PAPER II

Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
RELATIVITY		1
Examiner	• In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen. • Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness. • Write on this side only and between the margins. • Not more than one solution per sheet please.	Paper 1 / 2 / 3 / 4
Lecturer CHALLINOR		Section I / II

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(i) $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0.$$

Using $\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\kappa}(\partial_\nu g_{\kappa\rho} + \partial_\rho g_{\kappa\nu} - \partial_\kappa g_{\nu\rho})$,

have $$\frac{d^2x^\mu}{d\lambda^2} + \frac{1}{2}g^{\mu\kappa}(2\partial_\nu g_{\kappa\rho} - \partial_\kappa g_{\nu\rho})\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$$

$$\frac{dx^\nu}{d\lambda}\partial_\nu g_{\kappa\rho} = \frac{d}{d\lambda}g_{\kappa\rho}$$

$$\Rightarrow \frac{d^2x^\mu}{d\lambda^2} + g^{\mu\kappa}\left(\frac{d}{d\lambda}g_{\kappa\rho}\right)\frac{dx^\rho}{d\lambda} - \frac{1}{2}g^{\mu\kappa}(\partial_\kappa g_{\nu\rho})\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$$

[7] Multiply by $g_{\tau\mu}$ to get

$$g_{\tau\mu}\frac{d^2x^\mu}{d\lambda^2} + \left(\frac{d}{d\lambda}g_{\tau\rho}\right)\frac{dx^\rho}{d\lambda} = \frac{1}{2}(\partial_\tau g_{\nu\rho})\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda}\left(g_{\tau\mu}\frac{dx^\mu}{d\lambda}\right) = \frac{1}{2}(\partial_\tau g_{\nu\rho})\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}$$ NOTES

Relabelling indices gives

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right) = \frac{1}{2}(\partial_\mu g_{\nu\rho})\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} \qquad (*)$$

If $g_{\mu\nu}$ is independent of a coordinate, say x^τ, then

[3] $(*) \Rightarrow g_{\tau\mu}\frac{dx^\mu}{d\lambda}$ is constant, i.e., the τ component of the 4-momentum, p_τ, is constant → a conserved quantity.

Examples include p_0 (energy) in a stationary metric

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Comments

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(i) CONTINUED

and $r^2\dot\phi \sin^2\theta$ (angular momentum about the z-axis) in a spherically-symmetric system

—o—

[2] (ii) We have $\frac{\partial}{\partial x^0} g_{\mu\nu} = 0 \Rightarrow g_{0\nu}\frac{dx^\nu}{d\lambda} = \text{const}$. Since the metric is diagonal, $g_{00}\frac{dx^0}{d\lambda} = \underline{\underline{g_{00}\frac{dt}{d\lambda} = \text{const}}}$.

Setting $\mu = i$ in (*),

$$\frac{d}{d\lambda}\left(g_{i\nu}\frac{dx^\nu}{d\lambda}\right) = \frac{1}{2}(\partial_i g_{\nu\rho})\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right) = \frac{1}{2}(\partial_i g_{00})\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{2}\partial_i g_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}.$$

For a null curve, $ds^2 = 0 = g_{00}dt^2 + g_{ij}dx^i dx^j$,

so $\left(\frac{dt}{d\lambda}\right)^2 = -\frac{g_{ij}}{g_{00}}\left(\frac{dx^i}{d\lambda}\right)\left(\frac{dx^j}{d\lambda}\right)$, so

UNSEEN CALCULATION

[7]
$$\frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right) = \Bigg[\underbrace{-\frac{\frac{1}{2}(\partial_i g_{00})g_{jk}}{g_{00}} + \frac{1}{2}\partial_i g_{jk}}_{\frac{1}{2}\frac{\partial}{\partial x^i}(g_{jk}/g_{00})\,g_{00}}\Bigg]\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right) = \underline{\underline{\frac{1}{2}g_{00}\frac{\partial}{\partial x^i}(g_{jk}/g_{00})\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}}} \qquad (+)$$

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(ii) CONTINUED

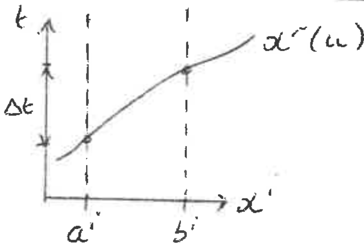
Now convert from λ to t using $\frac{d}{d\lambda} = \frac{dt}{d\lambda}\frac{d}{dt}$

$$= \frac{\text{const.}}{g_{00}}\frac{d}{dt}$$

[4] Follows that

$$\frac{1}{g_{00}}\frac{d}{dt}\left(\frac{g_{ij}}{g_{00}}\frac{d}{dt}x^j\right) = \frac{1}{2}g_{00}\frac{\partial}{\partial x^i}\left(g_{jk}/g_{00}\right)\frac{1}{(g_{00})^2}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\gamma_{ij}\frac{dx^j}{dt}\right) = \frac{1}{2}\frac{\partial}{\partial x^i}\left(\gamma_{jk}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} \qquad \left[\gamma_{ij} \equiv \frac{-g_{ij}}{g_{00}}\right]$$



[2] Since null, $dt^2 = \frac{-g_{ij}\,dx^i dx^j}{g_{00}}$

$$= \gamma_{ij}\frac{dx^i}{du}\frac{dx^j}{du}(du)^2$$

UNSEEN CALCULATION

It follows that $\Delta t = \int_0^1 \sqrt{\gamma_{ij}\frac{dx^i}{du}\frac{dx^j}{du}}\,du$

With $L = \sqrt{\gamma_{ij}\frac{dx^i}{du}dx^j/du}$, $\frac{\partial L}{\partial x^k} = \frac{d}{du}\frac{\partial L}{\partial(dx^k/du)}$ gives

[5]
$$\frac{1}{2L}(\partial_k\gamma_{ij})\frac{dx^i}{du}\frac{dx^j}{du} = \frac{1}{2}\frac{d}{du}\left(\frac{2}{L}\gamma_{kj}\frac{dx^j}{du}\right).$$

Now use $L = \frac{dt}{du}$ so $\frac{1}{L}\frac{d}{du} = \frac{d}{dt}$ and hence

geodesic eq (+)

$$\frac{L^2}{2L}(\partial_k\gamma_{ij})\frac{dx^i}{dt}\frac{dx^j}{dt} = L\frac{d}{dt}\left(\gamma_{kj}\frac{dx^j}{dt}\right) \Rightarrow \frac{d}{dt}\left(\gamma_{ij}\frac{dx^j}{dt}\right) = \frac{1}{2}\partial_i\gamma_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

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(i) Adopting the "Lagrangian"

$$L = \left(1-\frac{2\mu}{r}\right)c^2\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$ in the plane

$\theta = \pi/2$, the Euler-Lagrange equations, $\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right)$, give:

[2] $$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial L}{\partial \dot{t}} = \text{const.} \Rightarrow \left(1-\frac{2\mu}{r}\right)\dot{t} = k \;;$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const.} \Rightarrow r^2\dot{\phi} = h.$$ NOTES

The 4-momentum of the particle is $m\frac{dx^\mu}{d\tau} = p^\mu$. The 4-velocity of an observer at fixed r is

$$u^\mu = A\delta^\mu_0 \text{, where } g_{\mu\nu}u^\mu u^\nu = c^2 \text{ gives}$$

$$A^2c^2\left(1-\frac{2\mu}{r}\right) = c^2 \Rightarrow A = \left(1-\frac{2\mu}{r}\right)^{-\frac{1}{2}}.$$

The energy of the particle, as measured by the stationary observer, is $E = g_{\mu\nu}p^\mu u^\nu = A g_{0\mu}p^\mu = A g_{00}p^0$ EXTENSION OF NOTES.

$$= \left(1-\frac{2\mu}{r}\right)^{-\frac{1}{2}}c^2\left(1-\frac{2\mu}{r}\right) \times m\dot{t}$$

$$= \left(1-\frac{2\mu}{r}\right)^{-\frac{1}{2}}mkc^2$$

If the observed Lorentz factor is γ, it follows that

[4] $$\gamma = k\left(1-\frac{2\mu}{r}\right)^{-\frac{1}{2}} \to k \text{ as } r\to\infty$$ Since

$\gamma = (1-v^2/c^2)^{-\frac{1}{2}}$, follows that $\frac{v_\infty}{c} = \sqrt{1-k^{-2}}$

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Paper II

2 i) M, Spherical Symmetry, steady state, barotropic EOS (AFD I)

$\nabla p = c_s^2 \nabla \rho$

Continuity: $\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{0} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho (-u)) = 0$

$\Rightarrow r^2 \rho u = \text{const.}$

$\Rightarrow r^2 u \frac{\partial \rho}{\partial r} + r^2 \rho \frac{\partial u}{\partial r} + 2r\rho u = 0$

$\div r^2 u \rho \quad \Rightarrow \frac{1}{\rho}\frac{\partial \rho}{\partial r} + \frac{1}{u}\frac{\partial u}{\partial r} + \frac{2}{r} = 0$

$\Rightarrow \frac{d}{dr}\ln\rho + \frac{d}{dr}\ln u + \frac{2}{r} = 0$ ①

Momentum: $\underbrace{\cancel{\frac{\partial \underline{u}}{\partial t}}}_{0} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla P + \underline{g}$

$\Rightarrow u\frac{d}{dr}u = -\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{GM}{r^2}$

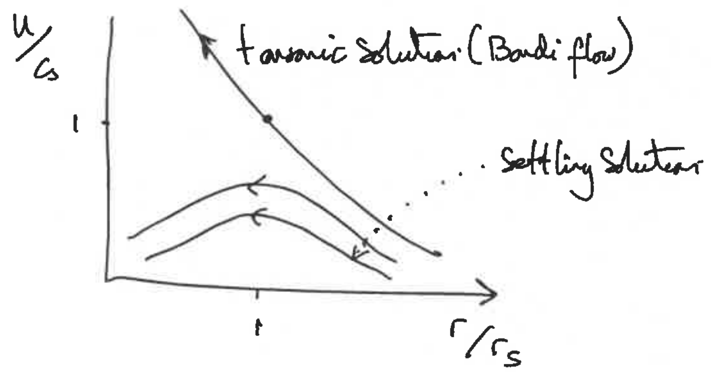
$\Rightarrow u^2 \frac{\partial \ln u}{\partial r} = -c_s^2 \frac{\partial \ln \rho}{\partial r} - \frac{GM}{r^2}$

Sub ① $\Rightarrow u^2 \frac{d}{dr}(\ln u) = c_s^2 \frac{d(\ln u)}{dr} + \frac{2c_s^2}{r} - \frac{GM}{r^2}$

$\Rightarrow (u^2 - c_s^2)\frac{d}{dr}(\ln u) = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$

$$\Rightarrow \left(\frac{u^2}{c_s^2}-1\right)\frac{d}{dr}(\ln u) = \frac{2}{r}\left(1-\frac{r_s}{r}\right)$$ 7

For accretion problem, need $u \to 0$ as $r \to \infty$. 1



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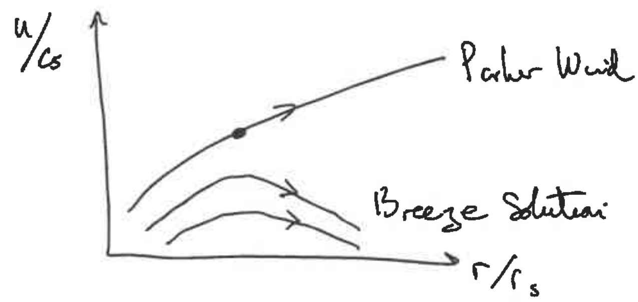
ii) Solar wind : Steady state, spherical Sym, isothermal

$\Rightarrow$ obeys same eqn. as found in Part i

Assuming subsonic at small r



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From part (i)

$$\left(\frac{u^2}{c_s^2}-1\right)\frac{d}{dr}(\ln u) = \frac{2}{r} - \frac{2r_s}{r^2}$$

$$\Rightarrow \int\left(\frac{u}{c_s^2}-\frac{1}{u}\right)du = \int \frac{2}{r} - \frac{2r_s}{r^2}\, dr$$

$$\Rightarrow \left[\frac{1}{2}\frac{u^2}{c_s^2} - \ln u\right]_{c_s}^{u} = \left[2\ln r + \frac{2r_s}{r}\right]_{r_s}^{r}$$

$$\Rightarrow \frac{1}{2}\frac{u^2}{c_s^2} - \ln u - \frac{1}{2} - \ln c_s = 2\ln r + \frac{2r_s}{r} - 2\ln r_s - 2$$

$$\Rightarrow \frac{u^2}{c_s^2} - 2\ln\left(\frac{u}{c_s}\right) = 4\ln\left(\frac{r}{r_s}\right) + \frac{4r_s}{r} - 3 \qquad 4$$

[or derive by writing down Bernoulli constant]

$T = 10^6\,K$

Isothermal sound speed: $c_s = (P/\rho)^{1/2}$

$$= \left(\frac{n k_B T}{\mu n m_p}\right)^{1/2} \qquad \mu = 0.5 \text{ for ionized H}$$

$$= \left(\frac{k_B T}{\mu m_p}\right)^{1/2}$$

$$\Rightarrow c_s \simeq 128\ \mathrm{km\,s^{-1}}$$

Sonic radius, $r_s = \frac{GM_\odot}{2c_s^2} \approx 4.02 \times 10^9$ m // 2/

Radius of Earth's orbit, $r_E = 1$au

$$\Rightarrow \frac{r_E}{r_s} \approx \frac{1.5 \times 10^{11}}{4.02 \times 10^9} \approx 37$$

Velocity given by Solution to

$$\left(\frac{u}{c_s}\right)^2 - 2\ln\left(\frac{u}{c_s}\right) = 4\ln\left(\frac{r_E}{r_s}\right) + 4\frac{r_s}{r_E} - 3$$

$$\Rightarrow \left(\frac{u}{c_s}\right)^2 - \ln\left(\frac{u}{c_s}\right)^2 \approx 4\ln 37 + 4/37 - 3$$

$$\Rightarrow x - \ln x \approx 11.6, \quad x \equiv (u/c_s)^2$$

Trial values of x

$x = 10 \Rightarrow 7.7$

$x = 15 \Rightarrow 12.3$

$x = 12 \Rightarrow 9.5$

$x = 14 \Rightarrow 11.4$

$x = 14.3 \Rightarrow 11.6$

So, estimate $x \approx 14.3$

$\therefore u \approx c_s\sqrt{14.3}$

$u \approx 480$ km s^{-1} // 4//

At very large distances from the Sun:

$\ln r/r_s$ dominates over r_s/r

$(u/c_s)^2$ dominates over $\ln(u/c_s)$

$$\Rightarrow \left(\frac{u}{c_s}\right)^2 \approx 4\ln(r/r_s)$$

$$u \approx 2c_s \ln(r/r_s)^{1/2}$$

Mass conservation gives $\dot{m} = 4\pi r^2 \rho u$

$$\Rightarrow \rho \propto 1/r^2 u$$

$$p = c_s^2 \rho \propto \frac{c_s^2}{r^2 u}$$

$$\therefore p \to 1/r^2 \ln(r/r_s)^{1/2} \quad \text{as } r \to \infty$$

4

Solar Breeze: $u \ll c_s$ at large r

$$\frac{d}{dr}(\ln u) \approx \frac{2}{r} \qquad \text{(using large } r\text{)}$$

$$= 2\frac{d}{dr}\ln r$$

$$\frac{d}{dr}\left(\ln u r^{-2}\right) = 0$$

$$ur^{-2} \to \text{const as } r \to \infty$$

$$\rho \to \text{const as } r \to \infty$$

$$p \to \text{const as } r \to \infty$$

The Solar wind flows out into the interstellar medium which has a very low pressure. So the appropriate Solution for Solar wind is the one for which $p \to 0$ as $r \to \infty$.

The breeze Solution is thus inappropriate whereas the Parker Wind is ok.

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Paper II   Cosmology

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COSMOLOGY		3X
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(ii) First consider νs when relativistic and in thermal equilibrium. Then, as

$$f_{eq}\, d^3p = \frac{d^3p}{h^3}\frac{1}{e^{E/k_BT}+1},$$

$$f_{eq} = \frac{1}{h^3}\frac{1}{e^{E/k_BT}+1}$$

If νs are highly relativistic when they decouple (at T_d, say), then the distribution function at decoupling is

[4]

$$\frac{1}{h^3}\frac{1}{e^{pc/k_BT_d}+1}$$

The distribution function then evolves according to the Collisionless Boltzmann equation, ie

$$\frac{\partial f}{\partial t} = \frac{\dot{R}}{R}p\frac{\partial f}{\partial p}$$

Try a solution $f \propto \dfrac{1}{e^{pc/k_BT_\nu(t)}+1}$ with $T_\nu = T_d$ at the decoupling time.

$$\frac{\partial f}{\partial t} \propto \frac{-1}{(e^{pc/k_BT_\nu}+1)^2}\, e^{\frac{pc}{k_BT_\nu}} \times \left(\frac{-pc}{k_BT_\nu^2}\right)\frac{dT_\nu}{dt}$$

$$\frac{\dot{R}}{R}p\frac{\partial f}{\partial p} \propto \left\{\quad\right\} \times \frac{pc}{k_BT_\nu}\frac{\dot{R}}{R} \Rightarrow \frac{\dot{T}_\nu}{T_\nu} = -\frac{\dot{R}}{R},$$

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[6] which is solved by $T_\nu \propto R^{-1}$ with $T_\nu = T_d$ at decoupling.

The comoving entropy $R^3[c^2\rho + P]/T$, of all the particles in thermal equilibrium is constant and is dominated by relativistic particles (massive annihilate) with $P \simeq \frac{1}{3}\rho c^2$.

When νs decouple, $T_\nu = T_\gamma$ initially. T_ν then continues to fall as $1/R$. Before e^+e^- annihilation,

$$\rho_{eq} c^2 = a T_\gamma^4 \Big[\underset{\gamma}{\frac{2}{2}} + \underbrace{\underset{\text{Fermions}}{\frac{7}{8}} \times \underset{\text{spin } \frac{1}{2}}{\frac{2}{2}}}_{e^+e^-} \times \underset{e^+ \& e^-}{2} \Big] = \frac{11}{4} a T_\gamma^4$$

After, $$\rho_{eq} c^2 = a T_\gamma'^4 \Big[\underset{\text{only } \gamma}{\frac{2}{2}} \Big] = a T_\gamma'^4$$

Hence $$\frac{R^3}{T_\gamma} \frac{11}{4} T_\gamma^4 = \frac{R'^3}{T_\gamma'} T_\gamma'^4 \Rightarrow R T_\gamma = \left(\frac{4}{11}\right)^{\frac{1}{3}} R' T_\gamma'$$

[5] But $R T_\nu = R' T_\nu'$ and $T_\nu = T_\gamma$ so $\underline{T_\nu' = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_\gamma'}$

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Before e^+e^- annihilation, have $e^+(g=2)$, $e^-(g=2)$, $\gamma\ (g=2)$, $\nu_i\ (g_i=1)$, $\bar{\nu}_i\ (g_i=1)$ for $i = e, \mu, \tau$. Also, $T_\gamma = T_\nu$.

↑only one helicity state

Hence $$g_{\rm eff} = \underbrace{\frac{7}{8}\times 2\times 2}_{e^+,e^-} + \underbrace{3\times\frac{7}{8}\times 2}_{\nu,\bar{\nu}} + \underbrace{2}_{\gamma} = \underline{\frac{43}{4}}$$

After, only have γ, ν_i, $\bar{\nu}_i$ and $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$.

Hence

[5] $$g_{\rm eff} = \underbrace{2}_{\gamma} + 3\times\frac{7}{8}\times 2\times\left(\frac{4}{11}\right)^{4/3} = \underline{3.36}$$

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(i) Particle horizon = max distance a signal can have propagated at time t. Consider a null geodesic,

$$ds^2 = 0 \Rightarrow c\,dt = R(t)\,d\chi \quad \text{(outgoing)}.$$

Have $\Delta\chi = \int_0^t \frac{c\,dt'}{R(t')}$, and so

[2] $$d_{ph}(t) = cR(t)\int_0^t \frac{dt'}{R(t')}$$

If $R \propto t^p$ $(p<1)$,

$$d_{ph}(t) = ct^p \int_0^t (t')^{-p}\,dt'$$

$$= \frac{c}{1-p} t^p \left[(t')^{1-p}\right]_0^t$$

$$= \frac{c}{1-p} t^p \left[t^{1-p}\right] \quad (p<1)$$

[1] $$= \frac{ct}{(1-p)}$$

If $P = w\rho c^2$, continuity equation gives

$$\dot\rho = -3\frac{\dot R}{R}(\rho + w\rho) \Rightarrow \frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot R}{R}$$

so $\rho \propto R^{-3(1+w)}$

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Stars Q2 Paper II

4i) Calculate mass loss prior to evolution off of main sequence

Combine eqs.

$$\log\left(\frac{dM}{dt}\right) \approx -12.76 + 1.30\left(0.78 + 2.76\log\left(\frac{M_i}{M_\odot}\right)\right)$$

$$= -11.746 + 3.59\log\left(\frac{M_i}{M_\odot}\right)$$

$$\frac{dM}{dt} = 10^{-11.746}\left(\frac{M_i}{M_\odot}\right)^{3.59}$$

$$dM = 10^{-11.746}\left(\frac{M_i}{M_\odot}\right)^{3.59} dt$$

$\boxed{1}$ $\therefore\ \Delta M = 10^{-11.746}\left(\frac{M_i}{M_\odot}\right)^{3.59}\Delta t$, where Δt is main sequence lifetime

Now, $\Delta t = 10^{7.72}\left(\frac{M_i}{M_\odot}\right)^{-0.66}$

so, $\Delta M = 10^{-4.026}\left(\frac{M_i}{M_\odot}\right)^{2.93}$

and $\frac{\Delta M}{M_i} = 10^{-4.026}\left(\frac{M_i}{M_\odot}\right)^{1.93}$

$M_i (M_\odot)$	$\Delta M/M_i \times 100$
25	4.7
40	11.6
60	25.5
85	49.9
120	97.0

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Stars Q2

For 85 $M_\odot$ Star, with convective core at 83% of mass

when is core exposed?

use eq. $\boxed{1}$ $\quad \Delta M = 10^{-11.746}\left(\frac{M_i}{M_\odot}\right)^{3.59} \Delta t$

Now $\frac{\Delta M}{M_i} = 0.17$

$$\therefore \quad 0.17 \cdot 10^{11.746}\left(\frac{M_i}{M_\odot}\right)^{-2.59} = \Delta t$$

$$\Delta t = 0.95 \times 10^6 \text{ years} \quad //$$

(2)

Such a Star would be classified as a WNL Star, i.e., a late Wolf-Rayet Star of the WN sub-type. //

(2)

Stars Q2

ii) 1

$$L(r) = -4\pi r^2 \frac{16\sigma}{3} \frac{T(r)^3}{\rho(r)\kappa(r)} \frac{dT(r)}{dr}$$

2

$$\kappa(r) \propto \rho(r) T(r)^{-3.5}$$

Show

$$L \propto T_{eff}^{4/5}$$

Need to determine scaling arguments, describing how parameters in 1, 2 scale with contracting radius of star.

2

$\rho(r)$ mean density of star mass M, radius R is

$$\bar{\rho} = \left[\frac{4}{3}\pi R^3\right]^{-1} M$$

under assumption of homology if $\bar{\rho}$ scales as R^{-3} so does the density at any radial distance r.

Hence

$$\rho(r) \propto R^{-3}$$

3

$p(r)$ For the temperature scaling in 1 use ideal gas law

$p(r) = \rho(r) k T(r) / \mu m_H$. In hydrostatic eqb.:

$$\frac{dp(r)}{dr} = -\frac{G m(r)\rho(r)}{r^2}$$

Assuming homology, a pressure interval $dp(r)$ scales with R in the same way as the core pressure, P_c

A radial interval dr scales in the same way as the radius R. Thus:

$$\frac{P_c}{R} \propto \frac{\rho}{R^2} \propto \frac{R^{-3}}{R^2} \propto R^{-5}$$

or, $P_c \propto R^{-4}$

By homology if the core pressure scales with the contracting radius as R^{-4}, so does the pressure at any radial distance. Hence

4/ $\quad p(r) \propto R^{-4}$

$T(r)$/ From the ideal gas law $T(r) \propto \frac{P(r)}{\rho(r)}$

5/ using 3/ & 4/ $\quad T(r) \propto R^{-1}$ // ④

$k(r)$/ from 2, with 3/ & 5/

$$k(r) \propto R^{-3} \cdot (R^{-1})^{-3.5}$$

$$\propto R^{0.5} \quad //$$ ①

$\frac{dT(r)}{dr}$/ A temperature increment $dT(r)$ scales with the contraction in the same way as $T(r)$. dr scales as R ∴

$$\frac{dT(r)}{dr} \propto \frac{R^{-1}}{R} \propto R^{-2} \quad //$$ ②

Applying Scalings derived to equation given for $L(r)$

$$L(r) \propto R^2 \cdot \frac{(R^{-1})^3}{(R^{-3})(R^{0.5})} \cdot R^{-2}$$

$$\propto R^{-0.5}$$

6/

By homology, if $L(r)$ Scales as $1/\sqrt{R}$, then so must the Surface luminosity of the Star, L // ①

Now, need to relate L to T_{eff}. For a black body $L = 4\pi\sigma R^2 T_{eff}^4$

From 6/ $R \propto L^{-2}$

$$\therefore L \propto (L^{-2})^2 T_{eff}^4$$

$$L^5 \propto T_{eff}^4$$

$$L \propto T_{eff}^{4/5}$$ // ④

the poor fit for $M \lesssim 2M_\odot$ suggest one of our assumptions must be less applicable to low mass Stars // ②

The assumption that fails is that energy transport is dominated by radiative diffusion rather than convection: the envelopes of lower mass stars experience more convection, hence they deviate more from the expected slope of 4/5 in the H-R diagram (log-log). / ④

Stat Phys - Paper 2, Question 5, Part (i)

- Micro canonical ensemble: closed system, fixed energy and number of particles (1)
- Canonical ensemble: system with fixed volume and number of particles in thermal contact with a heat reservoir at temperature T; heat can flow in/out and energy of system can fluctuate. (2)
- Grand canonical ensemble: as canonical ensemble, but also in contact with a particle reservoir; particles can flow in/out and so the number of particles can also fluctuate. (2)
- Choice of ensemble becomes irrelevant if fluctuations around the average energy and particle number are irrelevant, ie. for macroscopic systems (thermodynamic limit) as long as there are not long range correlations (so away from phase transitions) $N \to \infty$ (2)
- The canonical partition function $Z = \sum_i e^{-\beta E_i}$ where E_i is energy of state i
It defines the probability that a system occupies a given state, s $= \frac{1}{Z} e^{-\beta E_s}$ (3)
Thus the thermodynamic, or ensemble average of a quantity $\langle X \rangle = \frac{1}{Z} \sum_s X_s e^{-\beta E_s}$
eg, for total energy $\langle E \rangle = \frac{-1}{Z} \partial Z/\partial\beta = -\partial \ln Z/\partial\beta$

Stat Phys - Paper 2, Question 5, Part (ii)

not for marks

$$H = \underline{p}_1^2/2m + \underline{p}_2^2/2m + \Phi$$
$$= (\tfrac{1}{2}\underline{P} + \underline{p})^2/2m + (\tfrac{1}{2}\underline{P} - \underline{p})^2/2m + \Phi \quad , \text{ where } \underline{P} = \underline{p}_1 + \underline{p}_2 ,\ \underline{p} = (\underline{p}_1 - \underline{p}_2)/2$$
$$= \frac{1}{4m}\underline{P}^2 + \frac{1}{m}\underline{p}^2 + \Phi = \underline{\frac{1}{2M}\underline{P}^2 + \frac{1}{2\mu}\underline{p}^2 + \alpha|\underline{r}|} \quad , \text{ where } M = m_1 + m_2 = 2m,\ \mu = m/2$$

- H is a function of $\underline{R} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$, $\underline{r}$, $\underline{P}$ and $\underline{p}$ and is separable

$$\therefore Z = \frac{1}{(2\pi\hbar)^6}\int d^3\underline{R}\, d^3\underline{r}\, d^3\underline{P}\, d^3\underline{p}\, e^{-\beta H}$$
$$= \frac{1}{(2\pi\hbar)^3}\int d^3\underline{R}\, d^3\underline{P}\, e^{-\beta \underline{P}^2/2M} \cdot \frac{1}{(2\pi\hbar)^3}\int d^3\underline{r}\, d^3\underline{p}\, e^{-\beta H_{int}} \quad \text{where } H_{int} = \frac{1}{2\mu}\underline{p}^2 + \alpha|\underline{r}|$$
$$\underline{= Z_{trans} \cdot Z_{int}}$$ (3)

$$\therefore Z_{trans} = \left(\frac{V}{h^3}\right)\int d^3\underline{P}\, e^{-\beta \underline{P}^2/2M} = \left(\frac{V}{h^3}\right)\left[\sqrt{\pi 2M/\beta}\,\right]^3 \quad \text{(from hint)}$$
$$\underline{= V\,(2\pi M/\beta h^2)^{3/2}}$$

$$\therefore Z_{int} = \frac{1}{(2\pi\hbar)^3}\int d^3\underline{p}\, e^{-\beta \underline{p}^2/2\mu} \cdot \int_0^\infty 4\pi r^2 dr\, e^{-\beta\alpha r}$$
$$= 4\pi \cdot (2\pi\mu/\beta h^2)^{3/2} \cdot \left[\left[r^2\left(\cancel{-}\frac{1}{\beta\alpha}\right)e^{-\beta\alpha r}\right]_0^\infty - \int_0^\infty 2r\left(-\frac{1}{\beta\alpha}\right)e^{-\beta\alpha r}\,dr\right] \quad \text{integrating by parts}$$
$$= 4\pi\,(2\pi\mu/\beta h^2)^{3/2} \cdot \left(\frac{2}{\beta\alpha}\right)\left[\left[r\left(\cancel{-\frac{1}{\beta\alpha}}\right)e^{-\beta\alpha r}\right]_0^\infty - \int_0^\infty \left(-\frac{1}{\beta\alpha}\right)e^{-\beta\alpha r}\,dr\right]$$ (6)
$$\underline{= 8\pi \cdot (2\pi\mu/\beta h^2)^{3/2} \cdot (\beta\alpha)^{-3}}$$

- Average separation $$\langle r\rangle = \frac{1}{(2\pi\hbar)^6}\int r\, d^3\underline{R}\, d^3\underline{r}\, d^3\underline{P}\, d^3\underline{p}\, e^{-\beta H} / Z$$
$$= \int_0^\infty r^3 e^{-\beta\alpha r}\,dr \Big/ \int_0^\infty r^2 e^{-\beta\alpha r}\,dr$$
$$= [6/(\beta\alpha)^4] / [2/(\beta\alpha)^3] \quad \text{(from integration by parts as above)}$$
$$\underline{= 3/\alpha\beta}$$ (3)

- For N identical mesons, $Z_{tot} = \frac{1}{N!} Z_{single}^N$
where $Z_{single} = V\,(2\pi M/\beta h^2)^{3/2} \cdot 8\pi \cdot (2\pi\mu/\beta h^2)^{3/2} \cdot (\beta\alpha)^{-3}$
$= V \cdot 8\pi \cdot M^{3/2} \cdot \mu^{3/2} \cdot (2\pi/h^2\alpha)^3 \cdot \beta^{-6}$ (3)

- Free energy $F = -k_B T \ln Z$
$dF = d(E - TS) = -S\,dT - p\,dV$
$\therefore S = -\partial F/\partial T|_V$
$\therefore C_V = T\,\partial S/\partial T|_V = -T\,\partial^2 F/\partial T^2|_V$
Now $F = -k_B T \ln\left[\frac{1}{N!}(AT^6)^N\right] = -6Nk_B T\ln T + BT$
$\therefore \partial F/\partial T = -6Nk_B \ln T + C$
$\therefore \partial^2 F/\partial T^2 = -6Nk_B/T$
$\therefore \underline{C_V = 6Nk_B}$ (5)

Paper II

Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
PQM		6X

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- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Examiner CHALLINOR

Lecturer SKINNER

Paper 1 / 2 / 3 / 4

Section I / II

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Comments

(i) Let the perturbed state be

$$\psi_n = |n\rangle + \lambda \sum_{m\neq n} a_m^{(n)} |m\rangle + \lambda^2 \sum_{m\neq n} b_m^{(n)} |m\rangle + \cdots .$$

Then $(H+\lambda \Delta H)\psi_n = E_n(\lambda)\psi_n = (E_n + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots) \times \psi_n$

$$\Rightarrow E_n|n\rangle + \lambda \sum_{m\neq n} a_m^{(n)} E_m|m\rangle + \lambda^2 \sum_{m\neq n} b_m^{(n)} E_m |m\rangle$$

$$+ \lambda \Delta H|n\rangle + \lambda^2 \sum_{m\neq n} a_m^{(n)} \Delta H|m\rangle + \cdots \qquad (*)$$

$$= E_n|n\rangle + \lambda E_n \sum_{m\neq n} a_m^{(n)} |m\rangle + \lambda^2 E_n \sum_{m\neq n} b_m^{(n)} |m\rangle$$

$$+ \lambda E_n^{(1)}|n\rangle + \lambda^2 E_n^{(1)} \sum_{m\neq n} a_m^{(n)} |m\rangle + \lambda^2 E_n^{(2)} |n\rangle + \cdots$$

At 1st-order in λ,

$$\Delta H|n\rangle + \sum_{m\neq n} a_m^{(n)} E_m |m\rangle = E_n^{(1)}|n\rangle + E_n \sum_{m\neq n} a_m^{(n)}|m\rangle$$

Inner product of this with $|n\rangle \Rightarrow$ $\underline{E_n^{(1)} = \langle n|\Delta H|n\rangle}$.

Inner product with $|m\rangle \neq |n\rangle$ gives

$$\langle m|\Delta H|n\rangle + a_m^{(n)} E_m = E_n a_m^{(n)}$$

$$\Rightarrow a_m^{(n)} = \frac{\langle m|\Delta H|n\rangle}{E_n - E_m}$$

Page 1

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CHALLINOR		1 / 2 / 3 / 4
Lecturer		**Section**
SKINNER		I / II

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Taking the λ^2 terms in (*) and taking inner product with $|n\rangle$ gives

$$\underbrace{\sum_{m\neq n} a_m^{(n)} \langle n|\Delta H|m\rangle}_{\displaystyle\sum_{m\neq n} \frac{|\langle m|\Delta H|n\rangle|^2}{E_n - E_m}} = E_n^{(2)}$$

[10] (ΔH Hermitian)

/10

Comments

Page 2

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CHALLINOR		1 / 2 / 3 / 4
Lecturer		Section
SKINNER		I / II

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Comments

[5]

(ii) The 1st-order energy shifts are $\lambda\langle n|\frac{\hbar\omega}{L^3}X^3|n\rangle$.

This will vanish since the eigenstates of the unperturbed Hamiltonian have even or odd parity, but X^3 is odd under parity.

[Argument based on $X\sim A+A^\dagger$ and so $X^3\sim(A+A^\dagger)^3$ never involves equal numbers of raising and lowering operators so $\langle n|X^3|n\rangle=0$ also fine.]

At second order we have $E_n=(n+\frac{1}{2})\hbar\omega+\lambda^2\sum_{m\neq n}\frac{|\langle m|\frac{\hbar\omega}{L^3}X^3|n\rangle|^2}{E_n-E_m}$.

For $n=0$ (ground state), we have to evaluate

$$\langle m|\frac{X^3}{L^3}|0\rangle \quad (m\neq 0).$$

Now, $A+A^\dagger=\frac{2m\omega}{\sqrt{2m\hbar\omega}}X=\sqrt{\frac{2m\omega}{\hbar}}X=\frac{X}{L}$, so need

$\langle m|(A+A^\dagger)(A+A^\dagger)(A+A^\dagger)|0\rangle=\langle m|(A^2+A^\dagger A+AA^\dagger+A^\dagger A^\dagger)A^\dagger|0\rangle$

since $A|0\rangle=0$. Using $A^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$ and $A|n\rangle=\sqrt{n}|n-1\rangle$,

we have $\sqrt{1}\sqrt{1}\sqrt{0}\langle m|0\rangle+\sqrt{1}\sqrt{1}\sqrt{1}\langle m|1\rangle+\sqrt{1}\sqrt{2}\sqrt{2}\langle m|1\rangle$

$+\sqrt{1}\sqrt{2}\sqrt{3}\langle m|3\rangle=(1+2)\delta_{m1}+\sqrt{6}\,\delta_{m3}$

Page

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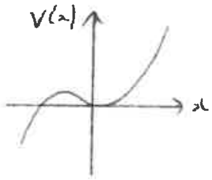
It follows that $E_0 = \frac{1}{2}\hbar\omega + (\hbar\omega)^2\lambda^2 \sum_{n\neq 0} \frac{(3\delta_{n1}+\sqrt{6}\delta_{n3})^2}{-n\hbar\omega}$

$$= \frac{1}{2}\hbar\omega - \hbar\omega\lambda^2\left(\frac{9}{1}+\frac{6}{3}\right)$$

[12] $$= \left(\frac{1}{2} - 11\lambda^2\right)\hbar\omega$$

For $\lambda \neq 0$, the potential tends to $-\infty$ at $x = \pm\infty$ (depending on sign of λ). For example, for $\lambda > 0$,



[3] Not all of the unperturbed states will transform to bound states and so do not expect perturbation theory to hold.

/20

Page 4

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Dynamics – Paper 2, Question 7, Part (i)

- Dynamical friction is the collective effect of a "sea" of light collisionless particles / stars on a massive object moving through that sea.

  If the massive object moves with velocity v that is larger than the velocity dispersion σ of the lighter particles, a wake forms behind the massive object, which exerts a backwards gravitational pull (2)

- A large angle deflection occurs for light particles with impact parameters $b < b_L$ where $GM/b_L \sim V^2$

  Assuming all such particles lose all their momentum to the massive object

  $$M\frac{dv}{dt} = \underbrace{\pi b_L^2}_{\text{interaction cross-section}} \times \underbrace{\rho V \cdot V}_{\text{momentum flux}}$$

  $$\propto \frac{M^2}{V^4}\rho V^2$$ (3)

  $$\therefore \underline{dV/dt \propto \rho M/v^2}$$

- Dynamical friction decelerates GCs orbiting in Galactic potential, causing them to sink deeper into the potential.

  Tidal effects lead to tidal stripping of stars from GCs and the formation of leading and trailing tidal tails (2)

- At a distance R_G from the centre of the Galaxy with mass M_G inside that distance, the tidal force at a distance r from the cluster is

  $$F_t \sim 2rGM_G/R_G^3 \quad *$$

  Tidal stripping occurs if this exceeds self gravity of cluster of mass M_c

  $$F_c \sim GM_c/r^2$$

  Equating these $\quad GM_c/r^2 = 2r\,GM_G/R_G^3$

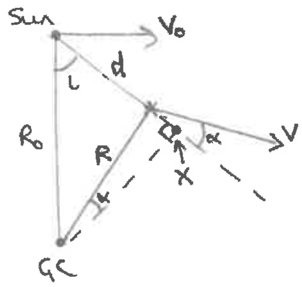
  $$\therefore M_c/r^3 \sim 2M_G/R_G^3$$ (3)

  $\therefore$ Mean density of cluster $\sim$ mean density of Galaxy inside orbit

* State or derive from acceleration to maintain circular orbit being $v^2/r = M/R^2$

tidal force at $R_G - r$ is $M_G/R_G^2 - M_G/(R_G-r)^2 = (M_G/R_G^2)[1-(1-r/R_G)^{-2}] \sim 2rM_G/R_G^3$

Dynamics – Paper 2, Question 7, Part (ii)

- 

2

$GC \to X$: $R_0 \sin \iota = R \cos\alpha$

$Sun \to X$: $R_0 \cos \iota = R \sin\alpha + d$

Radial component of V : $V_r = V\cos\alpha - V_0 \sin\iota$

$= V(R_0/R)\sin\iota - V_0 \sin\iota$ 2

$= (V/R - V_0/R_0) R_0 \sin\iota$

Tangential component of V : $V_t = V\sin\alpha - V_0\cos\iota$

$= V[(R_0/R)\cos\iota - d/R] - V_0\cos\iota$ 2

$= (V/R - V_0/R_0)R_0\cos\iota - (V/R)d$

- If $d/R_0 \ll 1 \Rightarrow R^2 = R_0^2 + d^2 - 2R_0 d\cos\iota$

$R = R_0[1 - 2(\frac{d}{R_0})\cos\iota + (\frac{d}{R_0})^2]^{1/2}$

$\simeq R_0 - d\cos\iota + O((d/R_0)^2)$

To first order $V/R = V_0/R_0 + (R-R_0)\frac{d}{dR}(V/R)\Big|_{R_0} + \ldots$ 3

$\therefore V/R - V_0/R_0 \simeq -d\cos\iota \frac{d}{dR}(V/R)\Big|_{R_0} = +d\cos\iota\left[\frac{V_0}{R_0^2} - \frac{1}{R_0}\frac{dV}{dR}\Big|_{R_0}\right]$

Thus $V_r \simeq +R_0 d\sin\iota\cos\iota\left[\frac{V_0}{R_0^2} - \frac{1}{R_0}\frac{dV}{dR}\Big|_{R_0}\right]$

$\simeq A d\sin 2\iota$ where $A = \frac{1}{2}\left[\frac{V_0}{R_0} - \frac{dV}{dR}\Big|_{R_0}\right]$ 2

And $V_t \simeq R_0 d\cos^2\iota\left[\frac{V_0}{R_0^2} - \frac{1}{R_0}\frac{dV}{dR}\Big|_{R_0}\right] - \left(\frac{V}{R}\right)d$

to 1^{st} order

$\simeq \frac{1}{2}d(1+\cos 2\iota)\left[\frac{V_0}{R_0} - \frac{dV}{dR}\Big|_{R_0}\right] - d\left[V_0/R_0 + d\cos\iota\left[\frac{V_0}{R_0^2} - \frac{1}{R_0}\frac{dV}{dR}\Big|_{R_0}\right]\right]$

$\simeq (A\cos 2\iota + B)d$ where $B = -\frac{1}{2}\left[\frac{V_0}{R_0} + \frac{dV}{dR}\Big|_{R_0}\right]$ 2

- Spherical $\Rightarrow V_c^2 = GM(<R)/R$

$M(<R) = \int_0^R 4\pi r^2 \rho_c (r_c/r)^\alpha \, dr$

$= \frac{4\pi\rho_c r_c^\alpha}{3-\alpha}\left[r^{3-\alpha}\right]_0^R$ 3

$\therefore V(R) = K R^{1-\alpha/2}$ where $K = \sqrt{\frac{4\pi G\rho_c r_c^\alpha}{3-\alpha}}$

$\therefore dV/dR = (1-\alpha/2)V/R$

$\therefore A = \frac{1}{2}\frac{V_0}{R_0}[1 - (1-\alpha/2)] = \frac{1}{4}\alpha V_0/R_0$

$B = -\frac{1}{2}\frac{V_0}{R_0}[1 + (1-\alpha/2)] = (\frac{1}{4}\alpha - 1)V_0/R_0$ 2

- At $\iota = 45°$, $V_r = Ad$ and $V_t = Bd \rightarrow V_r/V_t = A/B = -1$

$\therefore \frac{\alpha}{\alpha - 4} = -1$ 2

$\alpha = 2$

Topics – Paper 2, Question 8, Part (i)

- Stellar velocity dispersion is pumped up by scattering off discrete perturbers in a process known as relaxation

  There are a range of possible perturbers, including stars, spiral arms, molecular clouds. (3)

- $\sigma \propto t^{0.33} \Rightarrow t \propto \sigma^3$

  $\therefore \langle t_{HJ} \rangle / \langle t_{Nov} \rangle \triangleq (36/43)^3 \doteq 0.6$

  Observation explained if stars with HJs are younger than general field stars

  - Could be that formation of HJs was disfavoured at young Galactic ages due to lower metallicity
  - Or HJs form throughout all Galactic ages, but system evolution destroys HJs after Gyr
  - Or encounters that pump up velocity dispersion help to destroy HJs (7)

Topics - Paper 2, Question 8, Part (ii)

- If pericentre close to BH, $e \simeq 1 \rightarrow a = \frac{1}{2} \times \text{apocentre} = 0.1\,\text{pc}$

  $\therefore$ orbital period $= \sqrt{(0.1 \times \text{au/pc})^3 / 10^3} = 9.4 \times 10^4$ yr

  $\therefore$ over MS there are $4.5 \times 10^9 / 9.4 \times 10^4 = 4.8 \times 10^4$ pericentre passages (3)

- Total energy deposited is $4.8 \times 10^4\, G M_1^2 R_2^5 / a_{peri}^6$

  Internal gravitational binding energy $\frac{3}{5} G M_2^2 / R_2$ (where 3/5 is order unity factor)

  For these to be equal $(a_{peri}/R_2)^6 = \frac{5}{3} \times 4.8 \times 10^4 \,(M_1/M_2)^2$

  $\therefore$ $a_{peri} \simeq 66 R_\odot = 4.6 \times 10^{10}\,\text{m} = 0.3\,\text{au} = 1.5 \times 10^{-6}\,\text{pc}$ (6)

- $\dot{E}_{tidal} \simeq \left(\frac{3}{5} G M_2^2 / R_2\right) / (4.5 \times 10^9 \times 3.156 \times 10^7)$

  $= 1.6 \times 10^{24}\,\text{W} = 0.004\,L_\odot$

  This is negligible compared with energy generated by nuclear fusion and radiated by star $\rightarrow$ no effect on evolution. (3)

- $V_{peri} \simeq \sqrt{2G \times 10^3 M_\odot / a_{peri}} = 2400\,\text{km/s} = 8 \times 10^{-3}\,c$

  $\therefore$ no strong relativistic effect (2)

- Hill sphere of star is $\sim a_{peri} \left(M_\odot / 3 \times 10^3 M_\odot\right)^{1/3} \simeq 0.02\,\text{au}$

  So planets have to be hotter than $\sim 278/\sqrt{0.02} \simeq 2000\,\text{K}$

  $\therefore$ no habitable planets expected (2)

- $R_{rg} \simeq 1\,\text{au} > a_{peri} \rightarrow$ star would be destroyed in one pericentre passage although in practice this would happen progressively as star expanded, with outer envelope being stripped off and accreted onto BH (2)

- Observational signature of tidal disruption event - X-ray emission from accretion onto BH. (2)