Paper III

Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
RELATIVITY		1
Examiner	• In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen. • Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness. • Write on this side only and between the margins. • Not more than one solution per sheet please.	Paper: 1 / 2 / 3 / 4
Lecturer: CHALLINOR		Section: I / II

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(i) CONTINUED

Comments

Have Minkowski space as $r \to \infty$, so $r^2 \dfrac{d\phi}{dt} = b v_\infty$.

Follows that $\dfrac{r^2 \dot\phi}{\gamma_\infty} = b v_\infty$

UNSEEN (DONE FOR NULL CASE IN NOTES)

[4]

$$\Rightarrow b = \frac{h}{\gamma_\infty v_\infty} = \frac{h/c}{k\sqrt{1-1/k^2}} = \frac{h}{c\sqrt{k^2-1}}$$

—— o ——

(ii) We have

$$c^2 = \left(\frac{ds}{d\tau}\right)^2 = \left(1 - \frac{2\mu}{r}\right)c^2 \dot t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot r^2 - r^2\dot\phi^2$$

$$= \left(1-\frac{2\mu}{r}\right)^{-1}c^2k^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot r^2 - \frac{h^2}{r^2}.$$

[4]

It follows that $c^2\left(1-\frac{2\mu}{r}\right) = c^2k^2 - \dot r^2 - \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)$

$$\Rightarrow c^2(k^2-1) = \dot r^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu c^2}{r}$$

$$\Rightarrow \frac{1}{2}\dot r^2 + \underbrace{\frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right) - \frac{\mu c^2}{r}}_{V_{\text{eff}}(r)} = \frac{1}{2}c^2(k^2-1).$$

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Lecturer CHALLINOR		Section I / II

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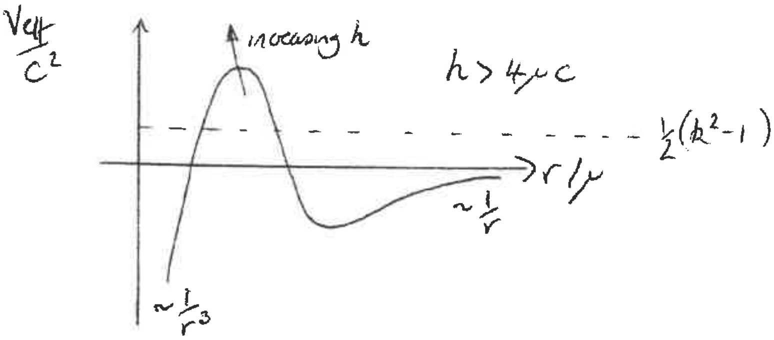
(ii) CONTINUED



[2] — NOTES

For any h, a particle with sufficient energy at ∞ (i.e., large enough k) will be able to cross the barrier at the maximum of V_{eff} and hence reach $r=0$. Note: very different to the Newtonian case, where V_{eff} does not turn over at small r and instead has a "centrifugal barrier"

[3] — INTERPRETATION UNSEEN

We have
$$\frac{dV_{eff}}{dr} = \frac{\mu c^2}{r^2} + \frac{h^2}{2}\left(\frac{-2}{r^3} + \frac{6\mu}{r^4}\right)$$
$$= \frac{\mu c^2}{r^2} - \frac{h^2}{r^3}\left(1 - \frac{3\mu}{r}\right).$$

The turning points are at
$$\frac{\mu c^2}{h^2} r^2 - \cancel{h^2} r + 3\mu \cancel{h^2} = 0$$
$$\Rightarrow \quad r = \frac{1 \pm \sqrt{1 - 12\left(\frac{\mu c}{h}\right)^2}}{2\mu c^2/h^2} = \frac{h}{2\mu c^2}\left(h \pm \sqrt{h^2 - 12(\mu c)^2}\right)$$
NOTES

At the extrema,
$$V_{eff} = \underbrace{-\frac{\mu c^2}{r}}_{-\frac{h^2}{r^2}\left(1-\frac{3\mu}{r}\right)} + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{h^2}{2r^2}\left(\frac{4\mu}{r} - 1\right)$$

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(ii) CONTINUED

For $V_\infty \ll c$, $k \to 1$ and the particle will reach $r=0$ if ~~...~~ $\max(V_{eff}) < 0$. The max of V_{eff} is at 0 when the maximum is at $r = 4\mu$, i.e. when

$$\frac{\mu c^2}{h^2} \times 16\mu^2 - 4\mu + 3\mu = 0 \Rightarrow \frac{4\mu c}{h} = 1 .$$

UNSEEN

So, as $k \to 1$, need $h < 4\mu c$ for capture, hence

$$b = \frac{h}{k V_\infty} = \frac{4\mu c}{k V_\infty} \to \frac{4\mu c}{V_\infty}, \text{ so need}$$

[7]

$$b < \frac{4\mu c}{V_\infty} .$$

For $V_\infty \to c$ (i.e., massless case), $k \gg 1$. The critical case now has ~~the~~ $h \gg 4\mu c$, in which case

$$r \text{ at the maximum} \to \mu\left[\frac{1-\left(1-6\left(\frac{\mu c}{h}\right)^2\right)}{2\left(\frac{\mu c}{h}\right)^2}\right] = 3\mu$$

UNSEEN AS LIMIT OF MASSIVE PARTICLE.

Follows that V_{eff} at max is $\simeq \frac{h^2}{27\mu^2}\left(\frac{4}{3}-1\right) = \frac{h^2}{54\mu^2}$

[4]

For capture, need $\frac{1}{2}(k^2-1)c^2 > \frac{h^2}{54\mu^2}$

$$\Rightarrow \frac{h^2}{c^2(k^2-1)} < 27\mu^2$$

$$\Rightarrow b < \sqrt{27}\mu \quad \text{(as for a photon).}$$

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Paper III

2 i) Geometrically thin disk around point mass M (AFD)

$$\frac{u_\phi^2}{R} = \frac{GM}{R^2} \Rightarrow u_\phi = \left(\frac{GM}{R}\right)^{1/2}$$

Angular velocity $\Omega = u_\phi/R = \left(\frac{GM}{R^3}\right)^{1/2}$

Start with momentum equation:

$$\frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3 \frac{\partial}{\partial R}\left(\frac{u_\phi}{R}\right)\right)$$

$$R\cdot\frac{1}{R^{1/2}}\frac{\partial\Sigma}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^{3/2}u_R\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\partial}{\partial R}\left(1/R^{3/2}\right)\right)$$

$$\Rightarrow \frac{\partial\Sigma}{\partial t} = -\frac{1}{R^{3/2}}\frac{\partial}{\partial R}\left(\Sigma R^{3/2}u_R\right) - \frac{3}{2}\frac{1}{R^{3/2}}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\cdot R^{-5/2}\right)$$

Sub in mass conv. eq.

$$-\frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = -\frac{1}{R^{3/2}}\frac{\partial}{\partial R}\left(\Sigma R^{3/2}u_R\right) - \frac{3}{2}\frac{1}{R^{3/2}}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)$$

$$\frac{\partial}{\partial R}(R\Sigma u_R) = R^{-1/2}\frac{\partial}{\partial R}\left(\Sigma R u_R\cdot R^{1/2}\right) + \frac{3}{2}R^{-1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)$$

$$\Rightarrow \cancel{\frac{\partial}{\partial R}(R\Sigma u_R)} = \frac{R^{-1/2}}{2}\cdot R^{-1/2}\,\Sigma R u_R + \cancel{\frac{\partial}{\partial R}(\Sigma R u_R)} + \frac{3}{2}R^{-1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)$$

$$0 = \frac{\Sigma u_R}{2} + \frac{3}{2} R^{-1/2} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$$

$$u_R = -\frac{3}{\Sigma R^{1/2}} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$$

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Local mass accretion rate

$$\dot{M} = 2\pi R \Sigma (-u_R)$$

$$= 2\pi R \Sigma \cdot \frac{3}{\Sigma R^{1/2}} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$$

$$= 6\pi R^{1/2} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$$

3

ii) Continuity equation is

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R \Sigma u_R\right) = 0$$

Substitute in for u_R from Part (i)

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma \cdot \left[-\frac{3}{R^{1/2}\Sigma}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right]\right) = 0$$

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right)$$

3

$\nu = \nu_0 R$, $x = R^{1/2}$

$\Rightarrow R = x^2$

$$\therefore \frac{\partial}{\partial x} = \frac{\partial R}{\partial x} \cdot \frac{\partial}{\partial R} = 2x \frac{\partial}{\partial R} = 2R^{1/2} \frac{\partial}{\partial R}$$

$$\Rightarrow \frac{\partial \Sigma}{\partial t} = \frac{3}{R^{3/2}} R^{1/2} \frac{\partial}{\partial R}\left[R^{1/2} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right]$$

$$= \frac{3}{4x^3} \frac{\partial^2}{\partial x^2}\left(\nu \Sigma x\right)$$

$$\Rightarrow x \nu \frac{\partial \Sigma}{\partial t} = \frac{3\nu}{4x^2} \frac{\partial^2}{\partial x^2}\left(\nu \Sigma x\right)$$

if $\nu = \nu(R)$, i.e., independent of t, then

$$\frac{\partial}{\partial t}\left(x \nu \Sigma\right) = \frac{3\nu}{4x^2} \frac{\partial^2}{\partial x^2}\left(x \nu \Sigma\right)$$

$$\Rightarrow \frac{\partial \Psi}{\partial t} = \frac{3\nu_0 x^2}{4x^2} \frac{\partial^2 \Psi}{\partial x^2} = \frac{3\nu_0}{4} \frac{\partial^2 \Psi}{\partial x^2}$$

where we have used $\nu = \nu_0 R = \nu_0 x^2$

and defined $\Psi = \nu \Sigma x = \nu_0 \Sigma x^3$ //

6 //

Mass accretion rate $\dot{M} = 6\pi R^{1/2} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$

$$= 3\pi . 2 R^{1/2} \frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$$

$$= 3\pi \frac{\partial}{\partial x}\left(\nu \Sigma x\right)$$

$$= 3\pi \frac{\partial \Psi}{\partial x}$$

3

Mass M, $R = R_0$, @ $t=0$

$$\Sigma(R, t=0) = \frac{M}{2\pi R_0}\,\delta(R - R_0)$$

$$\Rightarrow \Psi(x, t=0) = \nu_0 x_0^3 \cdot \frac{M}{2\pi x_0^2} \cdot \frac{\delta(x - x_0)}{|\partial R/\partial x|_{R_0}}$$ using $\Psi = \nu \Sigma x^3$

$$= \frac{1}{2}\nu_0 \cdot \frac{M}{2\pi}\,\delta(x - x_0)$$

$$= \frac{\nu_0 M}{4\pi}\,\delta(x - x_0)$$

Using Green's function, subsequent evolution given by:

$$\Psi(x,t) = \int \frac{M\nu_0}{4\pi}\,\delta(x - x_0)\,\frac{1}{\sqrt{4\pi D t}}\,e^{-(x-x_0)^2/4Dt}\,dx$$

$$= \frac{m\nu_0}{4\pi\sqrt{3\pi\nu_0}} \frac{1}{\sqrt{t}} e^{-(x-x_0)^2/3\nu t} \qquad \left(D \equiv \frac{3\nu_0}{4}\right)$$

Mass accretion rate at centre ($\ell = x = 0$) is

$$\dot{M}_0 = 3\pi \frac{\partial \varphi}{\partial x}\Big|_{x=0}$$

$$= \frac{m\nu_0 3\pi}{4\pi\sqrt{3\pi\nu_0}} \cdot \frac{1}{\sqrt{t}} \cdot 2(x_0 - x) \frac{1}{3\nu_0 t} \cdot e^{-(x-x_0)^2/3\nu t}\Big|_{x=0}$$

$$= \frac{m x_0}{2\sqrt{3\pi\nu_0}} \frac{1}{t^{3/2}} e^{-x_0^2/3\nu_0 t}$$

$$\therefore \dot{M}_0 = \frac{m \ell_0^{1/2}}{2\sqrt{3\pi\nu_0}} \frac{1}{t^{3/2}} e^{-\ell_0/3\nu_0 t}$$

6

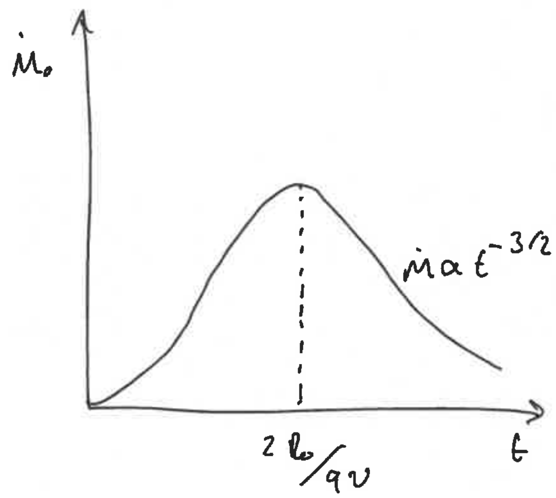
This is

$$\frac{\partial}{\partial t}\left(t^{-3/2} e^{-\ell_0/3\nu_0 t}\right) = 0$$

$$\Rightarrow \left(-\frac{3}{2} t^{-5/2} e^{-\ell_0/3\nu_0 t} + t^{-3/2} \cdot \frac{\ell_0}{3\nu_0} \frac{1}{t^2} e^{-\ell_0/3\nu_0 t}\right) = 0$$

$$\Rightarrow \frac{\ell_0}{3\nu_0} t^{-7/2} = \frac{3}{2} t^{-5/2}$$

$$\Rightarrow \quad \frac{2}{9}\frac{R_0}{v_0}$$



Paper III Cosmology

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COSMOLOGY		3X

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EFSTATHIOU	• Not more than one solution per sheet please.	I / II

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In a spatially-flat universe dominated by the matter,

$$H^2 = \frac{8\pi G}{3}\rho \Rightarrow \left(\frac{\dot R}{R}\right)^2 \propto R^{-3(1+w)}$$

$$\Rightarrow \dot R \propto R^{-\frac{1}{2}(1+3w)}$$

$$\Rightarrow \int_0^R dR' \, R'^{\frac{1}{2}(1+3w)} \propto t ,$$

So $R^{\frac{3}{2}(1+w)} \propto t$ and $R \propto t^{\frac{2}{3(1+w)}}$. Power-law expansion with $p = \frac{2}{3(1+w)}$, so $p = 1 \Rightarrow 1+w = \frac{2}{3}$

[5] $w = -\frac{1}{3}$.

The Friedmann equation, $\frac{\ddot R}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$
$= -\frac{4\pi G}{3}\rho(1+3w)$,

implies $\ddot R > 0$ if $w < -\frac{1}{3}$ given accelerated

[2] expansion, i.e., inflation. In inflation, the particle horizon can become much larger than the Hubble radius $c/H(t)$ solving the classical horizon problem.

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Lecturer EFSTATHIOU		**Section** I / II

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(ii) If $R(t) \propto e^{H_I t}$, particle horizon grows from t_i as

$$d_{ph}(t) = c\, e^{H_I t} \int_{t_i}^{t} e^{-H_I t'}\, dt'$$

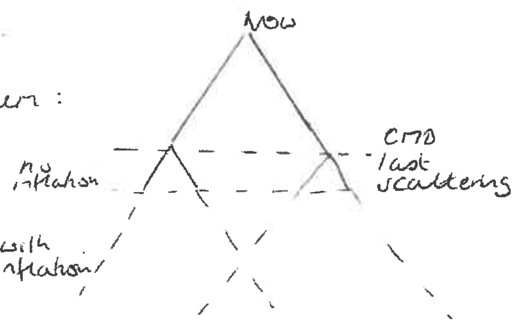
$$= \frac{c}{H_I} e^{H_I t} \left[e^{-H_I t'} \right]_t^{t_i}$$

$$= \frac{c}{H_I} \left[e^{H_I (t - t_i)} - 1 \right].$$

[5]

Horizon problem: in standard big bang cosmology, the particle horizon at recombination only subtends a few degrees on sky today, but CMB is very uniform. What can explain this? Inflation allows the particle horizon to grow quasi-exponentially, becoming much larger than the Hubble radius, allowing apparently causally disconnected regions at recombination actually to be causally connected.

Optional conformal diagram:



[5] Exponential expansion requires $w = -1$, but inflation occurs for $w < -\frac{1}{3}$.

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Comments

$$\phi = \sqrt{2p}\,\ln\left(t\sqrt{\frac{V_0}{p(3p-1)}}\right)$$

$$\Rightarrow \quad t = \sqrt{\frac{p(3p-1)}{V_0}}\, e^{\phi/\sqrt{2p}}, \text{ so}$$

$$V = V_0\left(\sqrt{\frac{V_0}{p(3p-1)}}\,t\right)^{-2} = \frac{p(3p-1)}{t^2}.$$

Have: $\dot{\phi} = \dfrac{\sqrt{2p}}{t}$, $\ddot{\phi} = -\dfrac{\sqrt{2p}}{t^2}$, and

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\times\frac{2p}{t^2} + \frac{p(3p-1)}{t^2}\right]$$

$$= \frac{p}{3t^2}\left(1 + 3p - 1\right) = \frac{p^2}{t^2},$$

[6] so $\ddot{\phi} + 3H\dot{\phi} = -\dfrac{\sqrt{2p}}{t^2} + \dfrac{3p}{t}\dfrac{\sqrt{2p}}{t} = (3p-1)\dfrac{\sqrt{2p}}{t^2}$

Compare to $-\dfrac{dV}{d\phi} = \sqrt{\dfrac{2}{p}}\,V = \sqrt{\dfrac{2}{p}}\,\dfrac{p(3p-1)}{t^2}$

$$= \sqrt{2p}\,(3p-1)/t^2 \;\checkmark$$

From $H = \dfrac{p}{t}$, it follows that $R \propto t^p$

[2] Since $\dfrac{\dot{R}}{R} = \dfrac{p}{t} \Rightarrow \ln R = p\ln t + \text{const}.$

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Lecturer: EFSTATHIOU

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$R \propto t^p \Rightarrow \frac{\ddot{R}}{R} = \frac{p(p-1)}{t^2}$ Inflation requires $\ddot{R} > 0$, i.e., $p(p-1) > 0$. We must have $p > 1/3$ for the solution to be valid, so inflation occurs for

[2] $\underline{p > 1}$.

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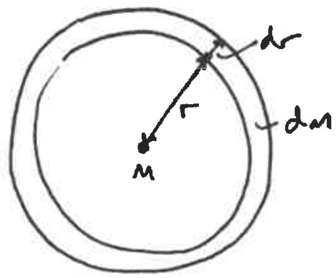
4 i) Star mass M, radius R, hydrostatic eqb. Derive exp. for U in terms of ρ



$dm = 4\pi r^2 \rho\, dr$, where ρ is mass density

$dU = -\frac{G M(r)}{r} dm = -\frac{G M(r)}{r} 4\pi r^2 \rho\, dr$, where $M(r)$ is the mass within radius r

Integration gives total gpe of star

$$U = -\int_0^R \frac{G M(r)}{r} . 4\pi r^2 \rho\, dr$$ (4)

If the density is uniform then

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$, at all radii

and

$$M(r) = \frac{4}{3}\pi r^3 \rho$$

Hence

$$U = -\int_0^R \frac{G \frac{4}{3}\pi r^3 \rho}{r} . 4\pi r^2 \rho\, dr$$

Stars Q3

$$U = -\int_0^R \frac{4}{3} \cdot 4\pi^2 G r^4 \rho^2 dr$$

$$= -\frac{16}{3}\pi^2 G \rho^2 \left[\frac{1}{5} r^5\right]_0^R$$

$$= -\frac{16}{15}\pi^2 G \rho^2 R^5$$

now, $M = \frac{4}{3}\pi R^3 \rho$

$$M^2 = \frac{16}{9}\pi^2 R^6 \rho^2$$

$$\therefore \; U = -\frac{3}{5}\frac{G M^2}{R}$$

2

Hence $U \propto \frac{1}{R}$

1

Show that total thermal energy of the Star, K, is given by

$$K = \int_0^R \frac{3}{2} p 4\pi r^2 dr$$

In a shell of dr, total particle number is

$$dN = 4\pi r^2 n \, dr$$

where n, is the particle number density

with the kinetic energy of each particle being $\frac{3kT}{2}$, the total kinetic energy in the shell

$$dK = \frac{3kT}{2} 4\pi r^2 n \, dr$$

Stars Q.3

$$K = \int_0^R \frac{3kT}{2}\, 4\pi r^2 n \, dr$$

noting that $p = nkT$

$$K = \int_0^R \frac{3}{2} p \, 4\pi r^2 n \, dr$$

3

ii) Assuming spherical symmetry throughout, the energy that goes into ejecting the mass is

$$\Delta E_\nu = \frac{3}{10} f G 1 M_\odot^2 \left(\frac{1}{r_{c,f}} - \frac{1}{r_{c,i}} \right)$$

$f = 0.01$

$r_{c,i}$ = core radius at start of collapse

$r_{c,f}$ = core radius after collapse

Energy needed to eject $9 M_\odot$ of material out of potential well of core of $1 M_\odot$ is

$$E_{ej} = \frac{G 9 M_\odot^2}{\langle r_g \rangle}$$

where $\langle r_g \rangle$ is the mass weighted radius of the $9 M_\odot$ envelope before core collapse

Equating the two expressions

$$\frac{1}{\langle r_g \rangle} = \frac{1}{30} f \left(\frac{1}{r_{c,f}} - \frac{1}{r_{c,i}} \right)$$

$$\frac{30}{\langle r_g \rangle} \frac{1}{f} + \frac{1}{r_{c,i}} = \frac{1}{r_{c,f}}$$

$$r_{c,f} = \left(\frac{30}{\langle r_g \rangle} \frac{1}{f} + \frac{1}{r_{c,i}} \right)^{-1}$$

Challenge is to find a sensible expression for $\langle r_g \rangle$

Stars Q3

For the initial pre-collapse core, we have $r_{c,i} \sim r_\oplus \sim 3000$ km

Possibilities for $\langle r_g \rangle$

a) assume uniform density in envelope

$\Rightarrow$ 0.5 of the mass in $0.5^{1/3}$ of envelope radius

~ 0.8

with the radius of a 10 $M_\odot$ Supergiant $\approx 100-200\,R_\odot$

can approximate $\langle r_g \rangle \sim 100\,R_\odot$

$$\therefore\ r_{c,f} = \left(\frac{30}{100 \cdot 7\times10^8} \cdot \frac{1}{0.01} + \frac{1}{3000\times10^3}\right)^{-1} \simeq 2660 \text{ km}$$

b) But, there will be a density gradient in the envelope

ρ will decrease outwards

$\Rightarrow \langle r_g \rangle$ lower than for uniform ρ

Conservative guess $\langle r_g \rangle \approx 1\,R_\odot$ for centrally condensed super giant

$$r_{c,f} = \left(\frac{30}{7\times10^8} \cdot \frac{1}{0.01} + \frac{1}{3\times10^6}\right)^{-1} \simeq 216 \text{ km}$$

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Both values of $r_{c,f}$ are significantly > $r_{c,f} \simeq 20$ km (from lecture)

The wrong assumption is that the energy liberated is 'just' enough to eject the remaining mass to ∞. SN remnants are observed with velocities > 1000 km s^{-1}

2

Velocity of ejecta if energy absorbed by envelope is 10^{51} erg

$$E_g = 10^{51} \times 10^{-7} = \frac{1}{2} M_{ej} \langle v \rangle^2$$

$$\langle v \rangle = \left(\frac{2 \times 10^{51} \times 10^{-7}}{M_{ej}} \right)^{\frac{1}{2}}$$

$$= 3340 \text{ km s}^{-1}$$

3

Type II SN are generally believed to be the end point of the evolution of stars $\gtrsim 10\,M_\odot$

Such stars have lifetimes $\lesssim 3 \times 10^7$ years

Globular clusters have ages $\gtrsim 3 \times 10^9$ years

$\therefore$ all massive stars in globular clusters should have long ago exploded as type II SN

3

Stat Phys - Paper 3, Question 5, Part (i)

• Relevant integral for summing over states is:

$$\int g(E)\,dE = g_s \frac{V}{(2\pi)^d} \int d^d k$$

$$= g_s \frac{V}{(2\pi)^d} \int S_{d-1} k^{d-1}\, dk \qquad 3$$

As $E = A(\hbar k)^\alpha$

$$\therefore\ dE = A\alpha\hbar^\alpha k^{\alpha-1}\, dk$$

$$\therefore\ \int g(E)\,dE = g_s \frac{V}{(2\pi)^d} \frac{1}{A\alpha\hbar^\alpha} S_{d-1} \int k^{d-\alpha}\, dE \qquad 2$$

But $k = (E/A)^{1/\alpha}\hbar^{-1}$

$$\therefore\ k^{d-\alpha} = A^{-\frac{1}{\alpha}(d-\alpha)}\hbar^{\alpha-d} E^{\frac{d}{\alpha}-1}$$

$$\therefore\ \int g(E)\,dE = g_s \frac{V}{(2\pi)^d} \frac{1}{\cancel{A}\alpha\cancel{\hbar^\alpha}} S_{d-1} A^{\cancel{1}-\frac{d}{\alpha}} \hbar^{\cancel{\alpha}-d} \int E^{\frac{d}{\alpha}-1}\, dE$$

$$= \int \frac{g_s S_{d-1} V}{\alpha (2\pi)^d A^{d/\alpha}\hbar^d} E^{\frac{d}{\alpha}-1}\, dE \qquad 3$$

• Bose-Einstein distribution 2

$$\langle n_r \rangle = \frac{1}{e^{\beta(E_r-\mu)} - 1}$$

Stat Phys – Paper 3, Question 5, Part (ii)

- μ has to be <0, as otherwise the grand partition function for Bose gas doesn't make sense (sum doesn't converge) (2)

- $p = -\partial F/\partial V$

  where $F = -\frac{1}{\beta}\ln Z$

  $\therefore p = \frac{1}{\beta}\partial \ln Z/\partial V$

  But $\ln Z = \int_0^\infty g(E)\,dE \ln[(1-e^{-\beta(E-\mu)})^{-1}]$

  and $g(E) = AVE^{\frac{d}{\alpha}-1}$ (from Part i)

  $$\therefore \ln Z = -AV\int_0^\infty E^{\frac{d}{\alpha}-1}\ln(1-e^{-\beta(E-\mu)})\,dE$$
  $$= -AV\left[\left[\tfrac{\alpha}{d}E^{d/\alpha}\ln(1-e^{-\beta(E-\mu)})\right]_0^\infty - \int_0^\infty\left(\tfrac{\alpha}{d}\right)\frac{E^{d/\alpha}\beta e^{-\beta(E-\mu)}}{1-e^{-\beta(E-\mu)}}\,dE\right]$$
  $$= \left(\tfrac{\beta\alpha}{d}\right)\int_0^\infty AVE^{\frac{d}{\alpha}-1}E/(e^{\beta(E-\mu)}-1)\,dE$$
  $$= \left(\tfrac{\beta\alpha}{d}\right)\int_0^\infty E\,g(E)/[e^{\beta(E-\mu)}-1]\,dE$$
  $$= \frac{\beta\alpha}{d}E$$

  $\therefore p = \frac{1}{\beta}\frac{\beta\alpha}{d}E/V$

  $\therefore \underline{pV = DE, \text{ where } D = \alpha/d}$ (9)

- Number of particles not in ground state (2)
  $$N = \int_0^\infty g(E)/[e^{\beta(E-\mu)}-1]\,dE$$
  $$= AV\int_0^\infty E^{d/\alpha-1}/[e^{\beta(E-\mu)}-1]\,dE$$

  Integrand $\to 0$ as $E\to\infty$ ✓

  For $E\to 0$, integrand $\to E^{d/\alpha-1}/E$

  for this to be finite $\Rightarrow d/\alpha - 1 - 1 > -1$

  $\therefore d > \alpha$

  Thus, get Bose-Einstein condensate at sufficiently low temperatures when $\underline{\alpha < d}$ (7)

Paper III

Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
PQM		6X

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(i) Total $j = (j_1+j_2), (j_1+j_2-1) \cdots |j_1-j_2|$ [2]

The eigenvalues of $\hat{\underline{J}}$ along any axis, given j, are

$m_j\hbar$ with $m_j = -j, -j+1 \cdots j-1, j$. [1]

Consider constructing the $j = 2j_1$ multiplet. The top state of this multiplet must be

$$|2j_1, 2j_1\rangle = |j_1\rangle \otimes |j_1\rangle$$

$\uparrow m_j$ value

Lowering repeatedly with $\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$ gives all other states in the multiplet. Since $\hat{J}_-$ is symmetric under exchange $1 \leftrightarrow 2$, the entire multiplet is symmetric.

Now consider the top state of the $2j_1-1$ multiplet. This is $|2j_1-1, 2j_1-1\rangle$ and must be constructed from $|j_1\rangle \otimes |j_1-1\rangle$ and $|j_1-1\rangle \otimes |j_1\rangle$. It has to be orthogonal to the $|2j_1, 2j_1-1\rangle$ state, which is of the form $\frac{1}{\sqrt{2}}(|j_1\rangle \otimes |j_1-1\rangle + |j_1-1\rangle \otimes |j_1\rangle)$ since symmetric under particle exchange. It follows that

$$|2j_1-1, 2j_1-1\rangle = \frac{1}{\sqrt{2}}\left(|j_1\rangle \otimes |j_1-1\rangle - |j_1-1\rangle \otimes |j_1\rangle\right)$$

and so the entire $2j_1-1$ multiplet is ~~odd~~ antisymmetric.

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Now consider the top state of the $j=2j_1-2$ multiplet

This will be of the form

$$|2j_1-2, 2j_1-2\rangle \propto |j_1\rangle \otimes |j_1-2\rangle + a|j_1-2\rangle\otimes|j_1\rangle + b|j_1-1\rangle\otimes|j_1-1\rangle \quad (+)$$

The $m_j = 2j_1-2$ state in the $j=2j_1$ multiplet is exchange symmetric, so must be

$$|2j_1, 2j_1-2\rangle \propto \frac{1}{\sqrt{2}}\left(|j_1\rangle\otimes|j_1-2\rangle + |j_1-2\rangle\otimes|j_1\rangle + c|j_1-1\rangle\otimes|j_1-1\rangle\right)$$

and in the $j=2j_1-1$ multiplet (exchange antisymmetric),

$$|2j_1-1, 2j_1-2\rangle \propto |j_1\rangle\otimes|j_1-2\rangle - |j_1-2\rangle\otimes|j_1\rangle$$

[7] As mutually orthogonal, $a=+1$ and $(+)$ is exchange symmetric.

$\overline{10}$

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(ii) On $j=1$ states, have $\hat{J}_-|1\rangle = \sqrt{2}\hbar|0\rangle$, $\hat{J}_-|0\rangle = \sqrt{2}\hbar|-1\rangle$ and $\hat{J}_+|-1\rangle = \sqrt{2}\hbar|0\rangle$ and $\hat{J}_+|0\rangle = \sqrt{2}\hbar|1\rangle$

In the $j=2$ multiplet, the top state is

$$|2,2\rangle = |1\rangle\otimes|1\rangle .$$

Applying $\hat{J}_- \Rightarrow \quad |2,1\rangle \propto |0\rangle\otimes|1\rangle + |1\rangle\otimes|0\rangle$,

and applying $\hat{J}_-$ again gives

$$|2,0\rangle \propto |-1\rangle\otimes|1\rangle + |0\rangle\otimes|0\rangle + |0\rangle\otimes|0\rangle + |0\rangle\otimes|-1\rangle$$

$$\propto |-1\rangle\otimes|1\rangle + |1\rangle\otimes|-1\rangle + 2|0\rangle\otimes|0\rangle$$

In the $j=1$ multiplet (exchange antisymmetric), we must have $|1,0\rangle \propto |-1\rangle\otimes|1\rangle - |1\rangle\otimes|-1\rangle$.

Finally, in the $j=0$ multiplet, we must have

$$|0,0\rangle \propto |-1\rangle\otimes|1\rangle + |1\rangle\otimes|-1\rangle + d\,|0\rangle\otimes|0\rangle$$

(exchange symmetric).

Orthogonality with $|2,0\rangle \Rightarrow d=-1$, hence (projectively)

(6) $$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle\otimes|-1\rangle + |-1\rangle\otimes|1\rangle - |0\rangle\otimes|0\rangle\right)$$

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Final spin state has $S = 2, 1,$ or 0 with $S=2$ and $S=0$ exchange symmetric. This will be combined with a spatial state describing the relative motion $\psi(\underline{r})$, where $\underline{r} = \underline{r}_1 - \underline{r}_2$. Under particle exchange $\underline{r}_1 \leftrightarrow \underline{r}_2$ and $\underline{r} \to -\underline{r}$. The final total angular momentum must be zero so an $S=2$ spin state can only combine with an $l=2$ $\psi(\underline{r})$, $S=1$ with $l=1$ and $S=0$ with $l=0$. (4)

However, the γs are bosons so the final state must be exchange symmetric. All cases satisfy this since, e.g., $S=2$ is symmetric and $l=2$ is exchange symmetric. [1]

Parity is restrictive though. The initial parity is -1 and the final parity is $(-1)^l$ for $\psi(\underline{r})$ and an orbital angular momentum state. Must have $\underline{l=1}$.

The final state is thus $S=1$, $l=1$ and zero total angular momentum. This is of the form constructed above, (4)

i.e.,

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle \otimes Y_{1,-1} + |-1\rangle \otimes Y_{1,1} - |0\rangle \otimes Y_{1,0}\right)$$

$\uparrow$ total spin azimuthal quantum number

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Required probability is thus

$$P = \frac{1}{3}\int_{\frac{\pi}{4}<\theta<\frac{3\pi}{4}} |Y_{1,-1}|^2 d\Omega = \frac{1}{3}\frac{\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \sin^2\theta \, d\cos\theta}{\int_{-1}^{1}\sin^2\theta \, d\cos\theta}$$

$$= \frac{1}{3}\frac{\left[x - \frac{1}{3}x^3\right]_0^{1/\sqrt{2}}}{\left[x-\frac{1}{3}x^3\right]_0^1}$$

[5]

$$= \frac{1}{3}\frac{\left[\frac{1}{\sqrt{2}} - \frac{1}{3}\times\frac{1}{2}\times\frac{1}{\sqrt{2}}\right]}{2/3}$$

$$= \frac{1}{2\sqrt{2}}\left(1-\frac{1}{6}\right) = \frac{5}{12\sqrt{2}}.$$

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(i) For $t>0$, $|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle$ and

at $t=0$, $|\psi(0)\rangle = |0\rangle$ so $c_n(0) = \delta_{n0}$.

Time-dependent S.E $\Rightarrow (H_0 + V(t)) \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle$

$$= i\hbar \sum_n \left(\dot{c}_n - i\frac{E_n}{\hbar} c_n \right) e^{-iE_n t/\hbar} |n\rangle.$$

Since $H_0|n\rangle = E_n|n\rangle$, we have

$$i\hbar \dot{c}_n e^{-iE_n t/\hbar} = \sum_m c_m(t) e^{-iE_m t/\hbar} \langle n|V|m\rangle$$

$$\Rightarrow \dot{c}_n = -\frac{i}{\hbar} \sum_m c_m(t) e^{i(E_n - E_m)t/\hbar} \langle n|V(t)|m\rangle$$

At zero order in V, have $c_n(t) = \delta_{n0}$. Iterating, at 1st order in V have

$$\dot{c}_n = -\frac{i}{\hbar} e^{i(E_n - E_0)t/\hbar} \langle n|V(t)|0\rangle$$

$$\Rightarrow c_n(t) = \delta_{n0} - \frac{i}{\hbar} \int_0^t e^{i(E_n - E_0)t'/\hbar} \langle n|V(t')|0\rangle dt'.$$

The probability of finding the particle in $|1\rangle$ at time t is

$$P = |\langle 1|\psi(t)\rangle|^2 = |c_1(t)|^2$$

$$= \frac{1}{\hbar^2} \left| \int_0^t e^{i(E_1 - E_0)t'/\hbar} \langle 1|V(t')|0\rangle dt' \right|^2$$

[10]

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Page 1

Dynamics – Paper 3, Question 7, Part (i)

- Collisionless Boltzmann equation: $\partial f/\partial t + (\partial f/\partial x_i)(dx_i/dt) + (\partial f/\partial v_i)(dv_i/dt) = 0$
  for phase space density distribution $f(\underline{x}, \underline{v})$
  Stationary $\Rightarrow \partial/\partial t = 0$
  Also $d\underline{x}/dt = \underline{v}$
  and $d\underline{v}/dt = -\nabla\Phi$ (2)
  So CBE $\Rightarrow \underline{v}\,\nabla f - \nabla\Phi\,\partial f/\partial\underline{v} = 0$

- If $f(\underline{x}, \underline{v}) = f(E)$ where $E = \frac{1}{2}\underline{v}^2 + \Phi(\underline{x})$
  then $\nabla f = (\partial f/\partial E)\nabla E = (\partial f/\partial E)\nabla\Phi$
  and $\partial f/\partial\underline{v} = (\partial f/\partial E)(\partial E/\partial\underline{v}) = (\partial f/\partial E).\underline{v}$ (2)
  $\therefore$ CBE $\Rightarrow \underline{v}(\partial f/\partial E)\nabla\Phi - \nabla\Phi(\partial f/\partial E).\underline{v} = 0$ ✓

- Spherical $\Rightarrow \Phi(r)$ is a monotonic function ~~of~~ r, and so is $\Psi(r)$
  $\therefore \rho(r) = \rho(\Psi) = \int m f(r, \underline{v})\, d^3\underline{v}$
  $= \int m f(r, \underline{v})\, 4\pi v^2 dv$ (2)

  Now $f(\mathcal{E}) \neq 0$ only if $\mathcal{E} = \Phi_0 - E = \Phi_0 - (\frac{1}{2}v^2 + \Phi) = \Psi - \frac{1}{2}v^2 > 0$
  $\therefore \rho(\Psi) = 4\pi m \int_0^{\sqrt{2\Psi}} f(\mathcal{E})\, v^2 dv$ (1)
  But $d\mathcal{E} = -v\,dv$
  and $v = 0 \Rightarrow \mathcal{E} = \Psi$
  and $v = \sqrt{2(\Psi - \mathcal{E})}$
  $\therefore \rho(\Psi) = 4\pi m\sqrt{2}\int_0^{\Psi} f(\mathcal{E}).(\Psi - \mathcal{E})^{1/2}\, d\mathcal{E}$ (3)

Dynamics – Paper 3, Question 7, Part (ii)

- Poissons eq $\nabla^2 \Phi = 4\pi G \rho$

  As $\Phi = \Phi_0 - \Psi \rightarrow \nabla^2 \Psi = -4\pi G\rho$

  Spherical $\rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) + 4\pi G \rho = 0$ 2

  $\therefore$ satisfied if $\rho = C\Psi^n$

- $\rho(\Psi) = 2^{5/2}\pi m \int_0^{\Psi} f(\epsilon)\sqrt{\Psi - \mathcal{E}}\, d\mathcal{E}$

  $= 2^{5/2}\pi m F \int_0^{\Psi} \mathcal{E}^{n-3/2}(\Psi - \mathcal{E})^{1/2} d\mathcal{E}$

  Let $\mathcal{E} = \Psi \cos^2\theta$ 4

  $d\mathcal{E} = -2\cos\theta \sin\theta\, \Psi\, d\theta$

  $\therefore \rho(\Psi) = 2^{5/2}\pi m F \int_{\pi/2}^{0} [\Psi \cos^2\theta]^{n-3/2}[\Psi(1-\cos^2\theta)]^{1/2}(-2)\cos\theta\sin\theta\,\Psi\, d\theta$

  $= 2^{7/2}\pi m F \Psi^n \int_0^{\pi/2} \sin^2\theta \cos^{2n-2}\theta\, d\theta$ 3

  $\therefore$ satisfied if $\underline{C = 2^{7/2}\pi m F \int_0^{\pi/2} \sin^2\theta \cos^{2n-2}\theta\, d\theta}$

- $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G C \Psi^n = 0$

  Let $s = kr$ where $k = \sqrt{4\pi G C \Psi_0^{n-1}}$

  $\therefore \frac{k^2}{s^2} k \frac{d}{ds}\left(\frac{s^2}{k^2} k \frac{d\Psi}{ds}\right) + k^2 \Psi^n / \Psi_0^{n-1} = 0$ 2

  $\therefore \frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\Psi'}{ds}\right) + \Psi'^n = 0$ where $\Psi' = \Psi/\Psi_0$

  For $n=5$, let $\Psi' = (1+\frac{1}{3}s^2)^{-1/2}$

  $s^2 d\Psi'/ds = s^2(-\frac{1}{2})\frac{2}{3}s(1+\frac{1}{3}s^2)^{-3/2} = -\frac{1}{3}s^3(1+\frac{1}{3}s^2)^{-3/2}$

  $\therefore \frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\Psi'}{ds}\right) = s^{-2}\left[-s^2(1+\frac{1}{3}s^2)^{-3/2} + \frac{1}{3}s^3\frac{3}{2}\frac{2}{3}s(1+\frac{1}{3}s^2)^{-5/2}\right]$

  $= (1+\frac{1}{3}s^2)^{-5/2}\left[-1-\frac{1}{3}s^2+\frac{1}{3}s^2\right]$ 2

  $\underline{= -(1+\frac{1}{3}s^2)^{-5/2} = -\Psi'^n}$ ✓

- $\rho(r) = C\Psi^5 = C\Psi_0^5(1+\frac{1}{3}k^2r^2)^{-5/2}$

  $\therefore \rho(r) > 0$ for $r = 0 \rightarrow \infty$ and as $r \rightarrow \infty$, $\rho \propto r^{-5}$ $\therefore$ mass is finite 2

- $M_{tot} = \int_0^{\infty} 4\pi r^2 \rho\, dr$

  $= 4\pi C \Psi_0^5 k^{-3}\int_0^{\infty} s^2(1+\frac{1}{3}s^2)^{-5/2} ds$

  $= \frac{4\pi C \Psi_0^5}{[4\pi G C \Psi^4]^{3/2}} I$

  where $I = \int_0^{\infty}[-s][-s(1+\frac{1}{3}s^2)^{-5/2}]ds$

  $= \int_0^{\infty}(-s)\frac{d}{ds}(1+\frac{1}{3}s^2)^{-3/2} ds$

  $= [-s(1+\frac{1}{3}s^2)^{-3/2}]_0^{\infty} + \int_0^{\infty}(1+\frac{1}{3}s^2)^{-3/2} ds$

  $= 0 + [s(1+\frac{1}{3}s^2)^{-1/2}]_0^{\infty}$

  $= \sqrt{3}$ 5

  $\therefore \underline{M_{tot} = \frac{\sqrt{3}}{2}\Psi_0^{-1}C^{-1/2}\pi^{-1/2}G^{-3/2}}$

Topics – Paper 3, Question 8, Part (i)

- Decay equation: $dN/dt = -\lambda N$

$$\int_{N_0}^{N} dN/N = \int_0^t -\lambda t$$
$$\therefore \quad N = N_0 e^{-\lambda t}$$

Given half-life $\rightarrow \frac{1}{2} = e^{-\lambda t_{1/2}} \rightarrow t_{1/2} = \ln 2/\lambda$ (3)

- Heating rate: $dH/dt = -\varepsilon dN/dt = \varepsilon \lambda N_0 e^{-\lambda t}$

$\therefore$ Total heating since formation at t_f: $H_{tot} = \int_{t_f}^{\infty} \varepsilon \lambda N_0 e^{-\lambda t} \, dt$

$$= \varepsilon \lambda N_0 \frac{1}{\lambda} [e^{-\lambda t}]_{\infty}^{t_f} = \varepsilon N_0 e^{-\lambda t_f}$$ (2)

- Heating required: $\Delta H_f . M_p$ where M_p = mass of planetesimal

Equating these: $\varepsilon N_0 e^{-\lambda t_f} = \Delta H_f M_p$

$$\therefore \lambda t_f = \ln[\varepsilon N_0 / \Delta H_f M_p]$$
$$\therefore t_f = t_{1/2} \ln[\varepsilon N_0 / \Delta H_f M_p] / \ln 2$$ (2)

- Initial number: $N_0 = M_p . X^{\circ}_{26Al} / (26 m_p)$

$$\therefore t_f = t_{1/2} \ln[\varepsilon X^{\circ}_{26Al} / 26 m_p \Delta H_f] / \ln 2$$
$$= 10^5 \ln[10^{-12} . 10^{-7} / 26 . 1.673 \times 10^{-27} . 10^6] / \ln 2$$ (2)
$$= 0.12 \text{ Myr}$$ ie within 0.12 Myr (independent of size)

- Has assumed that heat goes into melting rather than radiation, requiring time for diffusion to be longer than t_f

$t_{diff} \sim L^2/D$

Average radius of mass is $\frac{3}{4} R_p$ so $L \sim \frac{1}{4} R_p$

$\therefore t_{diff} \sim (0.25 \times 50 \times 10^3)^2 / 10^{-6} = 1.56 \times 10^{14}$ s $= 5$ Myr (1)

$\therefore \gg t_f$

Topics - Paper 3, Question 8, Part (ii)

- Gravitational potential energy from adding dM to planet of M, R is $dE = GM\,dM/R$

  Given $M \propto R^2 \rightarrow M = \left(\frac{M_\oplus}{R_\oplus^2}\right) R^2$

  $$\therefore dM = 2\left(\frac{M_\oplus}{R_\oplus^2}\right) R\,dR$$

  $$dE = G\left(\frac{M_\oplus}{R_\oplus^2}\right)R^2\, 2\left(\frac{M_\oplus}{R_\oplus^2}\right)R\,dR / R$$

  $$= 2G\left(M_\oplus/R_\oplus^2\right)^2 R^2\,dR$$

  $$\therefore E_{tot} = \int_0^R dE = \frac{2}{3} G\left(M_\oplus/R_\oplus^2\right)^2 R^3 \qquad (3)$$

- Energy for melting is $\Delta H_f . M = \Delta H_f \left(M_\oplus/R_\oplus^2\right)R^2$

  Equating these: $R = \frac{3}{2}\frac{\Delta H_f}{G} R_\oplus^2/M_\oplus \qquad (2)$

  $$= 150 \text{ km}$$

- Isothermal atmosphere $p = nk_BT = \rho k_B T/(\mu m_p)$

  Adding shell $\rightarrow$ $dp = -\rho g\,dz$

  $$= -\frac{\mu m_p}{k_B T} g p\,dz$$

  $$\int_{P_0}^{P_1} \frac{1}{p}dp = \int_0^z -\frac{\mu m_p}{k_B T} g\,dz \qquad (3)$$

  $$\therefore P = P_0\, e^{-(\mu m_p g/k_B T)z}$$

  But $P_0 = M_{atm}\, g / 4\pi R^2$

  $$g = GM/R^2 = GM_\oplus/R_\oplus^2 \simeq 10 \text{ m/s}^2 \qquad (3)$$

  $$M_{atm} = f_w M \rightarrow P_0 = f_w G\left(M_\oplus/R_\oplus^2\right)^2/4\pi$$

  Need to find z for $P = 1$ mbar

  $$\therefore z = \left(\frac{k_B T}{\mu m_p g}\right)\ln\left[f_w\, g\left(M_\oplus/R_\oplus^2\right)/4\pi P\right]$$

  $$= \left(\frac{1.381\times10^{-23}\times1000}{18\times1.673\times10^{-27}\times10}\right)\ln\left[0.01\times10\times\left(5.976\times10^{24}/(6.371\times10^6)^2\right)/4\pi\times100\right] \qquad (3)$$

  $$= 750 \text{ km}$$

- Sound speed $c_s = \sqrt{k_B T/\mu m_p} \simeq 700$ m/s

  Escape speed at surface: $V_{esc} = \sqrt{2GM/R}$

  $$= \sqrt{2G\left(\frac{M_\oplus}{R_\oplus^2}\right)R}$$

  For planet capable of melting $V_{esc} = \sqrt{3\Delta H_f} = 1{,}700$ m/s $\qquad (4)$

  As $V_{esc} > c_s$ atmosphere is bound (just) and would be more so for larger bodies $\rightarrow$ yes likely can retain atmospheres.

- Transit depth $\propto R_{obs}^2$

  Increase in transit depth $= (R+z)^2/R^2$

  $$= (R_\oplus + z)^2/R_\oplus^2 \qquad (2)$$

  $$= 1.25$$