UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 4 June 2018 13.30pm – 16.30pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 2X and 4X** *should be in one bundle and* **5Z and 6Z** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) Two particles, labelled A and B and with masses m_A and m_B, undergo a collision. In some inertial reference frame the components of their 4-momenta before the collision are p_A^μ and p_B^μ, respectively. Consider the quantity

$$s \equiv \eta_{\mu\nu}(p_A^\mu + p_B^\mu)(p_A^\nu + p_B^\nu)\,,$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

Explain why s is Lorentz invariant and, by considering its value in the zero-momentum frame, give a physical interpretation of $\sqrt{s}$.

If the energy of A in the zero-momentum frame is $E_{A,\mathrm{ZMF}}$, show that

$$E_{A,\mathrm{ZMF}} = \frac{sc^2 + m_A^2c^4 - m_B^2c^4}{2c\sqrt{s}}\,.$$

Hence, or otherwise, find the energy $E_{A,\mathrm{ZMF}}$ if the collision is inelastic and results in a single particle of mass M.

(ii) A particle moves along the x-axis of some inertial frame S with constant proper acceleration α. At $t = 0$, the particle is at rest at $x = c^2/\alpha$. Show that the worldline of the particle can be expressed as

$$ct(\tau) = \frac{c^2}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right)\,, \qquad x(\tau) = \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right)\,,$$

where τ is the particle's proper time synchronised such that $\tau = 0$ when $t = 0$.

A second particle performs a similar motion with proper acceleration α_2 with $\alpha_2 < \alpha$. This particle is also at rest in S at $t = \tau_2 = 0$ and has $x = c^2/\alpha_2$ at that time. We denote by S'_* the instantaneous rest frame of the first particle at the time of some event C on its worldline corresponding to $\tau = \tau_*$. Find the proper time τ_2 of the event D on the worldline of the second particle that is simultaneous in S'_* with event C.

Hence show that at event D the second particle is at rest in S'_* and is displaced from event C along the x'-axis of this frame by a constant distance that you should specify.

Question 2X - Astrophysical Fluid Dynamics

(i) Consider a spherically symmetric galaxy cluster composed of dark matter and intracluster gas, whose density distribution may be assumed uniform. The intracluster gas can be treated as fully ionized hydrogen. Assume that the system is in virial equilibrium, with a virial mass $M_{\rm vir} = 10^{15}\,{\rm M}_\odot$ and virial radius $R_{\rm vir} = 1\,{\rm Mpc}$. Further assume that the mean free path is set by Coulomb collisions so that it can be expressed as

$$\lambda = 20\,{\rm kpc}\left(\frac{T}{10^8\,{\rm K}}\right)^2\left(\frac{n}{10^{-3}\,{\rm cm}^{-3}}\right)^{-1},$$

where T is the gas temperature and n is gas number density. Derive the length-scale below which the gas is collisionless. Assume a baryon fraction of $f_{\rm B} = 0.17$.

Justify why intracluster gas can be treated as a fluid.

(ii) Consider an adiabatic shock where the gas is moving along the x-axis. At a given moment in time the shock discontinuity is located at $x = 0$ such that for $x < 0$ we have pre-shock fluid. Assume that the pre-shock fluid is threaded by a uniform $\mathbf{B}$ field and that the ideal MHD approximation applies. Further assume that the energy flux is given by

$$[\rho(e + \frac{1}{2}u^2) + p]\mathbf{u} + \frac{1}{\mu_0}\mathbf{B}\times(\mathbf{u}\times\mathbf{B}),$$

where ρ is the gas density, ρe is the internal energy per unit volume, $\mathbf{u}$ is the gas velocity, p is the gas pressure and μ_0 is the vacuum magnetic permeability. Derive shock jump conditions analogous to the three Rankine-Hugoniot relations if,

a) $\mathbf{B} = (B, 0, 0)$,

b) $\mathbf{B} = (0, B, 0)$.

Discuss the physical meaning of the jump condition related to the momentum equation in these two cases.

Determine the jump in the magnitude of $\mathbf{B}$ across the shock front in each case. Express your answer in terms of the pre- and post-shock fluid speeds.

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) Use the relativistic Doppler formula

$$\nu_0 = \nu_\mathrm{e} \sqrt{\frac{1 - v/c}{1 + v/c}}\,,$$

where ν_e and ν_0 are the photon emission and reception frequencies respectively, to derive the kinematic redshift of a light source moving with velocity v relative to the observer in the limit $v \ll c$.

Explain the difference between kinematic and cosmological redshifts.

When observing a distant galaxy, we measure a combination of kinematic and cosmological redshifts, as galaxies respond to the local gravitational field. Show that the combined redshift is $z_\mathrm{tot} = (1 + z_\mathrm{cosm})(1 + z_\mathrm{kin}) - 1$.

The typical rms velocity of galaxies relative to the uniform expansion is $\sqrt{\langle v_\mathrm{p}^2 \rangle} \approx 600\,\mathrm{km\,s^{-1}}$. For a present-day Hubble parameter $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, what is the minimum distance at which a galaxy must be for its redshift to give an estimate of its true distance accurate to better than 5%?

(ii) State the Friedman equations for a universe with pressure-less matter density ρ_mat and a cosmological constant $\Lambda \neq 0$.

Show that the Friedman equations for such a universe are equivalent to those for a universe with $\Lambda = 0$, $\rho = \rho_\mathrm{mat} + \rho_\mathrm{vac}$ and $p = p_\mathrm{vac} = -\rho_\mathrm{vac} c^2$, where ρ_vac and p_vac are the vacuum energy density and pressure.

Show that a positive Λ term can be interpreted as a repulsive force whose strength is proportional to distance.

Show that in such universes with positive Λ and zero or negative curvature the scale factor evolves as $a(t) \propto \exp(\alpha\, t)$ at late times and determine α.

Show that for such a universe with positive curvature there exists a static solution with $\Lambda = \Lambda_\mathrm{c}$ and calculate Λ_c.

Sketch the evolution with time of the scale factor $a(t)$ for a universe with $\Lambda = \Lambda_\mathrm{c} \pm \epsilon$, where $\epsilon \ll \Lambda_\mathrm{c}$.

Comment on the age of such a universe.

Question 4X - Structure and Evolution of Stars

(i) At the radius of the Earth's orbit, the Solar wind from the Sun produces a particle flux of $\mathcal{F}_{\rm w} \approx 6 \times 10^{12}\,{\rm m^{-2}s^{-1}}$. Derive an approximate formula for the ratio of the mass lost due to nuclear reactions in the core to that lost in the Solar wind in terms of the solar luminosity $L_\odot$, the solar wind flux $\mathcal{F}_w$, the radius of the Earth's orbit $R_{\rm AU}$ and relevant physical constants. Clearly state any additional assumptions necessary to derive this approximate formula.

Estimate the numerical value of the ratio of the mass lost due to nuclear reactions in the core to that lost in the Solar wind.

(ii) A star in hydrostatic equilibrium has a polytropic equation of state with the pressure P and density ρ related by $P = K\rho^{(n+1)/n}$, where K and n are constants. Show that the ratio of the change in P/ρ is related to the change in P by

$$d\left(\frac{P}{\rho}\right) = \frac{1}{(n+1)}\frac{dP}{\rho}.$$

Show that the gravitational potential energy of the star is

$$\Omega = -\frac{3}{(5-n)}\frac{GM^2}{R}.$$

Describe the behaviour of the polytrope as $n \to 5$ from below.

TURN OVER...

Question 5Z - Statistical Physics

(i) A macroscopic system has volume V and contains N particles. Let $\Omega(E, V, N; \delta E)$ denote the number of states of a system which have energy in the range $(E, E + \delta E)$, where $\delta E \ll E$. Define the entropy of the system and explain why the dependence of S on δE is usually negligible.

Define the temperature and pressure of the system, and hence obtain the fundamental thermodynamic relation.

(ii) A one-dimensional model of rubber consists of a chain of N links, each of length a. The chain lies along the x-axis with one end fixed at $x = 0$ and the other at $x = L$ where $L < Na$. The chain can "fold back" on itself so x may not increase monotonically along the chain. Let $N_\rightarrow$ and $N_\leftarrow$ denote the number of links along which x increases and decreases, respectively. All links have the same energy.

Show that $N_\rightarrow$ and $N_\leftarrow$ are uniquely determined by L and N.

Determine $\Omega(L, N)$, the number of different arrangements of the chain, as a function of $N_\rightarrow$ and $N_\leftarrow$.

Hence, show that if $N_\rightarrow \gg 1$ and $N_\leftarrow \gg 1$ then the entropy of the chain is

$$S(L, N) =$$

$$k_B N \left[\log 2 - \frac{1}{2}\left(1 + \frac{L}{Na}\right) \log\left(1 + \frac{L}{Na}\right) - \frac{1}{2}\left(1 - \frac{L}{Na}\right) \log\left(1 - \frac{L}{Na}\right) \right],$$

where k_B is Boltzmann's constant.

Let f denote the force required to hold the end of the chain fixed at $x = L$. This force does work $f dL$ on the chain if the length is increased by dL. Write down the fundamental thermodynamic relation for this system and hence calculate f as a function of L and the temperature T.

Assume that $Na \gg L$. Show that the chain satisfies Hooke's law $f \propto L$.

What happens if f is held constant and T is increased?

[*You may use Stirling's approximation:* $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$ *for* $n \gg 1$]

Question 6Z - Principles of Quantum Mechanics

(i) A one dimensional harmonic oscillator has Hamiltonian

$$H = \hbar\omega \left(A^\dagger A + \frac{1}{2} \right),$$

where $[A, A^\dagger] = 1$. Show that $A|n\rangle = \sqrt{n}|n-1\rangle$, where $H|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$.

(ii) The oscillator in Part (i) is perturbed by adding a new term λX^4 to the Hamiltonian. Given that

$$A = \frac{m\omega X - \mathrm{i}P}{\sqrt{2m\hbar\omega}},$$

where X and P are the position and momentum operators respectively, show that the ground state of the perturbed system is

$$|0_\lambda\rangle = |0\rangle - \frac{\hbar\lambda}{4m^2\omega^3} \left(3\sqrt{2}\,|2\rangle + \sqrt{\frac{3}{2}}\,|4\rangle \right),$$

to first order in λ.

[*You may use the fact that, in non–degenerate perturbation theory, a perturbation Δ causes the first–order shift*

$$|m^{(1)}\rangle = \sum_{n \neq m} \frac{\langle n|\Delta|m\rangle}{E_m - E_n}\,|n\rangle$$

in the m^{th} energy level.]

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Show that in a spherical potential $\Phi(r)$, where r is the radial distance from the centre, orbits can be described by

$$\frac{\mathrm{d}u^2}{\mathrm{d}\phi^2} + u = f(u),$$

for a suitable choice of coordinates (u, ϕ).

For which spherically symmetric potential $\Phi(r)$ is $r = a \exp(b\,\phi)$ a possible trajectory?

(ii) The density distribution of a spherical stellar system is given by

$$\rho(r) = \left(\frac{3M}{4\pi\, b^3}\right)\left(1 + \frac{r^2}{b^2}\right)^{-5/2},$$

where r is the distance to the centre and $M$and b are constants. Calculate the gravitational potential $\Phi(r)$.

Use the epicyclic approximation to calculate the rate $\Omega_\mathrm{P}(r)$ at radial distance r, at which nearly circular orbits in this potential are precessing in terms of the parameters M and b.

Are such orbits stable in this potential?

Question 8Y - Physics of Astrophysics

(i) Explain the orbital geometry of the Earth-Sun system on December 21, the winter solstice in the northern hemisphere.

Explain why the Earth's 'solar day' (time interval between successive points when the Sun is highest in the sky at a fixed point on the Earth's surface) is different from the Earth's 'stellar day' (the period of the Earth's spin on its own axis).

What is the sign of this difference and what does this imply for the rotation vectors in the Earth-Sun system?

The 24 hour day is based on the *mean* solar day averaged over the year. Suggest a reason why the solar day might vary around the Earth's orbit.

It is well known that, in the northern hemisphere, although the shortest day of the year is 21st December, the time of sunrise continues to be later each day for a few days afterwards. Suggest a reason for this and explain what can be deduced from this observation about the geometry of the Earth's orbit around the Sun.

(ii) The asteroid Uomuamua is a striking needle shaped object (dimensions $\sim$ 100 m $\times$ 100 m $\times$ 1 km) which recently passed within 0.25 au of the Sun. It is unique among asteroids in the solar system in that its orbit is highly inclined with respect to the ecliptic and that its orbit is unbound with respect to the Sun ($e \sim 1.2$). These latter properties have encouraged the view that Uomuamua is a interloper that originated outside the solar system.

On the assumption that this is the only extrasolar asteroid to have been observed after around a century of observation, estimate the density of free-floating asteroids in the interstellar medium that is required to explain the incidence of such events.

The local stellar density is a few stars per cubic parsec. Estimate the mass of asteroids (in Earth masses) that must be ejected into the interstellar medium in order to supply this density.

Comment on whether you consider this mass to be reasonable.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 5 June 2018 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 2

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Question 1X - Relativity

(i) A particle of mass m and charge q moves in Minkowski spacetime under the influence of an electromagnetic field. The equation of motion of the particle is,

$$m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu, \qquad \mu,\nu = (0,1,2,3) \qquad (\dagger)$$

where the components of the field-strength tensor are related to the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

Show that ($\dagger$) reduces to the expected Lorentz force law.

(ii) Explain briefly the physical interpretation of the energy–momentum tensor $T^{\mu\nu}$.

In a general spacetime, the energy–momentum tensor for the electromagnetic field has components

$$T^{\mu\nu}_{\text{em}} = -\frac{1}{\mu_0}\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right),$$

where μ_0 is the permeability of free space. Working in local inertial coordinates, show that

$$T^{00}_{\text{em}} = \frac{1}{2\mu_0}\left(|\mathbf{B}|^2 + \frac{|\mathbf{E}|^2}{c^2}\right)$$

and interpret this result physically.

Using Maxwell's equations

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu\,,$$
$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0\,,$$

where j^ν is the current 4-vector, show that

$$\nabla_\mu T^{\mu\nu}_{\text{em}} = -F^{\nu}{}_{\rho}j^\rho\,.$$

Explain the physical significance of the fact that $\nabla_\mu T^{\mu\nu}_{\text{em}} \neq 0$ in the presence of a non-zero current 4-vector.

Question 2X - Astrophysical Fluid Dynamics

(i) Explain what is meant by a viscous flow.

Starting from the Navier-Stokes equation and assuming that the shear viscosity η and mass density ρ are constant, the bulk viscosity is negligible, and the equation of state is barotropic, show that

$$\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{w}) + \frac{\eta}{\rho} \nabla^2 \mathbf{w} \, ,$$

where $\mathbf{w}$ is the fluid vorticity and $\mathbf{u}$ is the velocity.

Provide the physical interpretation of this equation when a) $\eta = 0$, and b) $\eta > 0$.

(ii) Consider a viscous flow in a horizontal pipe along the x-axis with a constant elliptical cross-section. Assume a no-slip boundary condition at the walls of the pipe, i.e. $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$, where a and b are the semi-axes of the elliptical cross-section. Further, assume that the length of the pipe ℓ greatly exceeds either of these semi-axes. Using the incompressible Navier-Stokes equation in Cartesian coordinates find a stationary solution for the velocity $\mathbf{v}(x, y, z)$ of the fluid flow in terms of a, b, η, ℓ, and the pressure difference between the ends of the pipe Δp.

By making the substitution $y = ar\cos\theta$ and $z = br\sin\theta$, calculate the total mass flow rate Q of a fluid of density ρ through this pipe.

For a given pressure gradient along the pipe, how does the maximum fluid velocity and mass flow rate compare to an analogous pipe with a circular cross-section with radius $R = a$?

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) The energy density of the Cosmic Microwave Background (CMB) is described by the blackbody distribution,

$$\rho_\gamma(\nu, T_\gamma)\, c^2\, \mathrm{d}\nu = \frac{8\pi h}{c^3} \frac{\nu^3\, \mathrm{d}\nu}{\left[\exp\left(\frac{h\nu}{k_\mathrm{B} T_\gamma}\right) - 1\right]} ,$$

where ν is the photon frequency, T_γ is the photon temperature of the CMB and k_B is the Boltzmann constant.

Assume a CMB temperature of $T_\gamma = 2.73\,\mathrm{K}$. Calculate the present day number density of CMB photons.

Use the Friedmann equations to calculate the energy density of a spatially flat universe with zero cosmolocical constant in terms of the Hubble parameter.

Assume a present-day Hubble parameter $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Show that the critical density corresponds to approximately 5 hydrogen atoms per cubic metre.

[You may find the following relation useful:

$$\frac{1}{(m-1)!} \int_0^\infty x^m \left[e^x - 1\right]^{-1} \frac{\mathrm{d}x}{x} = \sum_{n=1}^{\infty} n^{-m} \quad]$$

(ii) The energy density of the universe in radiation and relativistic particles can be written as $\rho c^2 = (g^*(T)/2)\, a\, T^4$, where $g^*(T)$ is the effective statistical weight at temperature T and a is the radiation density constant. Assume that the universe is as described in Part (i). Assume further that the universe also contains massless neutrinos that contribute with $g_{*,\nu} = \mathcal{N}_\nu\, (7/8)\, (g_\nu/2)$ to the effective statistical weight, where $\mathcal{N}_\nu$ is the number of neutrino species and g_ν is the statistical weight of the massless neutrinos. Using the fact that neutrinos decouple before electron-positron annihilation, calculate the present-day temperature $T_{\nu,0}$ of the neutrinos.

Assume that primordial black holes form when the universe has temperature $k_\mathrm{B} T = 200\,\mathrm{GeV}$ and $g^* \approx 100$. Calculate the redshift z_pbh when this happens.

Black holes lose mass by emitting Hawking radiation from the horizon surface with radius $r_\mathrm{h} = 2GM/c^2$. The Hawking radiation of a black hole with mass M can be approximated as a black-body with temperature

$$T_{\rm H} = \frac{\hbar\, c^3}{8\pi\, GM\, k_{\rm B}},$$

where $k_{\rm B}$ is Boltzmann's constant and $\hbar = h/2\pi$ is the reduced Planck constant. Estimate the minimum mass $m_{\rm min}$ of a primordial black hole formed in the early universe, if it is to survive to the present day, $t_0 \simeq 13.7 \times 10^9\,$yr.

Use the Friedmann equations to calculate the age of the universe at redshift $z_{\rm pbh}$ and hence show that at $z_{\rm pbh}$ the mass contained within the Hubble radius, ct, is larger than $m_{\rm min}$.

Calculate the fraction of the energy density of the universe that would have to collapse into primordial black holes at this redshift for them to account for the present-day matter density and comment on your result.

TURN OVER...

Question 4X - Structure and Evolution of Stars

(i) A team of astronomers have access to some preliminary data from the Gaia satellite and can measure the proper motions of stars with magnitude $m_V = 20$ to an accuracy of 4×10^{-5} arcseconds per year. The astronomers have identified a sample of horizontal branch stars, with luminosities of $100\,\mathrm{L}_\odot$ and distances of 100 kpc in the direction of the Galaxy's north pole. The stars are moving independently within the gravitational field of the Galaxy and the astronomers wish to test their model for the Galaxy in which the mass interior to 100 kpc is $5 \times 10^{11}\mathrm{M}_\odot$. Demonstrate whether or not Gaia measurements of the horizontal branch stars will allow the astronomers to verify their prediction for the mass of the Galaxy, presenting the calculations needed to support your conclusion.

(ii) A binary system consists of two stars with masses M_1 and M_2 in a circular orbit with separation a. Suppose that one of the stars expands to fill its Roche lobe, allowing mass transfer to occur. Further assume that the orbital angular momentum is conserved and the total mass of the system is constant during mass transfer. Determine the behaviour of the orbital separation on the mass ratio M_1/M_2 as mass transfer proceeds.

Comment on the implications for the evolution of mass transfer binary systems.

How does your answer relate to the properties of the bright eclipsing binary star Algol? Algol is a binary system with a period of 69 hours consisting of a $3.7\,\mathrm{M}_\odot$ primary star on the main sequence and a $0.8\,\mathrm{M}_\odot$ secondary star at the bottom of the red-giant branch.

Question 5Z - Statistical Physics

(i) Starting from the canonical ensemble, derive the Maxwell-Boltzmann distribution for the velocities of particles in a classical gas of atoms of mass m.

Derive the distribution of speeds v of the particles.

Calculate the most probable speed.

(ii) A certain atom emits photons with frequency ω_0. A gas of these atoms is contained in a box. A small hole is cut in a wall of the box so that photons can escape in the $+x$ direction where they are received by a detector. The frequency of the photons received is Doppler shifted according to the formula

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right),$$

where v_x is the x-component of the velocity of the atom that emits the photon and c is the speed of light. Let T be the temperature of the gas.

Calculate the mean value $\langle\omega\rangle$ of ω.

Calculate the standard deviation $\sqrt{\langle(\omega - \langle\omega\rangle)^2\rangle}$.

Show that the relative number of photons received with frequency between ω and $\omega + d\omega$ is $I(\omega)d\omega$ with

$$I(\omega) \propto \exp(-a(\omega - \omega_0)^2),$$

where a is a constant.

Calculate the coefficient a.

Hence explain how observations of the radiation emitted by the gas can be used to measure its temperature.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Explain what is meant by *parity* and what is the condition under which parity is a symmetry of a quantum system.

Explain what is meant by the *intrinsic parity* of a particle.

How is the total parity of a system of particles related to the intrinsic parities of the particles.

(ii) In each of the decay processes below, parity is conserved. A deuteron (d^+) has intrinsic parity $\eta_d = +1$ and spin $s = 1$. The ground state of a hydrogenic 'atom' formed from a deuteron and a negatively charged pion (π^-) of spin $s = 0$ decays to two identical neutrons (n), each of spin $s = \frac{1}{2}$ and parity $\eta_n = 1$. Deduce the intrinsic parity of the pion.

The Δ^- particle has spin $s = \frac{3}{2}$ and decays as

$$\Delta^- \;\to\; \pi^- + n\,.$$

What are the allowed values of the orbital angular momentum?

In the centre of mass frame, the vector $\mathbf{r}_\pi - \mathbf{r}_n$ joining the pion to the neutron makes an angle θ to the $\hat{\mathbf{z}}$-axis. The final state is an eigenstate of J_z and the spatial probability distribution is proportional to $\cos^2\theta$. Deduce the intrinsic parity of the Δ^-.

[The first three Legendre polynomials are given by

$$P_0(x) = 1\ , \quad P_1(x) = x\ , \quad P_2(x) = \frac{1}{2}(3x^2 - 1)\ . \ ]$$

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) A sun-like star is moving on an orbit with tangential velocity $v_{\rm t}$ and radial velocity $v_{\rm r}$ at distance r_0 relative to a black hole with mass $M = 10^6 M_\odot$.

Calculate the minimum tangential velocity $v_{\rm t,0} = v_{\rm t}(r = r_0)$ for which the star can avoid tidal disruption by the black hole.

Estimate the value of $v_{\rm t,0}^{\rm min}$ if $r_0 = 1$pc and the radial velocity of the star away from the black hole is $v_{r,0} = 50$ km/s.

Assume that the black hole is located at the centre of a galaxy. Is the star likely to be disrupted?

(ii) A black hole of mass M is embedded in the centre of an infinite, homogeneous, three-dimensional sea of test particles. Far from the black hole the test particles have a Maxwellian velocity distribution,

$$f_0(v) = \frac{n_0}{(2\pi\,\sigma^2)^{3/2}} \exp\left[-\frac{1}{2}v^2/\sigma^2\right].$$

Show that the fraction of test particles that are not bound to the black hole is

$$n(r)/n_0 = 2\sqrt{r_{\rm h}/(\pi r)} + \exp\left[r_{\rm h}/r\right]\left[1 - {\rm erf}\left(\sqrt{r_{\rm h}/r}\right)\right],$$

where $r_{\rm h} \equiv GM/\sigma^2$.

Discuss the asymptotic behaviour of $n(r)$ close to the black hole.

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) A binary star system with a circular orbit and separation d consists of equal mass stars, each of mass m_*. The binary passes on a parabolic trajectory close to a black hole of mass $M_{\rm bh} >> m_*$. The minimum distance r to the black hole is just small enough for the binary star to be tidally disrupted.

Show that $r \sim (M_{\rm bh}/m_*)^{1/3}\, d$ and that one of the stars ends up in a bound orbit around the black hole with semi-major axis $a \sim (M_{\rm bh}/m_*)^{1/3}\, r$.

Show that the ratio of the period of the bound star in its orbit around the black hole to the original orbital period of the binary is $\sim (M_{\rm bh}/M_*)^{1/2}$.

[You may set factors of order unity to one.]

(ii) Massive stars drive strong stellar winds whose line emission has a characteristic P-Cygni profile with strong emission in the red wing of the line and absorption in the blue wing. Explain why this is the case.

The Galactic Centre contains a population of massive stars known as S stars which are believed to have become bound to the central supermassive black hole of mass $\sim\ 10^6 M_\odot$ by the process outlined in Part (i) above. Assume that one of these S stars was originally in a circular binary containing two stars of mass $10 M_\odot$ separated by 1 au. Assume further that both stars drive a spherical wind with terminal velocity 700 km s^{-1}.

Sketch the variation of the profile of the line emission from this binary in the CIV 1550 A line for different orbital phases of the binary orbit, providing an indication of the wavelength scale.

Estimate the radial velocity of the component of the binary captured by the black hole and determine whether the star's orbital motion should be readily detectable given the width of the P-Cygni feature.

Given that the Galactic Centre is 8.5 kpc from the Earth, what is the required accuracy for locating the position of the star on the sky in order to detect its orbital motion about the black hole on a reasonable timescale?

When the S star ultimately explodes as a supernova, asymmetry in the explosion provides a kick of ~ 500 km s^{-1} to the remnant neutron star. Discuss whether you would expect pulsars to be detected in the region occupied by the S stars.

[You may use any of the formulae stated in Part i) without proof.]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 7 June 2018 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 2X and 4X** *should be in one bundle and* **5Z and 6Z** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) The Schwarzschild line element for the vacuum spacetime outside a spherical body of mass M centred on $r = 0$ is

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) ,$$

where $\mu = GM/c^2$. Show that the geodesic equations for a massless particle moving with non-zero orbital angular momentum in the plane $\theta = \pi/2$ can be written in the form

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = \frac{1}{bc} ,$$
$$r^2 \dot{\phi} = 1 ,$$
$$\dot{r}^2 + \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{1}{b^2} ,$$

where b is a constant and overdots denote differentiation with respect to a suitably-chosen affine parameter.

By considering the motion as $r \to \infty$, or otherwise, give a physical interpretation of the constant b.

(ii) A photon is emitted outwards at coordinate radius $2\mu < r_{\text{em}} < 3\mu$ in the Schwarzschild spacetime of Part (i). By considering the effective potential for the radial motion, or otherwise, show that the photon will escape to infinity if $b < \sqrt{27}\mu$.

If a stationary observer at constant radius r_{em} measures the photon to be emitted at an angle α (with $0 < \alpha < \pi/2$) to the outward radial direction, show that

$$\sin\alpha = b\,[U(r_{\text{em}})]^{1/2} ,$$

where $U(r) = r^{-2}(1 - 2\mu/r)$.

Hence show that the photon will escape to infinity if

$$\sin\alpha < \sqrt{27}\mu\,[U(r_{\text{em}})]^{1/2} .$$

Comment on the limiting values of α as $r_{\text{em}} \to 2\mu$ and $r_{\text{em}} \to 3\mu$.

Question 2X - Astrophysical Fluid Dynamics

(i) An incompressible fluid is uniformly rotating as a whole. Starting from the standard equation of motion in a rotating frame, show that in the rotating frame the momentum equation becomes,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p_{\text{eff}} + 2\mathbf{v} \times \mathbf{\Omega}\,, \quad (*)$$

where $p_{\text{eff}} = p - \frac{1}{2}\rho(\mathbf{\Omega} \times \mathbf{r})^2$, $\mathbf{v}$ is the fluid velocity, ρ is the fluid density, p the pressure, and $\mathbf{\Omega}$ is the angular velocity which you can assume to be along the z-axis.

Consider now a steady motion in this rotating fluid where l and u are the characteristic length and velocity of the problem, respectively. In the limit of rapidly rotating fluid derive the equation of steady motion and show that

$$\frac{\partial v_{\text{x}}}{\partial x} + \frac{\partial v_{\text{y}}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial v_{\text{z}}}{\partial z} = 0.$$

Provide a physical description of this dynamics.

(ii) Consider an incompressible fluid uniformly rotating as a whole with angular velocity $\mathbf{\Omega}$ with the direction along the z-axis. Using (*) from Part (i), or otherwise, show that for a small pressure perturbation in this fluid,

$$\frac{\partial \nabla \times \mathbf{v}}{\partial t} = 2\Omega \frac{\partial \mathbf{v}}{\partial z},$$

$$\frac{\partial^2}{\partial t^2}\nabla^2 p' + 4\Omega^2 \frac{\partial^2 p'}{\partial z^2} = 0\,,$$

where p' is the pressure perturbation and $\mathbf{v}$ is the velocity perturbation.

Derive the dispersion relation for this system.

Show that the resulting waves are transverse and circularly polarised.

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) Before recombination at $z \simeq 1100$, the main reaction responsible for maintaining hydrogen and radiation in equilibrium is $\mathrm{p} + \mathrm{e}^- \rightleftharpoons \mathrm{H} + \gamma$. Assume that the particle densities are given by the non-relativistic Maxwell-Boltzmann formula

$$n_i = A\, g_i\, m_i^{3/2}\, T^{3/2}\, \exp\left[(\mu_i - m_i c^2)/(k_\mathrm{B} T)\right],$$

where A is a constant, k_B is Boltzmann's constant, T is the temperature and g_i, μ_i and m_i are the number of spin states, the chemical potential and the mass of particle species i, respectively.

Show that

$$n_\mathrm{e} n_\mathrm{p}/n_\mathrm{H} \approx A\, m_\mathrm{e}^{3/2}\, T^{3/2}\, \exp\left[-B_\mathrm{H}/(k_\mathrm{B} T)\right],$$

where $B_\mathrm{H}/k_\mathrm{B} = 158\,000\mathrm{K}$ is the binding energy of the hydrogen atom.

Show that for the ionization fraction $X = n_\mathrm{e}/(n_\mathrm{e} + n_\mathrm{H})$,

$$X^2/(1-X) \approx \left[(1-Y)\, \rho_\mathrm{bar}/m_\mathrm{p}\right]^{-1}\, A\, m_\mathrm{e}^{3/2}\, T^{3/2}\, \exp\left[-B_\mathrm{H}/(k_\mathrm{B} T)\right],$$

where Y is the helium mass fraction and ρ_bar is the mean baryon mass density.

Why does hydrogen recombination in the early universe occur at significantly lower temperature than B_H/k_B?

(ii) The optical depth to Thomson scattering along a line of sight to redshift z is

$$\tau(z) = \int_{t(z)}^{t_0} \sigma_T\, n_\mathrm{e}\, c\, \mathrm{d}t,$$

where t_0 is the time at the present-day, n_e is the mean density of free electrons, and σ_T is the Thomson scattering cross-section. Assume the universe to be Einstein-de Sitter with matter density $\Omega_\mathrm{mat} = 1$ and zero cosmological constant. Show that if the intergalactic medium is fully reionised at redshift z and remains fully ionised to the present-day,

$$\tau(z) \approx \frac{H_0 \sigma_T c}{4\pi G m_\mathrm{p}} \left(1 - \frac{Y}{2}\right) \Omega_{\mathrm{bar},0} \left[(1+z)^{3/2} - 1\right],$$

where $\Omega_{\mathrm{bar},0}$ is the present-day baryon density parameter, H_0 is the present-day value of the Hubble parameter, and Y is the helium mass fraction.

Estimate the redshift of reionisation required to produce an optical depth $\tau = 0.1$. Discuss the likely sources of the ionising photons if reionisation occurred at this redshift? Explain the reasoning behind your answer.

Briefly describe the observational evidence that the universe is almost completely ionised by $z = 6$ and has remained so to the present day.

TURN OVER...

Question 4X - Structure and Evolution of Stars

(i) An analysis of the spectrum and light curve of an eclipsing spectroscopic binary reveals a period of 8.6 years. The maximum Doppler shift of the Hydrogen Balmer Hα line (6562.8 Å) is 0.35 Å for star A and 0.068 Å for star B. Estimate the mass ratio of the two stars.

Assume that the orbital axis is perpendicular to the line of sight to the stars. Estimate the orbital velocity for star A, its orbital radius in astronomical units and the masses of the stars in units of Solar mass.

(ii) Write an essay describing the evolution of stars, starting at the end of the main sequence through to the creation of white dwarfs, neutron stars and black holes. Include estimates of the mass of stars on the main sequence that eventually produce each type of remnant.

What observational evidence is there for the existence of each type of remnant?

Question 5Z - Statistical Physics

(i) A system of non-interacting bosons has single particle states $|i\rangle$ with energies $\epsilon_i \geq 0$. Show that the grand canonical partition function is

$$\log \mathcal{Z} = -\sum_i \log\left(1 - e^{-\beta(\epsilon_i - \mu)}\right),$$

where $\beta = 1/(k_{\mathrm{B}}T)$, k_{B} is Boltzmann's constant, and μ is the chemical potential.

What is the maximum possible value for μ?

(ii) A system of $N \gg 1$ bosons has one energy level with zero energy and $M \gg 1$ energy levels with energy $\epsilon > 0$. The number of particles with energy 0 is N_0 and with energy ϵ is N_ϵ.

Write down expressions for $\langle N_0 \rangle$ and $\langle N_\epsilon \rangle$ in terms of μ.

At temperature T what is the maximum possible number $N_\epsilon^{\max}$ of bosons in the state with energy ϵ?

What happens for $N > N_\epsilon^{\max}$?

Calculate the temperature T_{B} at which Bose condensation occurs.

For $T > T_{\mathrm{B}}$, show that $\mu = \epsilon\,(T_{\mathrm{B}} - T)/T_{\mathrm{B}}$.

For $T < T_{\mathrm{B}}$ show that

$$\mu \approx -\frac{k_{\mathrm{B}}T}{N}\frac{e^{\epsilon/(k_{\mathrm{B}}T)} - 1}{e^{\epsilon/(k_{\mathrm{B}}T)} - e^{\epsilon/(k_{\mathrm{B}}T_{\mathrm{B}})}}.$$

Calculate the mean energy $\langle E \rangle$ for $T > T_{\mathrm{B}}$ and for $T < T_{\mathrm{B}}$.

Hence show that the heat capacity of the system is

$$C = \begin{cases} \frac{1}{k_{\mathrm{B}}T^2}\frac{M\epsilon^2}{(e^{\beta\epsilon}-1)^2} & T < T_{\mathrm{B}} \\ 0 & T > T_{\mathrm{B}} \end{cases}.$$

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) A quantum system is prepared in the ground state $|0\rangle$ at $t = 0$. It is subjected to a time–varying Hamiltonian $H = H_0 + \Delta(t)$. Show that, to first order in $\Delta(t)$, the system evolves as

$$|\psi(t)\rangle = \sum_k a_k(t)\, \mathrm{e}^{-\mathrm{i}E_k t/\hbar}|k\rangle\,,$$

where $H_0|k\rangle = E_k|k\rangle$ and

$$a_k(t) = \frac{1}{\mathrm{i}\hbar}\int_0^t \langle k|\Delta(t')|0\rangle\, \mathrm{e}^{\mathrm{i}(E_k - E_0)t'/\hbar}\, \mathrm{d}t'\,.$$

(ii) A large number of hydrogen atoms, each in the ground state, are subjected to an electric field

$$\mathbf{E}(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \hat{\mathbf{z}}\, \mathcal{E}_0 \exp(-t/\tau) & \text{for} \quad t > 0\,, \end{cases}$$

where $\mathcal{E}_0$ is a constant. Using results from Part (i) or otherwise, show that the fraction of atoms found in the state $|n, \ell, m\rangle = |2, 1, 0\rangle$ is, after a long time and to lowest non-trivial order in $\mathcal{E}_0$,

$$\frac{2^{15}}{3^{10}} \frac{a_0^2 e^2 \mathcal{E}_0^2}{\hbar^2(\omega^2 + 1/\tau^2)},$$

where $\hbar\omega$ is the energy difference between the $|2, 1, 0\rangle$ and $|1, 0, 0\rangle$ states.

What fraction of atoms lie in the $|2, 0, 0\rangle$ state?

[*Hint: You may assume the hydrogenic wavefunctions*

$$\langle \mathbf{r}|100\rangle = \frac{2}{\sqrt{4\pi}} \frac{1}{a_0^{3/2}} \exp\left(-\frac{r}{a_0}\right) \quad \textit{and} \quad \langle \mathbf{r}|210\rangle = \frac{1}{\sqrt{4\pi}} \frac{1}{(2a_0)^{3/2}} \frac{r}{a_0} \cos\theta \exp\left(-\frac{r}{2a_0}\right).$$

and the integral

$$\int_0^\infty r^m \mathrm{e}^{-\alpha r}\, \mathrm{d}r = \frac{m!}{\alpha^{m+1}}$$

for m a positive integer.]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Let f be the distribution function of a colisionless stellar system with density ρ moving in a gravitational potential ϕ. Let us use angled brackets to define averages over the distribution function in velocity space, for example

$$\langle v_i \rangle = \frac{1}{\rho} \int v_i \, f \mathrm{d}\, v^3.$$

Derive the Jeans equation in the form

$$\rho \frac{\partial \langle v_j \rangle}{\partial t} + \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\rho \frac{\partial \phi}{\partial x_j} - \frac{\partial (\rho \, \sigma_{ij}^2)}{\partial x_i},$$

where the velocity dispersion tensor $\sigma_{ij}^2 = \langle \langle v_i - \langle v_i \rangle \rangle \langle v_j - \langle v_j \rangle \rangle \rangle$.

Give a physical interpretation of σ_{ij}.

(ii) The Hohmann interplanetary travel orbit is the ellipse which is in contact with the circular orbit of the Earth at pericentre and the planet at apocentre. Show that the time of flight of a satellite on a Hohmann orbit from the Earth to a planet with semi-major a is

$$T = \frac{1}{4\sqrt{2}} \left[1 + \frac{a}{a_\oplus} \right]^{3/2} \mathrm{yr},$$

where $a_\oplus$ is the Earth's semi-major axis.

Show using Kepler's second law or otherwise that

$$h^2 = G M_\odot \, a \, (1 - e^2),$$

where h is the angular momentum and e is the ellipticity of the orbit.

If $V_\oplus$ is the circular velocity of the Earth, then the additional velocity required to place a satellite on a Hohmann orbit is $V_{\mathrm{add}} = V_{\mathrm{Hoh}} - V_\oplus$, where V_{Hoh} is the velocity of the Hohmann orbit at pericentre.

Show that

$$V_{\mathrm{add}} = V_\oplus \left[\sqrt{2} \left(\frac{a}{a + a_\oplus} \right)^{1/2} - 1 \right].$$

Mars has semi-major axis $a = 1.524$ au. Calculate the eccentricity of the Hohmann orbit, the time of flight and the additional velocity.

[Hint: The period P of a test particle in an elliptic orbit of semi-major axis a around a point mass M is $P = 2\pi \, a^{3/2}/(GM)^{1/2}$.]

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) Sources which all have the same intrinsic luminosity are distributed with uniform number density around the Sun. Show that $n(m)$, the number of sources per unit interval of apparent magnitude m, is proportional to $10^{3m/5}$. Sketch $n(m)$ as a function of m.

Now consider the case where the sources are embedded in a dusty medium with uniform density and uniform optical properties. Sketch on the same plot, without detailed calculation, the form of $n(m)$ with and without reddening.

Explain qualitatively the difference between the two cases.

(ii) Assume a point mass of mass M passes through a medium of test particles with velocity v. Estimate the impact parameter $b_{\rm crit}$, where test particles undergo a change of velocity of order v.

Assume that the dynamical effect on the point mass is dominated by material with impact parameter close to $b_{\rm crit}$. Give an approximate expression for the drag force acting on the point mass when it passes through a medium with mass density ρ made of particles with mass $<< M$.

Consider the case where the point mass is a star passing perpendicularly through a gas disc with mass per unit area Σ, rotating about the z axis. Show that the passage through the disc causes the star to acquire an angular momentum in the z direction of $\sim (GM/v^2)^2\, \Sigma\, r\, v_\phi$, where r is the distance from the disc's rotation axis and v_ϕ is the local tangential velocity of the disc.

Now assume that the disc is embedded in a cluster of such stars passing perpendicularly through the disc with number density n and total number N. Show that the timescale on which the disc loses angular momentum is of order N crossing times. You may assume that both the gas in the disc and the stars in the cluster are subject to a gravitational field mainly produced by the cluster stars.

A disc in the core of a galaxy contains a gas mass of $10^3 M_\odot$ within a radius of 1 pc of the centre of the galaxy. Estimate the number of stars in the star cluster at the centre of the galaxy if the angular momentum transfer from disc to cluster is sufficiently fast for the disc to form a central black hole of mass $\sim 10^3 M_\odot$ over a Hubble time. Assume that the mean mass of the stars is $1 M_\odot$.

State what evolutionary effects would occur within the star cluster within a Hubble time.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 8 June 2018 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

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Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) The line element for a homogeneous and isotropic cosmological model with positive spatial curvature takes the form

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + S_K^2(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] , \qquad (*)$$

where $S_K(\chi) = \sin(\chi\sqrt{K})/\sqrt{K}$. Using the Friedmann equations, show that in the absence of matter but in the presence of a positive cosmological constant Λ, the scale factor $a(t)$ takes the form

$$a(t) = A\cosh(ct/\alpha),$$

for a suitable choice of the origin of time, where A is a constant.

Specify the constant α in terms of Λ.

Express K in terms of A and Λ and hence show that the line element $(*)$ can be written as

$$\mathrm{d}s^2 = \alpha^2\left\{\mathrm{d}u^2 - \cosh^2 u\left[\mathrm{d}\tilde{\chi}^2 + \sin^2\tilde{\chi}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right]\right\} , \qquad (\dagger)$$

where the relation of the coordinates u and $\tilde{\chi}$ to t and χ should be given.

(ii) In one spatial dimension, the analogue of the line element $(\dagger)$ from Part (i) is

$$\mathrm{d}s^2 = \alpha^2\left(\mathrm{d}u^2 - \cosh^2 u\,\mathrm{d}\psi^2\right) , \qquad (\ddagger)$$

where $0 \leq \psi < 2\pi$ and $-\infty < u < \infty$. Show that this is the induced line element on the hyperboloid $(x^1)^2 + (x^2)^2 - (x^0)^2 = \alpha^2$ embedded in the 3D Minkowski space with line element

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 .$$

Given that the Riemann tensor R_{abcd} in 2D has only one independent component, verify by direct calculation that for the space with line element $(\ddagger)$

$$R_{abcd} = -\alpha^{-2}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) ,$$

where g_{ab} is the metric tensor.

Question 2X - Astrophysical Fluid Dynamics

(i) A spherically symmetric star in hydrostatic equilibrium is well described by a barotropic equation of state, $p_0 = K\rho_0^\gamma$, where p_0 is the gas pressure, ρ_0 is the gas density, K is a constant and γ is the adiabatic index. Consider a uniform and infinitesimally small expansion of the star such that the position of a fluid element at r_0 becomes $r_0(1+\delta)$, where δ is a constant such that $0 < \delta \ll 1$. Show that the density and pressure of the perturbed state satisfy $\rho = \rho_0(1 - 3\delta)$ and $p = p_0(1 - 3\gamma\delta)$ to first order accuracy.

Using these results find a condition on γ such that the star is stable to radial oscillations.

(ii) In the centre of our Galaxy, the luminous source SgrA* is powered by a supermassive black hole with mass $M = 4 \times 10^6\,\mathrm{M}_\odot$. Assume that the black hole is embedded in a hot isothermal medium of fully ionized hydrogen with temperature $T_\infty = 10^7\,\mathrm{K}$ and proton number density $n_\infty = 1\,\mathrm{cm}^{-3}$ at large distances from the black hole. Assume that this medium is undergoing spherically-symmetric Bondi accretion into the black hole. Starting from the fluid equations derive the steady-state mass accretion rate $\dot{M}$ as a function of T_∞ and n_∞.

How long does it take for the black hole to double its mass?

How does the mass-doubling timescale compare with the age of the Universe (approximately 13.7 Gyr)? Comment on the physical significance of this comparison.

Assume that the black hole emits at a luminosity $L = \epsilon_\mathrm{r}\dot{M}c^2$, with $\epsilon_\mathrm{r} = 0.1$ being the radiative efficiency and c the speed of light. Calculate the luminosity of SgrA* assuming Bondi accretion and compare this with the luminosity of the Galaxy.

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) Give a short account of what is meant by

(a) strong gravitational lensing,

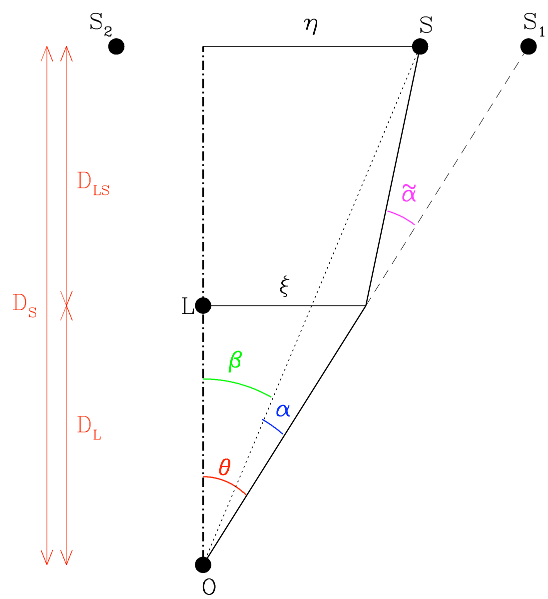
(b) weak gravitational lensing,

(c) and microlensing,

giving examples of the observational signatures of each.

Discuss the main applications of each in astrophysics.

(ii) Consider the diagram below, where S, L and O denote, respectively, a source, a gravitational lens and an observer. Explain the meaning of the remaining quantities in the diagram and derive the basic gravitational lens equation, $\beta = \theta - \alpha(\theta)$. State the necessary assumptions in your discussion



.

The light from a source at impact parameter b gravitationally lensed by a point mass M is deflected by an angle $4\,GM/(b\,c^2)$. Give the definition of the *Einstein radius* and explain its significance in gravitational lensing.

Show that the gravitational lens equation can be written in terms of the Einstein radius $\theta_{\rm E}$ as,

$$\beta = \theta - \frac{\theta_{\rm E}^2}{\theta} \,.$$

Show that lensing by a point mass produces a magnification,

$$\mu \equiv \frac{\theta}{\beta}\frac{{\rm d}\theta}{{\rm d}\beta} = \left[1 - \left(\frac{\theta_{\rm E}}{\theta}\right)^4\right]^{-1} \,.$$

Give a physical explanation for the case when $\theta < \theta_{\rm E}$.

The pair of quasars 1146+111B,C are separated by 157 arcseconds on the sky. Both quasars have magnitude $m_{\rm V} = 18.5$, and their spectra exhibit broad Mg II $\lambda 2798$ emission lines at the same redshift $z = 1.012 \pm 0.001$ and with the same Full Width at Half Maximum, FWHM $= 64 \pm 4$ Å.

Discuss the alternative possibilities that this pair is,
(a) the same quasar gravitationally lensed into two images, or
(b) two separate quasars.

Propose a set of observations that could distinguish between the two possibilities.

[The angular diameter distance at z=1.012 is approximately 1650 Mpc.]

TURN OVER...

Question 4X - Structure and Evolution of Stars

(i) Draw a sketch of the Hertzsprung-Russell diagram, labelling the axes carefully. Include in your sketch the approximate positions of: the Sun, the main sequence, white dwarfs, the horizontal branch, red giants and the Hayashi track.

(ii) A group of homogeneous radiative stars all have the same chemical composition, with electron scattering as the main opacity source (so $\kappa = \text{const}$) and energy production via the CNO cycle ($\epsilon = \epsilon_0 \rho T^{18}$). Neglecting radiation pressure, show using homology arguments that

$$L \sim M^3,$$

and

$$R \sim M^{17/21}.$$

Determine how the effective temperature T_* of the stars scales with the luminosity L.

For a second group of homogeneous radiative stars with the same chemical composition, the opacity source is of the Kramers type with $\kappa = \kappa_0 \rho T^{-7/2}$, while the energy source is via the proton-proton chain, with $\epsilon \approx \epsilon_0 \rho T^4$. Assuming homology, show that for these stars we would expect

$$R \sim M^{1/13}.$$

In fact, the observed relationship for such stars is $R \sim M^{0.7}$. Give possible reasons why the homology arguments may fail.

Question 5Z - Statistical Physics

(i) The one-dimensional Ising model consists of a set of N spins s_i with Hamiltonian

$$H = -J\sum_{i=1}^{N} s_i s_{i+1} - \frac{B}{2}\sum_{i=1}^{N}(s_i + s_{i+1}),$$

where periodic boundary conditions are imposed so $s_{N+1} = s_1$. J is a positive coupling constant and B is an external magnetic field. Define a 2×2 matrix M with elements

$$M_{st} = \exp\left[\beta J st + \frac{\beta B}{2}(s+t)\right],$$

where indices s, t take values ± 1 and $\beta = (k_{\mathrm{B}}T)^{-1}$, where k_{B} is Boltzmann's constant and T is temperature.

Prove that the partition function of the Ising model can be written as $Z = \mathrm{Tr}(M^N)$. Calculate the eigenvalues of M and hence determine the free energy in the thermodynamic limit $N \to \infty$. Explain why the Ising model does not exhibit a phase transition in one dimension.

(ii) Consider the case of zero magnetic field $B = 0$. The *correlation function* $\langle s_i s_j\rangle$ is defined by

$$\langle s_i s_j\rangle = \frac{1}{Z}\sum_{\{s_k\}} s_i s_j e^{-\beta H},$$

where Z, H, s_i, β are as defined in Part (i). Show that, for $i > 1$,

$$\langle s_1 s_i\rangle = \frac{1}{Z}\sum_{s,t} st(M^{i-1})_{st}(M^{N-i+1})_{ts}.$$

By diagonalizing M, or otherwise, calculate M^p for any positive integer p. Hence show that

$$\langle s_1 s_i\rangle = \frac{\tanh^{i-1}(\beta J) + \tanh^{N-i+1}(\beta J)}{1 + \tanh^{N}(\beta J)}.$$

In the thermodynamic limit, the *correlation length* ξ is defined by $\langle s_i s_j\rangle \sim e^{-|i-j|/\xi}$. Use the above result to determine ξ, and discuss how it behaves for high and low temperatures.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) The spin operators obey the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$. Let $|s,\sigma\rangle$ be an eigenstate of the spin operators S_z and $\mathbf{S}^2$, with $S_z|s,\sigma\rangle = \sigma\hbar\,|s,\sigma\rangle$ and $\mathbf{S}^2|s,\sigma\rangle = s(s+1)\hbar^2\,|s,\sigma\rangle$. Show that

$$S_{\pm}|s,\sigma\rangle = \sqrt{s(s+1)-\sigma(\sigma\pm 1)}\,\hbar\,|s,\sigma\pm 1\rangle\,,$$

where $S_{\pm} = S_x \pm \mathrm{i}S_y$.

(ii) Using results from Part (i) when $s = 1$, derive the explicit matrix representation

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

in a basis in which S_z is diagonal.

A beam of atoms, each with spin 1, is polarised to have spin $+\hbar$ along the direction $\mathbf{n} = (\sin\theta, 0, \cos\theta)$. This beam enters a Stern–Gerlach filter that splits the atoms according to their spin along the $\hat{\mathbf{z}}$-axis. Show that $N_+/N_- = \cot^4(\theta/2)$, where $N_{\pm}$ is the number of atoms emerging from the filter with spins parallel / anti-parallel to $\hat{\mathbf{z}}$.

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) A homogeneous sphere has density ρ_0 for radius $r \leq r_0$, and density zero elsewhere. Determine the equation of motion for a collisionless particle with $r < r_0$.

Compare the orbits qualitatively to those in the potential of a point mass.

Derive an integral equation linking the particle's polar coordinates r and ϕ.

(ii) A spherical galaxy with finite total mass has the potential

$$\Phi(r) = -\frac{G\,M}{r + a},$$

where a is a length scale and r is the distance to the centre. The galaxy can be assumed to be in steady state.

Find the rotation curve of the galaxy.

Using Poisson's equation, find the mass density $\rho(r)$ of the galaxy.

Verify that the galaxy has a distribution function

$$f(E, L) = \frac{C\,E^2}{L},$$

where E is the energy and L is the angular momentum.

Determine the constant C in terms of a and M.

What is the ratio of the radial velocity dispersion to the tangential velocity dispersion in the model?

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) Meteorites in the solar system are found to contain a range of chemical species that are daughter products of short-lived radio-isotopes which are best explained as contaminants from the supernova ejecta of a neighbouring star of mass $m \gtrsim 25M_\odot$. Assume that the distribution of stellar masses of the birth cluster of the Sun can be calculated by randomly drawing N stellar masses from the Salpeter mass function. For what value of N is the probability of a cluster containing a star as massive as $25M_\odot$ equal to 50%?

The fraction of birth clusters with a number of stars in the range N to $N+\mathrm{d}N$ has been estimated to be proportional to $N^{-2}\mathrm{d}N$ for $N > 50$ and zero for $N < 50$. What fraction of stellar birth clusters have a chance of containing a star as massive as $25M_\odot$ of at least 50%?

[For a Salpeter initial mass function the mass contained in stars in the mass range m to $m + \mathrm{d}m$ scales as $m^{-1.35}\mathrm{d}m$ for stellar masses $> 0.1M_\odot$.]

(ii) Gas accretes radially onto an OB star of radius 0.1 au which ionises the inflowing gas out to a radius $r_I = 100$ au. The radius and velocity of the ionised gas are given by,

$$n = n_I \left(\frac{r}{r_I}\right)^{-3/2}, \qquad v = v_I \left(\frac{r}{r_I}\right)^{-1/2},$$

where $n_I = 3 \times 10^{13}$ m^{-3} and $v_I = 20$ km s^{-1}.

Estimate the mass of the OB star assuming that the velocity is roughly the free-fall velocity and explain why this is a reasonable assumption.

Calculate the ionising luminosity of the OB star assuming all ionising photons have an energy of 13.6 eV and compare it to the luminosity due to the accretion of the gas. Comment on whether it is plausible that the ionising radiation is due to the accretion.

Assume that the bubble of ionised gas is optically thick to its thermal bremstrahlung emission at a wavelength of 2 cm. Estimate the luminosity of the ionised bubble at this wavelength in W Hz^{-1}.

Assess whether this would be detectable by a radio telescope with a sensitivity of 1mJy, if the OB star is at a distance of 450 pc from Earth.

[For hydrogen the coefficient for recombinations to excited electronic states is $\alpha = 3 \times 10^{-19}$ m^3 s^{-1}. The black body power per unit emitting area per unit frequency (W m^{-2} Hz $^{-1}$) is $dF_\nu/d\nu = 2\pi h/((\lambda^2 \exp(h\nu/kT) - 1))$. 1 mJy corresponds to the radio flux at the detector of a source of luminosity 2×10^5 W Hz^{-1} at a distance of 1 pc.]

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