K \rangle$. For M-B distribution of velocities:

2/ $$\frac{1}{2} m V_{rms}^2 = \frac{3}{2} k \langle T \rangle$$, m is particle mass
V_{rms} is the rms velocity

Multiply eq 2/ by number of particles, N

Expressing N in terms of the mass, M.

pure He: $N_{He} = \frac{M}{4m_p}$ $\quad$ $N_{electron} = \frac{M}{2m_p}$

pure H: $N_H = \frac{M}{m_p}$

m_p is the proton mass (assuming He is ^{4}He)

3/ ∴ $$E_{th} = \frac{3}{2} NkT = \frac{3}{2}\frac{M}{m_p}\left(\frac{1}{4}+\frac{1}{2}\right)k\langle T \rangle = \frac{9}{8}\frac{M}{m_p}k\langle T \rangle$$

Equate 1/ and 3/ $$\frac{9}{8}\frac{M}{m_p}k\langle T \rangle = \frac{3}{10}\frac{GM^2}{R}$$

$$\langle T \rangle = \frac{4}{15}\frac{GMm_p}{Rk}$$

$\therefore \langle T \rangle \approx 3.06 \times 10^{8}\ \mathrm{K}$

⑧

observational tests:

a) spectral slope - rising steeply towards shorter wavelengths

But, a blackbody at 10^{8} K would peak in the x-ray regime

(Wien's displacement $\lambda_{max} \sim \frac{10^{-3}}{T} \sim 10^{-11}\ \mathrm{m}$)

∴ the spectral energy distribution in the visible would not be a good tracer of such high T.

b) Better option is to observe ionisation of ejected envelope (planetary nebula) This will exhibit emission lines from high ionisation stages of He, O, Ne. The strengths, in relative terms, of these transitions would allow, via the Saha equation, inference that the source has a very high T_{eff}.

⑥

Measured WD temperatures are in the range $200000 \gtrsim T_{eff} \gtrsim 3000$ K i.e., significantly lower than the value deduced above. Two main reasons for this:

a) The T_{eff} measured is the effective temperature of the thin insulating non-degenerate envelope, rather than the kinetic temperature of the degenerate core.

b) More importantly, once nuclear burning has ceased, the stellar remnant slowly cools with time. Most white dwarfs are old stars

④

Stars Q4

There are no white dwarfs with temperatures < 3000 K (where T is Teff, the black body temperature), because the cooling time is longer than the age of the Milky Way. //

②

Stat Phys – Paper 4, Question 5, Part (i)

- $F = E - TS$

  $\therefore dF = dE - TdS - SdT$

  1st law $dE = dQ + dW$

  $= TdS - PdV$

  $\therefore dF = -PdV - SdT$

  $\therefore P = -(\partial F/\partial V)_T$

  and $S = -(\partial F/\partial T)_V$

  $\therefore (\partial P/\partial T)_V = -(\partial(\partial F/\partial V)_T/\partial T)_V$

  $(\partial S/\partial V)_T = -(\partial(\partial F/\partial T)_V/\partial V)_T$

  $\therefore \underline{(\partial P/\partial T)_V = (\partial S/\partial V)_T}$ ⑤

- Consider a system in contact with reservoir such that total energy E_{total} is fixed (fixed V and N)

  $S_{total}(E_{total}) = S_R(E_{total} - E) + S(E)$

  $\simeq S_R(E_{total}) - (\partial S_R/\partial E_{total})E + S(E)$

  $= S_R(E_{total}) - \frac{1}{T}E + S(E)$

  $= S_R(E_{total}) - F/T$

  $\therefore$ <u>S_{total} is maximized by minimising F</u> ③

Stat Phys – Paper 4, Question 5, Part (ii)

- First order phase transition: two phases co-exist and there's a discontinuity in the first derivative of a thermodynamic potential (eg. volume, entropy)

  Second order phase transition: first derivatives are continuous, but second derivatives are discontinuous (eg. heat capacity) ③

- $F(T,m) = F_0(T) + \frac{a}{2}(T-T_c)m^2 + \frac{b}{4}m^4 + \frac{c}{6}m^6$

  $\therefore \partial F/\partial m = a(T-T_c)m + bm^3 + cm^5$

  $= m[cm^4 + bm^2 + a(T-T_c)]$

  $\partial^2 F/\partial m^2 = a(T-T_c) + 3bm^2 + 5cm^4 \quad *$

  Equilibrium → minimise F → $\partial F/\partial m = 0$ and $\partial^2 F/\partial m^2 > 0$

  $\therefore$ $m = 0$, which is a minimum for $T > T_c$ (from * as $a > 0$), but maximum for $T < T_c$

  or $m^2 = -\frac{b}{2c} \pm \frac{1}{2c}\sqrt{b^2 - 4ac(T-T_c)}$, which gives turning points if real and +ve

  As $b < 0$ → real if $b^2 - 4ac(T-T_c) > 0 \Rightarrow T < T_c + \frac{b^2}{4ac}$

  + is always +ve, but – only if $T > T_c$

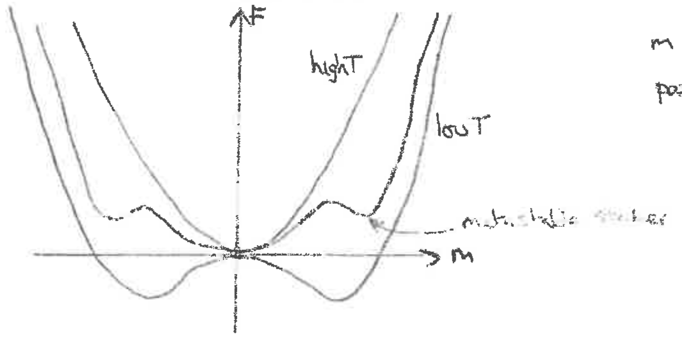
  $\therefore$ 3 regimes: $T < T_c$: $m = \pm\sqrt{-\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2 - 4ac(T-T_c)}}$

  $T_c < T < T_c + \frac{b^2}{4ac}$: $m = 0$ and $\pm\sqrt{-\frac{b}{2c} \pm \frac{1}{2c}\sqrt{b^2 - 4ac(T-T_c)}}$

  $T > T_c + \frac{b^2}{4ac}$: $m = 0$

  The sketch shows the – is a max. as $F \to \infty$ as $|m| \to \infty$

  ⑧

- 

  m changes discontinuously as T passes through T_c → 1st order phase transition

  ④

- For $b = 0$, the intermediate regime is not present

  $T > T_c \Rightarrow m = 0$, $F(T) = F_0(T)$

  $\therefore S = -\partial F/\partial T = -\partial F_0/\partial T$

  $C = -T\partial^2 F/\partial T^2 = -T\partial^2 F_0/\partial T^2$

  $T < T_c \Rightarrow m^4 = \frac{a}{c}(T_c - T)$, $F(T) = F_0(T) + \frac{a}{2}(T-T_c)\sqrt{\frac{a}{c}}(T_c-T)^{1/2} + \frac{c}{6}\left(\frac{a}{c}\right)^{3/2}(T_c-T)^{3/2}$

  $= F_0(T) - \frac{1}{3}\sqrt{\frac{a^3}{c}}(T_c-T)^{3/2}$

  $\therefore S = -\partial F_0/\partial T + \frac{1}{2}\sqrt{\frac{a^3}{c}}(T_c-T)^{1/2}$

  $C = T[-\partial^2 F_0/\partial T^2 - \frac{1}{4}\sqrt{\frac{a^3}{c}}(T_c-T)^{-1/2}]$

  ⑤

Paper IV

Course

PQM

NST Astrophysics Part II: 2020-21

Model Solution

Question Number

6X

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Examiner

CHALLINOR

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

SKINNER

Section

I / [✓ II]

Draft Mark Scheme

Comments

(ii) From (i), have

$$\dot{c}_0 = -\frac{i}{\hbar}\Big[c_0(t)\, e^{i\times 0 t/\hbar} \underbrace{\langle 0|V(t)|0\rangle}_{0} + c_1(t)\, e^{i(\omega_0-\omega_1)t}\underbrace{\langle 0|V(t)|1\rangle}_{\hbar v e^{i\omega t}}\Big] \qquad (t>0)$$

$$= -ivc_1(t)\, e^{i(\omega_0-\omega_1+\omega)t} ,$$

and

$$\dot{c}_1(t) = -\frac{i}{\hbar}\Big[c_0(t)\, e^{i(\omega_1-\omega_0)t} \underbrace{\langle 1|V(t)|0\rangle}_{\hbar v e^{-i\omega t}} + c_1(t)\underbrace{\langle 1|V(t)|1\rangle}_{0}\Big] \qquad (t>0)$$

$$= -iv\, c_0(t)\, e^{-i(\omega_0-\omega_1+\omega)t}$$

Follows that $\ddot{c}_1 = -iv\,\dot{c}_0\, e^{-i(\omega_0-\omega_1+\omega)t}$

$$\mp v(\omega_0-\omega_1+\omega)\, v\, c_0\, e^{-i(\omega_0-\omega_1+\omega)t}$$

$$= -v^2 c_1 - i(\omega_0-\omega_1+\omega)\,\dot{c}_1$$

$$\Rightarrow \ddot{c}_1 + i(\omega_0-\omega_1+\omega)\,\dot{c}_1 + v^2 c_1 = 0 .$$

Page

2

Please do not write below this line

Course

PQM

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

6X

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Examiner

CHALLINOR

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

SKINNER

Section

I / [✓ II]

Draft Mark Scheme

Comments

Trial solution $e^{\lambda t}$:

$$\lambda^2 + i\lambda(\omega - \underset{\substack{\uparrow \\ \omega_1-\omega_0}}{\Delta\omega}) + v^2 = 0$$

$$\Rightarrow \lambda = \frac{-i(\omega-\Delta\omega) \pm \sqrt{-(\omega-\Delta\omega)^2 - 4v^2}}{2}$$

$$= \frac{-i}{2}(\omega-\Delta\omega) \pm i\underbrace{\sqrt{v^2 + \tfrac{1}{4}(\omega-\Delta\omega)^2}}_{\equiv \Omega \text{, say}}$$

Initial conditions at $t=0$ $C_1 = 0$ and $C_1 = -iv$ (since $C_0 = 1$ initially).

Follows that $C_1(t) = A e^{-\frac{i}{2}(\omega-\Delta\omega)t} \sin\Omega t$, $(t>0)$

and $C_1 = -iv \Rightarrow A\Omega = -iv \Rightarrow A = -iv/\Omega$.

$$C_1(t) = \frac{-iv}{\Omega} e^{-\frac{i}{2}(\omega-\Delta\omega)t} \sin\Omega t,$$

so Probability of being in $|1\rangle$ at time t is $|C_1|^2$, i.e.,

[12]

$$P = \frac{v^2}{\Omega^2}\sin^2\Omega t \qquad (\dagger)$$

Course

PQM

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

6X

Examiner

CHALLINOR

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

1 / 2 / 3 / [✓ 4]

Lecturer

SKINNER

Section

I / [✓ II]

Draft Mark Scheme

Comments

The result in (i) gives

$$P \simeq \frac{1}{\cancel{\hbar}^2}\left|\underbrace{\int_0^t e^{i\Delta\omega t'}\cancel{\hbar}\, v\, e^{-i\omega t'}\,dt'}_{\frac{v}{i(\Delta\omega-\omega)}\left[e^{i(\Delta\omega-\omega)t}-1\right]}\right|^2$$

$$= \frac{v}{i(\Delta\omega-\omega)}\, e^{\frac{1}{2}i(\Delta\omega-\omega)t}\left[2i\sin\tfrac{1}{2}(\Delta\omega-\omega)t\right]$$

$$\therefore\ P \simeq \frac{4v^2}{(\Delta\omega-\omega)^2}\sin^2\tfrac{1}{2}(\Delta\omega-\omega)t$$

Now, $\Omega = \cancel{v\left(1+\tfrac{1}{4}(\omega-\Delta\omega)^2\right)^{\frac{1}{2}}}\ \tfrac{1}{2}|\omega-\Delta\omega| + O(v^2)$,

$\gg v$

[4] So, at leading order, $P \simeq \dfrac{4v^2}{(2\Omega)^2}\sin^2\Omega t$,

consistent with the exact result (†).

For a given ω (hence Ω), P is periodic in t with max value

[4] $\dfrac{v^2}{\Omega^2}$. This is maximised when Ω is minimised, i.e., when

$\omega = \Delta\omega = \underline{\omega_1 - \omega_0}$.

Please do not write below this line

Page

Course

RELATIVITY

NST Astrophysics Part II: 2020-21
Model Solution

Question Number

1

Examiner

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Paper

[✓ 1] / 2 / 3 / 4

Lecturer

CHALLINOR

Section

[✓ I] / II

Draft Mark Scheme

Comments

(i) The symmetry of the radiation pattern means that in $d\tau$ (proper time = coordinate time in IRF) the particle radiates no net 3-momentum, but energy $dE_{IRF} = P_0 \, d\tau$. It follows that

$$dp^\mu = d\tau \left(\frac{P_0}{c}, \vec{0} \right)$$

Lorentz transforming to a frame in which the particle is (instantaneously) moving at speed v, have

$$\frac{dE}{c} = \gamma \frac{dE_{IRF}}{c} = \frac{\gamma}{c} P_0 \, d\tau .$$

[6] However, the coordinate time in this frame corresponding to $d\tau$ is $\gamma d\tau$ (time dilation), hence

$$\frac{dE}{dt} = \underline{\underline{\frac{\gamma P_0 \, d\tau}{\gamma \, d\tau}}} = P_0 \quad \rightarrow \text{LORENTZ INVARIANT.}$$

Relate $|\vec{a}_{IRF}|^2$ to acceleration 4-vector:

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma \frac{d}{dt}(\gamma [c, \vec{u}]) = \gamma \frac{d\gamma}{dt} [c, \vec{u}] + \gamma^2 [0, \vec{a}]$$

Have $\gamma^{-2} = 1 - \vec{u}\cdot\vec{u}/c^2 \Rightarrow -2\gamma^{-3}\frac{d\gamma}{dt} = -2\vec{u}\cdot\vec{a}/c^2 \Rightarrow \frac{d\gamma}{dt} = \gamma^3 \frac{\vec{u}\cdot\vec{a}}{c^2}$.

Follows that $a^\mu = \gamma^4 \frac{\vec{u}\cdot\vec{a}}{c^2}[c, \vec{u}] + [0, \gamma^2 \vec{a}]$. In IRF,

[4] $a'^\mu = [0, \vec{a}_{IRF}]$ so $|\vec{a}_{IRF}|^2 = -a_\mu a^\mu$. Follows that

$$\underline{\underline{\mathcal{P} = -\frac{\mu_0 q^2}{6\pi c} a_\mu a^\mu .}}$$

Please do not write below this line

Page

1

Dynamics – Paper 4, Question 7, Part (i)

- Collisionless Boltzmann Equation : $\partial f/\partial t + (dx_i/dt)(\partial f/\partial x_i) + (dv_i/dt)(\partial f/\partial v_i) = 0$

  In cylindrical polars : $\partial f/\partial t + \dot{R}\,\partial f/\partial R + \dot{\phi}\,\partial f/\partial \phi + \dot{z}\,\partial f/\partial z + \dot{v}_R\,\partial f/\partial v_R + \dot{v}_\phi\,\partial f/\partial v_\phi + \dot{v}_z\,\partial f/\partial v_z = 0$

  ②

  Axisymmetric $\rightarrow \partial/\partial\phi = 0$

  Substitute given expressions for $\dot{v}_R, \dot{v}_\phi, \dot{v}_z$

  $$\partial f/\partial t + v_R\,\partial f/\partial R + v_z\,\partial f/\partial z + (v_\phi^2/R - \partial\Phi/\partial R)\,\partial f/\partial v_R - \tfrac{1}{R}(v_R v_\phi)\,\partial f/\partial v_\phi - (\partial\Phi/\partial z)\,\partial f/\partial v_z = 0$$

  ②

- Steady state $\rightarrow \partial/\partial t = 0$

  Take first moment with v_z

  $$\therefore \int \Big[\underbrace{v_z v_R\,\partial f/\partial R}_{\boxed{1}} + \underbrace{v_z^2\,\partial f/\partial z}_{\boxed{2}} + \underbrace{v_z(v_\phi^2/R - \partial\Phi/\partial R)\,\partial f/\partial v_R}_{\boxed{3}} - \underbrace{\frac{v_z}{R}(v_R v_\phi)\,\partial f/\partial v_\phi}_{\boxed{4}} \overbrace{- v_z(\partial\Phi/\partial z)\,\partial f/\partial v_z}^{\boxed{5}}\Big]\, dv_R\, dv_\phi\, dv_z = 0$$

  ②

  $\boxed{1}$ $\frac{\partial}{\partial R}\int v_z v_R f\, d^3\underline{v} = \frac{\partial}{\partial R}(\nu\,\overline{v_z v_R})$

  $\boxed{2}$ $\frac{\partial}{\partial z}\int v_z^2 f\, d^3\underline{v} = \frac{\partial}{\partial z}(\nu\,\overline{v_z^2})$

  $\boxed{3}$ $\int v_z(v_\phi^2/R - \partial\Phi/\partial R)\,\partial f/\partial v_R\; dv_R\, dv_\phi\, dv_z = 0$ as $[f]_{v_R=-\infty}^{v_R=\infty} = 0$

  $\boxed{4}$ $\int -\frac{v_z v_R}{R}\underbrace{v_\phi \frac{\partial f}{\partial v_\phi}\, dv_\phi}_{[v_\phi f]^{0} - \int f\, dv_\phi} dv_R\, dv_z = \int \frac{v_z v_R}{R} f\, dv_R\, dv_\phi\, dv_z = \nu\,\overline{v_R v_z}/R$

  $\boxed{5}$ $\int -(\partial\Phi/\partial z)\underbrace{v_z\,\partial f/\partial v_z\, dv_z}_{[v_z f]^{0} - \int f\, dv_z} dv_R\, dv_\phi = \int \frac{\partial\Phi}{\partial z} f\, dv_R\, dv_\phi\, dv_z = \nu\,\partial\Phi/\partial z$

  $$\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}+\boxed{5} \rightarrow \underline{\frac{\partial}{\partial R}(\nu\,\overline{v_z v_R}) + \frac{\partial}{\partial z}(\nu\,\overline{v_z^2}) + \nu\,\overline{v_R v_z}/R + \nu\,\partial\Phi/\partial z = 0}$$

  ④

Dynamics – Paper 4, Question 7, Part (ii)

- $$\underset{\boxed{1}}{\frac{\partial}{\partial R}\left[\nu\,\overline{V_R V_z}\right]} + \underset{\boxed{2}}{\frac{\partial}{\partial z}\left[\nu\,\overline{V_z^2}\right]} + \underset{\boxed{3}}{\nu\,\frac{\overline{V_R V_z}}{R}} + \underset{\boxed{4}}{\nu\,\frac{\partial \Phi}{\partial z}} = 0$$

  Assume characteristic scales R_0, z_0

  Highly flattened → $z_0/R_0 \ll 1$ ③

  $\partial/\partial R \sim \frac{1}{R_0}$, $\partial/\partial z \sim \frac{1}{z_0}$

  $\boxed{1}$ $\partial[\nu\,\overline{V_R V_z}]/\partial R \sim \nu\,\overline{V_R V_z}/R_0$

  $\boxed{2}$ $\partial[\nu\,\overline{V_z^2}]/\partial z \sim \nu\,\overline{V_z^2}/z_0$

  $\boxed{3}$ $\nu\,\overline{V_R V_z}/R \sim \nu\,\overline{V_R V_z}/R_0$

  $\therefore \boxed{1} \sim \boxed{3} \sim \left(\frac{\overline{V_R V_z}}{\overline{V_z^2}}\right)\left(\frac{z_0}{R_0}\right)\boxed{2} \ll \boxed{2}$

  $\therefore \partial/\partial z\left(\nu\,\overline{V_z^2}\right) + \nu\,\partial\Phi/\partial z \simeq 0$ ④

- Poisson's eq $\nabla^2\Phi = 4\pi G\rho$

  $\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \simeq \frac{\partial^2}{\partial z^2}$ for $z_0 \ll R_0$

  $\therefore \partial^2\Phi/\partial z^2 = 4\pi G\rho$ ④

  $\therefore \underline{\partial/\partial z\left[\frac{1}{\nu}\frac{\partial}{\partial z}\left(\nu\,\overline{V_z^2}\right)\right] = -4\pi G\rho}$

- Mass dominated by central mass ⇒ $\Phi = -GM/\sqrt{R^2+z^2}$

  $\therefore \partial\Phi/\partial z = GMz\,(R^2+z^2)^{-3/2} \simeq GMz/R^3$

  $\therefore \partial^2\Phi/\partial z^2 \simeq GM/R^3$ ②

  Given $\nu(z) = \nu_0 e^{-\frac{1}{2}z^2/z_0^2}$ and that $\overline{V_z^2} = \sigma_z^2$ is independent of z

  $\therefore \frac{\partial}{\partial z}\left(\nu\,\overline{V_z^2}\right) + \nu\,\partial\Phi/\partial z = 0$

  $\rightarrow \sigma_z^2\,\partial\nu/\partial z + \nu\,GMz/R^3 = 0$

  $\therefore \sigma_z^2\,\nu\,(-z/z_0^2) + \nu\,GMz/R^3 = 0$ ③

  $\therefore \underline{\sigma_z^2 = GMz_0^2/R^3}$

- If interpreted as self-gravitating

  $\partial/\partial z\left[\frac{1}{\nu}\frac{\partial}{\partial z}\left(\nu\,\overline{V_z^2}\right)\right] = -4\pi G\rho$

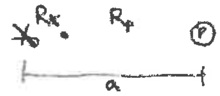
  $\therefore$ Mass surface density within z_0 is $\int_0^{z_0}\rho\,dz = \frac{1}{4\pi G}\left[\frac{1}{\nu}\frac{\partial}{\partial z}\left(\nu\,\overline{V_z^2}\right)\right]_{z_0}^{0}$

  $= \frac{\sigma_z^2}{4\pi G}\left[-z/z_0^2\right]_{z_0}^{0}$

  $= \frac{\sigma_z^2}{4\pi G z_0}$ ④

  Or mean mass density is $\underline{\sigma_z^2/4\pi G z_0^2}$

Topics – Paper 4, Question 8, Part (i)

- 

$M_* R_* = M_p R_p$ and $R_* + R_p = a$

$\therefore R_* = a M_p / (M_* + M_p)$

Kepler's law: $n^2 a^3 = G(M_* + M_p)$

Stellar velocity: $V_* = n R_*$

$= \sqrt{G(M_* + M_p)/a^3} \, . \, a M_p / (M_* + M_p)$

$= M_p \sqrt{G / a(M_* + M_p)}$

$\simeq M_p \sqrt{G / a M_*}$

④

- Doppler effect: $f' = f / (1 + v/c)$

$\lambda = c/f$

$\therefore \lambda' = \lambda (1 + v/c)$

$\therefore \Delta\lambda = \lambda' - \lambda = \lambda v / c$

Maximum radial velocity if edge-on $\Rightarrow \Delta\lambda = \lambda M_p \sqrt{G/aM_*} \; / c$

$\therefore M_p = \frac{\Delta\lambda}{\lambda} \sqrt{\frac{aM_*}{G}} \times c$

Kepler's law with $n = 2\pi/T \rightarrow a = \left(\frac{GM_* T^2}{4\pi^2}\right)^{1/3}$

$\therefore M_p = \frac{\Delta\lambda}{\lambda} G^{-1/3} M_*^{2/3} T^{1/3} c / (2\pi)^{1/3}$

$= \frac{10^{-3}}{500} (6.7 \times 10^{-11})^{-1/3} \, . \, (2 \times 10^{30})^{2/3} \, . \, (10 \times 24 \times 60 \times 60)^{1/3} \, . \, 3 \times 10^{8} / (2\pi)^{1/3}$

$= 1.2 \times 10^{28}$ kg

$\simeq 6.1 \, M_{\text{Jupiter}}$

⑤

- $R = \lambda / \Delta\lambda$

$\therefore$ Can detect $\Delta\lambda > \lambda / R \simeq 500\text{ nm} / 10^5 = 0.005\text{ nm}$

But $\Delta\lambda = 0.001\text{ nm} \rightarrow$ no

①

Topics - Paper 4, Question 8, Part (ii)

- 

Consider parcel of gas at r from star at height z

Radial component of equation of motion : $-(GM_*/r^2) - \frac{1}{\rho}dp/dr + V_{\phi,g}^2/r = 0$ ④

$\therefore V_{\phi,g}^2 = (GM_*/r)[1 + \frac{r^2}{\rho GM_*}dp/dr]$

$\therefore \underline{V_{\phi g} = V_K[1-\eta]^{1/2}}$ where $\eta = -\frac{r^2}{\rho GM_*}dp/dr = \frac{-r}{\rho V_K}\frac{dp}{dr}$

But for locally isothermal gas $p = c_s^2\rho$

and given $P \propto r^{-n} \rightarrow dp/dr = -np/r = -nc_s^2\rho/r$ ③

$\therefore \underline{\eta} = -\frac{r}{\rho V_K}(-nc_s^2\rho/r) = \underline{nc_s^2/V_K^2}$

- Vertical component of eq. of motion : $-(GM_*z/r^3) - \frac{1}{\rho}dp/dz = 0$

Isothermal $\rightarrow$ $-(V_K^2/r^2)z = c_s^2\frac{1}{\rho}d\rho/dz$

$\therefore [\ln\rho]_{\rho_0}^{\rho} = [-\frac{1}{2}(\frac{V_K}{c_s})^2(\frac{z}{r})^2]_0^z$

$\therefore \rho = \rho_0 e^{-\frac{1}{2}(\frac{V_K}{c_s})^2(\frac{z}{r})^2}$

$\therefore \Sigma = \int_{-\infty}^{\infty}\rho\, dz = \rho_0\int_{-\infty}^{\infty}e^{-x^2}dx\,(dz/dx)$

where $\frac{1}{2}(\frac{V_K}{c_s r})^2z^2 = x^2$ $\therefore dz/dx = \sqrt{2}c_s r/V_K$

$\therefore \Sigma = \rho_0\sqrt{2\pi}(c_s r/V_K) \propto r^{-3/2}$ ⑥

$\therefore p_0 = c_s^2\rho_0 \propto c_s^2 r^{-3/2}V_K/rc_s$

$\propto c_s r^{-3}$

As $c_s \propto r^{-\alpha}$, $p_0 \propto r^{-3-\alpha}$ $\rightarrow dp/dr \propto r^{-4-\alpha}$

- For bump to trap dust $\rightarrow dp/dr > 0$ over width Δr

$\therefore dp/dr \sim p/\Delta r$ whereas before it was $\sim -p/r$

$\therefore$ radial drift velocity, which is $\propto \eta V_K \propto dp/dr$ is faster by $r/\Delta r$ factor

$\therefore$ If infall time $t_i \sim r/V_r$ and accumulation time $t_a \sim \Delta r/V_r'$ ⑤

then $\underline{t_a/t_i} \simeq (\frac{\Delta r}{r})(\frac{V_r}{V_r'}) \simeq \underline{(\frac{\Delta r}{r})^2}$

- Pressure bump prevents drift of dust into inner regions $\rightarrow$ prevents planet growth inside, but accumulation of dust in bump increases collisions and so possible growth to planets. ②