NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 30 May 2016 13.30pm – 16.30pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (i) *and Part* (ii)*, which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i)*. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z***, according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 7Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) The four-velocity of a particle is defined as $u^\mu = dx^\mu/d\tau$, where x^μ are the spacetime coordinates of the particle ($\mu = 0$–3) and τ is the proper time. Derive the components of the four-velocity in an inertial frame in which the particle is moving with three-velocity $\boldsymbol{u}$.

Two particles have three-velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in some inertial frame. Show that their relative speed, $\beta_{\mathrm{rel}}c$, satisfies

$$(1-\beta_{\mathrm{rel}}^2)^{-1/2} = \gamma(\boldsymbol{u}_1)\gamma(\boldsymbol{u}_2)\left(1 - \frac{\boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{c^2}\right),$$

where $\gamma(\boldsymbol{u}) = (1-\boldsymbol{u}^2/c^2)^{-1/2}$ is the Lorentz factor for a three-velocity $\boldsymbol{u}$ and c is the speed of light.

(ii) High energy protons can interact with photons of the cosmic microwave background to produce pions via the Δ^+ resonance:

$$\gamma_{\mathrm{CMB}} + \mathrm{p} \to \Delta^+ \to \mathrm{p} + \pi^0.$$

Show that for ultra-relativistic protons, the threshold energy E_{p} in the laboratory frame for the production of pions is

$$E_{\mathrm{p}} \approx \frac{m_{\mathrm{p}} m_\pi c^4}{2E_\gamma}\left(1 + \frac{1}{2}\frac{m_\pi}{m_{\mathrm{p}}}\right),$$

where E_γ is the energy of the photon and m_{p} and m_π are the proton and pion rest masses, respectively.

Estimate E_{p}, taking the temperature of the cosmic microwave background to be $2.73\,\mathrm{K}$ and the pion rest mass $m_\pi = 135\,\mathrm{MeV}/c^2$.

Question 2Y - Astrophysical Fluid Dynamics

(i) Write down the definition of a conservative force and show why this means that such a force can be written as the gradient of a scalar potential.

Explain what is meant by the *Gaussian surface* method. Apply this method to a sphere of radius $R_{\rm out}$, which is hollow in the centre up to some radius $R_{\rm in} < R_{\rm out}$, and has a mass M. Thus derive the magnitude and orientation of the gravitational acceleration $\boldsymbol{g}$, both inside $R_{\rm in}$ and outside $R_{\rm out}$.

Using the relevant timescales derive an approximate expression for the Jeans length and explain the physical reasoning behind the choice of these timescales. List astrophysical examples where Jeans instability plays a fundamental role.

(ii) Consider a spherical homogeneous Giant Molecular Cloud (GMC) with density ρ, radius λ and temperature T, comprised of gas with a mean molecular weight μ and adiabatic index γ. Write down an expression for the thermal energy K as a function of these parameters.

Under the assumption of virial equilibrium derive the Jeans length and Jeans mass of this GMC and comment on the physical meaning of the dependences on ρ, T and μ.

Small perturbations to the GMC result in a new mean density and temperature, $\rho_{\rm n}$ and $T_{\rm n}$, resulting in a new Jeans mass $M_{\rm J,n}$. If the temperature of the cloud depends on its density as $T \propto \rho^{\gamma-1}$, determine the values of γ for which fragmentation ensues.

Discuss the possibility of fragmentation in the adiabatic and isothermal limit and so discuss briefly how radiative gas cooling may promote star formation within GMCs.

TURN OVER...

Question 3X - Physical Cosmology

(i) The distance modulus is defined by $m - M = 5\log_{10}(d_{\rm L}/10\,{\rm pc})$, where m is the apparent magnitude, M is the absolute magnitude, and $d_{\rm L}$ is the luminosity distance. Given that the galaxy Cam 300 has a distance modulus of $+30$, what is its distance in parsecs?

Knowing that the Sun has an absolute magnitude in the B-band of $M_{\rm B} \simeq +5$, estimate the approximate apparent magnitude $m_{\rm B}$ of the galaxy Cam 300 if its stellar mass is similar to that of the Milky Way, stating any assumptions made.

Explain what is meant in astronomy by the terms 'standard candle' and 'standard ruler'. Explain their application to cosmological distance determinations.

(ii) Give two examples of standard candles of particular importance for cosmology, highlighting their main advantages and limitations.

Derive expressions for the angular diameter distance and luminosity distance as a function of redshift z for a spatially-flat Friedmann–Robertson–Walker universe filled with pressureless matter and with $\Lambda = 0$. You should express your answers in terms of the present-day Hubble parameter H_0.

The surface brightness Σ of an astronomical object is defined as its observed flux f divided by its observed angular area $(\delta\theta)^2$, so that $\Sigma = f/(\delta\theta)^2$. For a class of objects that are both standard candles and standard rulers, deduce the functional dependence of Σ on redshift z.

Explain generally whether observing the surface brightness of such a class of objects can be used to determine cosmological parameters such as $\Omega_{\rm m,0}$, $\Omega_{\rm k,0}$, and $\Omega_{\Lambda,0}$ (where $\Omega_{\rm m,0}$, $\Omega_{\rm k,0}$, and $\Omega_{\Lambda,0}$ are the present-day contributions to the critical density by, respectively, matter, curvature, and the cosmological constant)?

Question 4Z - Structure and Evolution of Stars

(i) Sketch the behaviour of the radius of the Sun as a function of age. Set the origin as the time the Sun first appears on the Hertzsprung-Russell diagram. Annotate the plot to indicate the evolutionary phase corresponding to significant changes in the Solar radius.

(ii) A star on the red-giant branch possesses an isothermal core of radius $r_\mathrm{c} \ll R_\mathrm{star}$ and a density distribution as a function of distance from the centre, r, given by

$$\rho(r) = \rho_0(1 - r/R_\mathrm{star}),$$

where ρ_0 is a constant and R_star is the stellar radius.

Show that the mass of the core is given by

$$M_\mathrm{c} \simeq \frac{4\pi r_\mathrm{c}^3}{3}\rho_0.$$

Show that the pressure at the centre of the core is

$$P_0 = P_\mathrm{c} + \frac{2\pi}{3}G\rho_0^2 r_\mathrm{c}^2,$$

where G is the gravitational constant and P_c is the pressure at the core boundary.

Assuming that $P_0 \gg P_\mathrm{c}$ and that the equation of state is that for an ideal gas, show that the temperature at the centre of the core is

$$T_0 \simeq \frac{2\pi G\mu}{3\mathcal{R}} r_\mathrm{c}^2 \rho_0,$$

where μ is the mean molecular mass and $\mathcal{R}$ is the gas constant.

TURN OVER...

Question 5Z - Statistical Physics

(i) Consider an ideal quantum gas with one-particle states $|i\rangle$ of energy ϵ_i. Let $p_i^{(n_i)}$ denote the probability that state $|i\rangle$ is occupied by n_i particles. Here, n_i can take the values 0, 1 for fermions and any non-negative integer for bosons. The entropy of the gas is given by

$$S = -k_{\mathrm{B}} \sum_i \sum_{n_i} p_i^{(n_i)} \ln p_i^{(n_i)} , \qquad (*)$$

where k_{B} is Boltzmann's constant. Write down three constraints that must be satisfied by the probabilities if the average energy $\langle E\rangle$ and average particle number $\langle N\rangle$ are kept at fixed values.

Use variation of the entropy to show that a maximal S implies

$$p_i^{(n_i)} = \frac{1}{Z_i} e^{-(\beta\epsilon_i+\gamma)n_i} , \qquad (**)$$

where β and γ are Lagrange multipliers, and you should provide an expression for Z_i.

(ii)

Use the results from Part (i), the expression $(*)$, and the first law of thermodynamics to interpret the meaning of the Lagrange multipliers in equation $(**)$.

Use the probabilities $p_i^{(n_i)}$ from equation $(**)$ to calculate the average occupation number $\langle n_i\rangle = \sum_{n_i} n_i p_i^{(n_i)}$ for Fermi-Dirac and for Bose-Einstein gases.

Question 6Z - Principles of Quantum Mechanics

(i) A particle in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$ whose eigenstates obey

$$\langle x|x'\rangle = \delta(x-x')\,, \qquad \langle p|p'\rangle = \delta(p-p')\,, \qquad \langle x|p\rangle = (2\pi\hbar)^{-1/2} e^{ixp/\hbar}\ .$$

Given a state $|\psi\rangle$, define the momentum-space and position-space wavefunctions $\tilde{\psi}(p)$ and $\psi(x)$, respectively, and show how each of these can be expressed in terms of the other.

Write down the translation operator $U(\alpha)$ with the property that $U(\alpha)|x\rangle = |x+\alpha\rangle$ and relate the momentum-space and position-space wavefunctions for $U(\alpha)|\psi\rangle$ to $\tilde{\psi}(p)$ and $\psi(x)$, respectively.

(ii) Consider a harmonic oscillator of mass m and frequency ω with normalised energy eigenstates $|n\rangle$ and corresponding wavefunctions $\psi_n(x)$ and $\tilde{\psi}_n(p)$ with $n = 0, 1, 2, \ldots$. Using results from Part (i) or otherwise, express $\psi_0(x-\alpha)$ and $\psi_1(x-\alpha)$ in terms of the wavefunctions $\psi_n(x)$.

Show also that if $\tilde{\psi}_n(p) = f_n(p)\,\tilde{\psi}_0(p)$ for polynomials f_n, then

$$e^{-i\alpha p/\hbar} = e^{-m\omega\alpha^2/(4\hbar)} \sum_{n=0}^{\infty} \left(\frac{m\omega}{2\hbar}\right)^{n/2} \frac{\alpha^n}{\sqrt{n!}}\, f_n(p)\ .$$

[You may quote standard results regarding the eigenstates of a harmonic oscillator with annihilation and creation operators

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\,, \qquad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\,.$$

You may also use, without proof, the result

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

for any operators A and B which both commute with $[A, B]$.]

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Starting with Newton's Second Law, prove that energy is conserved for an orbit in a steady state potential.

Consider a Keplerian orbit satisfying the following equation in polar coordinates (r, ϕ)

$$\frac{1}{r} = \frac{GM}{h^2}[1 + e\cos(\phi - \phi_0)],$$

where G is the gravitational constant, M is the mass of the central body, h is the specific angular momentum, e is the eccentricity of the orbit, and ϕ_0 is a constant. Show that the orbital energy is $E = -GM/(2a)$, where a is the semi-major axis size.

(ii) The gravitational potential

$$\Phi = -\frac{GM}{b + \sqrt{b^2 + r^2}},$$

where G, M and b are constants and r is the distance from the origin, is called the *isochrone* potential. Show that in the isochrone potential, the energy of a circular orbit is given by $E = -GM/(2a)$, where $a = \sqrt{b^2 + r^2}$.

Let the angular momentum of this orbit be $L_c(E)$. Show that

$$L_c = \sqrt{GMb}(x^{-1/2} - x^{1/2}),$$

where $x \equiv -2Eb/(GM)$.

The isochrone potential is generated by matter with a density $\rho(r)$. Determine $\rho(r)$ in the limits $r \to 0$ and $r \gg b$, and comment on how this compares to observations of real globular clusters.

Question 8Y - Physics of Astrophysics

(i) Two stars, A and B, are separated on the sky by 0.2 arcsec in the Orion star forming region which is 450 pc from Earth. Ten years later the projected separation of the pair is 0.21 arcsec and their separation vector has rotated through an angle of 3° on the sky. If the masses of the two stars are $1M_\odot$ and $0.5M_\odot$, determine whether the pair is definitely gravitationally bound, unbound or indeterminate on the basis of the information provided.

What other data is relevant to deciding whether the pair is unbound or not?

(ii) The young star RW Auriga A is surrounded by a disc of dusty gas of radius 60 au from which extends a tidal arm. This arm is believed to be composed of marginally bound disc material that was stripped off during a close encounter (at interstellar separation $\sim$ 60 au) between this star and a low mass companion star, RW Auriga B. Provide a rough estimate of the expansion velocity of the tidal arm material with respect to RW Auriga A.

Recently RW Auriga A became rapidly dimmer over a period of 20 days and remained in this state for 6 months. It has been suggested that this dimming event was due to a dense blob of gas in the tidal arm temporarily occulting the star. Evaluate whether this is a plausible explanation and, if so, what size of dense blob is required to explain these observations.

During the dim phase, observations of the 5897.5 Å NaI D_1 line showed a deep absorption feature at a wavelength of 5896.3 Å. Determine whether the blob of gas mentioned above is likely to be responsible for this absorption.

Estimate the maximum relative velocity of the two stars at closest approach if another close encounter is to be expected between them during the next 10^4 years.

The opacity of neutral sodium (NaI) at the centre of the D_1 line is $4 \times 10^8\,\mathrm{m^2\,kg^{-1}}$. Given that the line is opaque at its centre, estimate the minimum column density of hydrogen in the absorbing material. You may assume that the relative atomic mass of sodium is 23 and that the ratio of Na to H in the gas is 10^{-9} by number.

[You may assume RW Auriga A has a mass $1M_\odot$ and radius $1.6R_\odot$.]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 31 May 2016 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 2

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Question 1X - Relativity

(i) The equation for the parallel transport of a vector A^i along a path $x^i(u)$ is

$$\frac{dA^i}{du} = -\Gamma^i_{jk} A^j \frac{dx^k}{du},$$

where Γ^i_{jk} are the Christoffel symbols. Now consider the parallel transport of the vector A^i around a small closed contour $\mathcal{C}$ in the neighourhood of a point P, with coordinates x^i_P, that lies on the contour. By expanding A^i and Γ^i_{jk} to first order in the displacement $x^i(u) - x^i_P$,

$$\begin{aligned} A^i(u) &= A^i(u_P) - \Gamma^i_{jk}|_P A^j(u_P)[x^k(u) - x^k_P], \\ \Gamma^i_{jk}[x(u)] &= \Gamma^i_{jk}|_P + \partial_l \Gamma^i_{jk}|_P [x^l(u) - x^l_P], \end{aligned}$$

show that the change in the vector A^i when parallel transported from P around the closed contour $\mathcal{C}$ is

$$\Delta A^i = -(\partial_l \Gamma^i_{jk} - \Gamma^i_{mk}\Gamma^m_{jl})|_P A^j(u_P) \oint x^l \, dx^k.$$

Hence show that

$$\Delta A^i = -\frac{1}{2} R^i{}_{jkl}|_P A^j(u_P) \oint x^k \, dx^l,$$

where $R^i{}_{jkl}|_P$ is the curvature tensor at point P.

Give a geometrical interpretation of this result.

(ii) Consider a two-sphere of radius a with metric

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2,$$

where θ and ϕ are spherical polar coordinates. Show that the only non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \cot\theta. \end{aligned}$$

Consider the parallel transport of a vector V^i along a curve defined by $\theta = \theta_0 =$ const. on the two-sphere. Show that

$$\begin{aligned} V^\theta(\phi) &= A\cos\alpha\phi + B\sin\alpha\phi, \\ V^\phi(\phi) &= C\cos\alpha\phi + D\sin\alpha\phi, \end{aligned}$$

where A, B, C and D are constants and $\alpha = \cos\theta_0$.

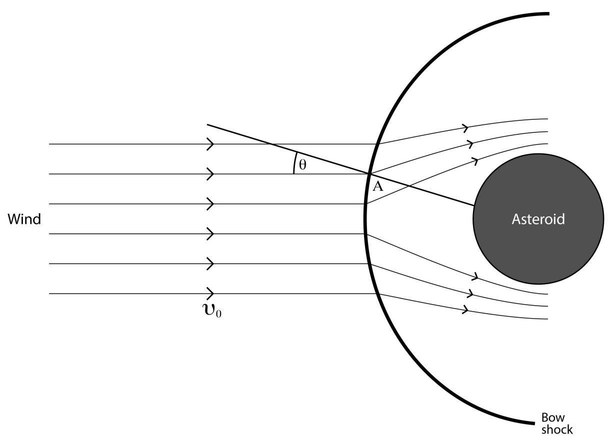
Discuss the solutions if $(V^\theta, V^\phi) = (U, 0)$ at $\phi = 0$, contrasting the behaviour of the solutions for $\theta_0 = \pi/2$ and $\theta_0 \approx 0$.

Question 2Y - Astrophysical Fluid Dynamics

(i) For a steady-state fluid with a barotropic equation of state, derive Bernoulli's constant and explain its physical meaning.

Consider an incompressible fluid filling an open cylindrical tank of radius R_1 up to a height h on the Earth's surface. A small nozzle of radius $R_2 \ll R_1$ is opened at the bottom of the tank. Assuming steady state find the velocity u at which fluid passes through the nozzle as a function of h and the gravitational acceleration g.

(ii) A spherical asteroid of radius r is embedded in the Solar wind which hits the asteroid with speed u_0. The wind is supersonic with a Mach number $\gg 1$ and has an adiabatic index $\gamma = 5/3$. As a result a bow shock forms as sketched below:



Consider the point on the bow shock A whose normal to the asteroid's surface forms an angle θ with respect to the incident direction of the wind. Determine the Mach number in the post-shock flow, $M_{\rm post}$, as a function of θ.

For which values of θ is the flow behind the bow shock subsonic?

What is the sound speed in terms of u_0 at the location where $M_{\rm post} = 1$?

Show by explicit substitution of the quantities derived above that the third Rankine-Hugoniot condition is satisfied along each streamline both in the pre-shocked and post-shocked gas.

[You may assume the second Rankine-Hugoniot condition in the form: $M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}$.]

TURN OVER...

Question 3X - Physical Cosmology

(i) Show that for a pressureless matter-dominated universe with $\Lambda = 0$,

$$\rho_0 = \frac{3H_0^2}{4\pi G} q_0,$$

and

$$\frac{kc^2}{a_0^2} = (2q_0 - 1)H_0^2,$$

where $q \equiv -a\ddot{a}/\dot{a}^2$ is the deceleration parameter, a is the scale factor, k is the curvature, H is the Hubble parameter, ρ is the density, G is the gravitational constant, and c is the speed of light. The subscript 0 indicates the present time.

(ii) Show that in the model of Part (i), at a general time

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[2q_0 \left(\frac{a_0}{a}\right)^3 + (1 - 2q_0)\left(\frac{a_0}{a}\right)^2\right]. \qquad (*)$$

Consider the case of an open universe ($0 \leq q_0 \leq 1/2$). By introducing conformal time η, with $d\eta = dt/a$, or otherwise, derive the following parametric solution of $(*)$,

$$\frac{a}{a_0} = \frac{q_0}{1 - 2q_0}(\cosh w - 1),$$
$$H_0 t = \frac{q_0}{(1 - 2q_0)^{3/2}}(\sinh w - w),$$

where the parameter $w \geq 0$.

Calculate the current age of such a universe with $q_0 = 0.15$ in units of $1/H_0$.

Find the asymptotic form of $a(t)$ at late times and calculate the asymptotic value of the deceleration parameter. Comment on your result.

Question 4Z - Structure and Evolution of Stars

(i) Show that under suitable approximations (which you should state) the mean molecular mass in a stellar interior is given by

$$\mu = \frac{4}{(3+5X)},$$

where X is the mass fraction of hydrogen.

Derive an approximate expression for the relation between the hydrogen mass fraction and the number of free electrons per nucleon within a stellar interior in which material is fully ionized.

Assuming that the Sun is halfway through its lifetime on the main sequence, what is the approximate value of the hydrogen mass fraction within the nuclear burning core?

(ii) For a star in hydrostatic equilibrium it can be shown that the central pressure $P_{\rm c}$, central density $\rho_{\rm c}$ and the mass of the star M are related by $P_{\rm c} \simeq 0.5GM^{2/3}\rho_{\rm c}^{4/3}$, where G is the gravitational constant. If the total pressure is due to the contributions of gas pressure $P_{\rm g}$ and radiation pressure $P_{\rm r}$, such that $P_{\rm g} = \beta P_{\rm c}$ and $P_{\rm r} = (1-\beta)P_{\rm c}$, where β is a dimensionless parameter, derive an expression that shows how the contributions of $P_{\rm g}$ and $P_{\rm r}$ depend on the stellar mass.

Adopting a value of $\beta = 0.5$ to indicate when radiation pressure limits the mass of a star, estimate the maximum mass of a star.

For the most massive stars on the zero-age main sequence the luminosity exceeds the Eddington luminosity and significant mass loss occurs. For such stars it is found that their masses scale linearly with their radii. Assume that the radiation pressure associated with a fraction f of the luminosity L accelerates material in the photosphere to the escape velocity $v_{\rm esc}$, at which point the material escapes from the star. Derive an expression that shows the dependence of the mass-loss rate $\dot{M}$ on the luminosity, mass and radius of the star.

TURN OVER...

Question 5Z - Statistical Physics

(i) Briefly describe the canonical ensemble.

Consider a system in the canonical ensemble with temperature T that can be in states $|n\rangle$, $n = 0, 1, 2, \ldots$ with energies E_n. Write down the partition function for this system and the probability $p(n)$ that the system is in state $|n\rangle$.

The average of a quantity A of the system is defined as

$$\langle A \rangle = \sum_n p(n) A_n \,.$$

Derive expressions for the average energy $\langle E \rangle$, the specific heat $\partial \langle E \rangle / \partial T$, and the entropy $\langle S \rangle = -k_{\mathrm{B}} \sum_n p(n) \ln p(n)$, in terms of the partition function, where k_{B} is Boltzmann's constant.

(ii) Consider an anharmonic oscillator with energy levels

$$E_n = \hbar\omega \left[\left(n + \frac{1}{2} \right) + \delta \left(n + \frac{1}{2} \right)^2 \right], \qquad (n = 0,\, 1,\, 2,\, \ldots),$$

where ω is a constant and $0 < \delta \ll 1$ is a small constant. Let the oscillator be in contact with a reservoir at temperature T. All following calculations are to be performed to linear order in δ. Show that the partition function Z_1 for a single oscillator is given by

$$Z_1 = \frac{c_1}{\sinh(\frac{x}{2})} \left[1 + c_2 \delta x \left(1 + \frac{2}{\sinh^2(\frac{x}{2})} \right) \right],$$

where $x = \hbar\omega/(k_{\mathrm{B}}T)$, and c_1 and c_2 are constants to be determined.

Show that the average energy of a system of N uncoupled oscillators of this type is given by

$$\langle E \rangle = \frac{N\hbar\omega}{2} \left\{ c_3 \coth\frac{x}{2} + \delta \left[c_4 + \frac{c_5}{\sinh^2(\frac{x}{2})} \left(1 - x \coth\frac{x}{2} \right) \right] \right\},$$

where c_3, c_4, c_5 are constants to be determined.

Question 6Z - Principles of Quantum Mechanics

(i) Let $|\, j\, m\,\rangle$ be standard angular momentum eigenstates with labels specifying eigenvalues for $\boldsymbol{J}^2$ and J_3, where $\boldsymbol{J}$ is the total angular momentum and J_3 is the 3-component of the angular momentum. Taking units in which $\hbar = 1$, and quoting any angular momentum commutation relations that you require, verify the formulae

$$J_{\pm}|\, j\, m\,\rangle = [\,(j{\mp}m)(j{\pm}m{+}1)\,]^{1/2}|\, j\, m{\pm}1\,\rangle\ .$$

You need not derive restrictions on the values of j and m.

(ii) Two particles, each of spin $s > 0$, have combined spin states $|\,J\, M\,\rangle$. Find expressions for all such states with $M = 2s{-}1$ in terms of product states.

Suppose now that these particles move about their centre of mass with a spatial wavefunction that is a spherically-symmetric function of relative position. If the particles are identical, what spin states $|\,J\ 2s{-}1\,\rangle$ are allowed? Justify your answer.

Now consider two particles of spin 1 that are not identical and are both at rest. If the 3-component of the spin of each particle is zero, what is the probability that their total combined spin is zero?

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) What is meant by the epicyclic approximation?

Starting with the equation of motion in an axi-symmetric potential $\Phi(R, z)$, and assuming that circular orbits exist, derive the following expressions for the radial and vertical frequencies, respectively, for orbits near the plane of symmetry at $z = 0$:

$$\kappa^2 = \left(R\frac{d\Omega^2}{dR} + 4\Omega^2 \right)_{R_\mathrm{g}} ,$$

$$\gamma^2 = \left(\frac{\partial^2 \Phi}{\partial z^2} \right)_{(R_\mathrm{g},0)} ,$$

where $\Omega(R)$ is the circular frequency and R_g is the guiding radius.

(ii) Let $\Phi(R, z)$ be the axi-symmetric Galactic potential. At the Solar location, $(R, z) = (R_0, 0)$, prove that

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G\rho_0 + 2(A^2 - B^2),$$

where G is the gravitational constant, ρ_0 is the density in the Solar neighborhood and A and B are Oort's constants.

The Oort constants are $A = 14.5\,\mathrm{km\,s^{-1}\,kpc^{-1}}$ and $B = -12\,\mathrm{km\,s^{-1}\,kpc^{-1}}$. Explain why these values imply that the density in the Galaxy falls off as $\sim R^{-2}$.

Consider one of the Jeans equations in cylindrical polar coordinates (R, ϕ, z):

$$\frac{\partial(\nu\overline{v_z})}{\partial t} + \frac{\partial(\nu\overline{v_R v_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{\nu\overline{v_R v_z}}{R} + \nu\frac{\partial \Phi}{\partial z} = 0,$$

where ν is the density of stellar tracers, v_R and v_z are, respectively, the radial and vertical velocities, and Φ is the potential. Explaining your assumptions carefully, describe how kinematic measurements of stars in the Solar neighborhood can be used to determine the local total matter density ρ_0.

Question 8Y - Physics of Astrophysics

(i) A black hole of mass $10^8 M_\odot$ is ejected at $1000\,\mathrm{km\,s^{-1}}$ from a galactic nucleus. One possible mechanism for its ejection is that of a three-body slingshot involving three black holes of similar mass at a separation of 1 pc. Another involves the coalescence of two black holes in a roughly equal mass binary which spiral in from a separation of 10^{-3} pc as a result of gravitational wave emission. It is believed that asymmetric gravitational wave emission can result in the merger product acquiring kinetic energy of up to 1% of the binary's initial binding energy. Determine whether either or both of these mechanisms are viable in this system.

If the black hole's host galaxy is located at the centre of a rich cluster of galaxies containing 1000 galaxies of radius 20 kpc within a radius of 30 Mpc, what is the probability that the ejected black hole passes through another galaxy? You may assume that gravitational focussing can be ignored.

(ii) An ejected black hole with parameters as given in Part (i) passes through the disc of a galaxy with stellar surface density $50\,\mathrm{pc^{-2}}$. On average how many stars will be directly swallowed during its passage through the disc?

Estimate how much energy must be dissipated for a main sequence star in the galactic disc to end up bound to the black hole. Is it plausible that this could be achieved via the dissipation of tides raised in the star?

The change in velocity of a star of mass m_* encountering a point mass $M \gg m_*$ with impact parameter b and velocity at infinity v_∞ is given by $\Delta v = 2GM/(bv_\infty)$, where G is the gravitational constant. Assuming that the galaxy has parameters similar to the Milky Way, make a rough estimate of the size of the patch of disc over which stars are torn out of the galaxy by the passage of the black hole. Hence estimate the number of stars lost from the galaxy per black hole encounter.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 2 June 2016 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 7Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) A photon trajectory in the Schwarzschild geometry of a point mass M satisfies

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2,$$

where $u \equiv 1/r$, r is the radial coordinate, ϕ is the azimuthal angle, and G and c are the gravitational constant and speed of light, respectively. By perturbing about the straight-line path $b = r\sin\phi$, where b is the impact parameter, show that light from a distant source is deflected by the mass M through an angle

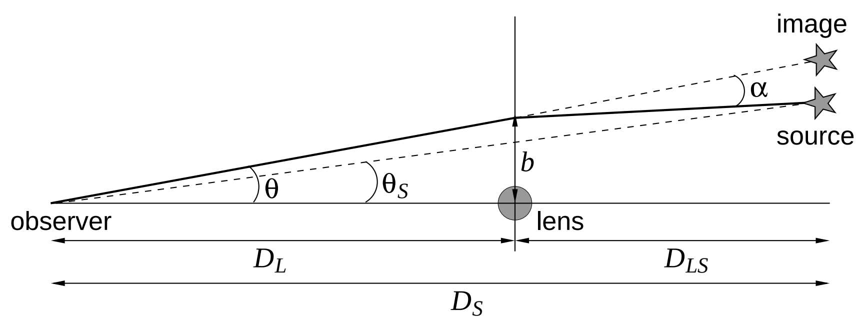
$$\Delta\phi = \frac{4GM}{c^2b}. \qquad (*)$$

How does this answer differ from the deflection predicted by Newtonian gravity?

(ii) Gravitational lensing of a distant source by a point mass M is described by the lensing equation

$$\alpha D_{LS} + \theta_S D_S = \theta D_S, \qquad (**)$$

where the distances and small angles are defined in the figure below. Specifically, θ is the *observed* angle of the source away from the centre of the lens, θ_S would be the angle of the source with no lensing, α is the deflection angle for the impact parameter b of the deflected light ray, D_L and D_S are the distances to the lens and source, respectively, and D_{LS} is the distance from lens to source.



For a point mass M, the deflection angle α is given by $(*)$ from Part (i). Show that the lens equation $(**)$ can be written as

$$\theta_E^2/\theta = \theta - \theta_S, \qquad (\dagger)$$

where

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_S D_L}}.$$

Writing $x = \theta/\theta_E$ and $y = \theta_S/\theta_E$, show that if $y \neq 0$, the solution of (†) predicts two images located at

$$x = \frac{1}{2}\left(y \pm \sqrt{y^2+4}\right).$$

Discuss the solution for $y = 0$.

For a circularly-symmetric extended lens that is thin compared to D_L and D_{LS}, one can take the deflection angle at impact parameter ξ from the lens centre to be

$$\alpha(\xi) = \frac{4GM(\xi)}{c^2\xi},$$

where $M(\xi)$ is the projected mass contained within radius ξ. If the lens is a uniform mass sheet of surface density

$$\Sigma_{\rm crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L},$$

show that such a lens focuses rays from an on-axis source ($\theta_S = 0$) to the observer for any θ, i.e., it behaves as a perfect lens.

Question 2Y - Astrophysical Fluid Dynamics

(i) Explain the physical meaning of shear viscosity and describe the conditions for which it occurs.

Consider a very thin ring of matter orbiting a star of mass M at a distance R_0. Qualitatively explain what happens to this ring as a function of time for an ideal incompressible fluid.

Assume now that the same ring has a kinematic viscosity ν. Sketch the evolution of the surface density of the ring Σ versus distance to the star R. What happens to the mass and angular momentum in the ring on very long timescales?

Consider a short cylinder of radius $r_{\rm c}$ filled with an incompressible fluid of density $\rho_{\rm f}$ and kinematic viscosity $\nu_{\rm f}$. Assuming that the fluid is in solid body rotation with the cylinder which spins at angular velocity $\Omega_{\rm c}$, calculate how pressure $p_{\rm f}$ changes with radius r, if the pressure at the centre of the cylinder is $p_{\rm f,0}$.

(ii) Consider an ideal fluid subject to a spherically symmetric radial body force which for time $t > 0$ maintains the flow

$$\boldsymbol{u}(r,t) = \begin{cases} -\omega r(\omega t+1)^{-2}\,\hat{\boldsymbol{r}}, & \text{for } r < R, \\ -\omega R^3 r^{-2}(\omega t+1)^{-2}\,\hat{\boldsymbol{r}}, & \text{for } r > R, \end{cases}$$

where ω and R are constants, and r is the distance from the origin in the direction of the unit vector $\hat{\boldsymbol{r}}$. Calculate how the density ρ changes as a function of time within $r < R$, given that the density in this region always remains uniform and is $\rho = \rho_0$ at $t = 0$.

For $r > R$ the density can be expressed as $\rho(r,t) = \exp(g(t) + k(r))$, where the functions $g(t)$ and $k(r)$ depend only on t and r, respectively. Find the solution which ensures continuity across $r = R$. Is this continuity condition physically necessary?

Compute and sketch the trajectory $r(t)$ of a fluid element which crosses $r = R$ at some time $t = \tau$.

[You may assume that the divergence of a spherically symmetric vector field $\boldsymbol{u} = u_r\hat{\boldsymbol{r}}$ is $\nabla.\boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)$.]

Question 3X - Physical Cosmology

(i) Define the comoving particle horizon in cosmology.

Show that in a spatially-flat universe composed of pressure-free matter and a cosmological constant, the comoving particle horizon at redshift z is

$$r_{\mathrm{H}}(z) = \int_{z}^{\infty} \frac{c\,dz'}{H_0\,\left[\Omega_{\mathrm{m},0}\,(1+z')^3 + \Omega_{\Lambda,0}\right]^{1/2}}\,, \qquad (*)$$

where $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ are the present-day contributions to the critical density by, respectively, matter and the cosmological constant, H_0 is the present-day value of the Hubble parameter, and c is the speed of light. You should take the present-day scale factor to be $a_0 = 1$.

(ii) Evaluate the integral $(*)$ of Part (i) in the case of an Einstein-de Sitter universe ($\Omega_{\mathrm{m},0} = 1$ and $\Omega_{\Lambda,0} = 0$).

If $\Omega_{\Lambda,0} \geq 0$, the universe will continue to expand so that the scale factor $a \to \infty$ and the redshift $z \to -1$. Sketch the comoving particle horizon $r_{\mathrm{H}}(z)$ in the interval $-1 \leq z \leq 2$ for the following two cosmological models:

(a) an Einstein-de Sitter universe; and

(b) our own Universe, with parameters $\Omega_{\mathrm{m},0} \simeq 0.3$, and $\Omega_{\Lambda,0} \simeq 0.7$. (You do not need to evaluate the integral $(*)$ explicitly for this case.)

You should assume that the quantity $H_0\sqrt{\Omega_{\mathrm{m},0}}$ takes the same value in the two models.

For both models, discuss the main features of $r_{\mathrm{H}}(z)$ that you have sketched.

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) The extinction due to the presence of dust in the interstellar medium of the Galaxy is given by $A_\lambda = 0.8\lambda^{-1}\,\mathrm{mag\,pc^{-1}}$ where λ is the wavelength of light in nm. Calculate the extinction in the U, B and V bands experienced by light from a star at a distance of 5 kpc from an observer. The effective wavelengths of the U, B and V bands may be taken as 365, 430 and 550 nm, respectively.

The effect of interstellar extinction is to shift the positions of stars in a U–B versus B–V two-colour diagram. Sketch the shift in the (U–B, B–V) two-colour plane, specifying the slope and indicating the direction of increasing extinction.

Discuss under what circumstances the position of a star in the (U–B, B–V) two-colour plane can be used to infer the intrinsic colour of the star in the absence of extinction.

(ii) A double-lined spectroscopic binary system has a period $P = 10\,\mathrm{yr}$ and shows peak-to-peak wavelength shifts of 1.2 Å and 0.2 Å measured using the Hα line which has a rest wavelength of 6563 Å. The stars are in circular orbits and the system is observed edge-on. Calculate the orbital velocities and the masses of the two stars.

The binary is observed to have a bolometric magnitude $m = 15.00$ out of eclipse and shows a primary eclipse with minimum brightness $m = 15.50$ and a secondary eclipse with minimum brightness $m = 15.01$. Both eclipses have a maximum duration of 5.0 days and both show a flat-bottomed central minimum of duration 4.0 days. Calculate the radii of the two stars.

If the low-mass star has an effective temperature $T_{\rm eff} = 10^4\,\mathrm{K}$, calculate the effective temperature of the high mass star. The effects of limb darkening may be neglected.

Question 5Z - Statistical Physics

(i) Consider an ideal gas consisting of N particles of mass m moving freely with Hamiltonian $H = \boldsymbol{p}^2/(2m)$, where $\boldsymbol{p}$ is the momentum. Derive the partition function

$$Z_{\text{ideal}} = \frac{V^N}{\lambda^{3N} N!} ,$$

in the canonical ensemble, where V is the volume and the thermal de Broglie wavelength $\lambda = AT^{-1/2}$ is a function of the temperature T and A is a constant. Determine the proportionality factor A. You may use that $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$.

By using the free energy $F = -k_{\text{B}} T \ln Z_{\text{ideal}}$, where k_{B} is Boltzmann's constant, derive the ideal gas equation of state for the pressure $p(N, T, V)$.

(ii) A monatomic gas of interacting particles is described by a modification of the ideal gas where any pair of two particles with separation r interact through a potential energy term $U(r)$. The partition function for this gas can be written as

$$Z = Z_{\text{ideal}} \left[1 + \frac{N}{2V} \int f(r) d^3 r \right]^N ,$$

where $f(r) = e^{-\beta U(r)} - 1\,, \quad \beta = 1/(k_{\text{B}} T)$, where k_{B} is Boltzmann's constant. Use the free energy as in Part (i) to show that the coefficient $B_2(T)$ in the virial expansion for small densities (N/V),

$$\frac{p}{k_{\text{B}} T} = \frac{N}{V} + B_2(T) \frac{N^2}{V^2} + \mathcal{O}\left(\frac{N^3}{V^3} \right) ,$$

is given by

$$B_2(T) = -2\pi \int_0^\infty f(r) r^2 dr \,.$$

The Lennard-Jones potential is

$$U(r) = \epsilon \left(\frac{r_0^{12}}{r^{12}} - 2 \frac{r_0^6}{r^6} \right) ,$$

where ϵ and r_0 are constants. Find the root σ such that $U(\sigma) = 0$ and the location $r_{\min}$ of the minimum of this potential in terms of r_0 and sketch the graph $U(r)$.

Calculate $B_2(T)$ for the Lennard-Jones potential using the approximations that $\exp(-\beta U) \approx 0$ for $r \le \sigma$ and $\beta U \ll 1$ for $r > \sigma$.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) A three-dimensional oscillator of mass m and a characteristic frequency ω has Hamiltonian

$$H = \frac{1}{2m}(\,\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2\,) \;+\; \frac{1}{2}m\omega^2(\,\alpha^2\hat{x}_1^2 + \beta^2\hat{x}_2^2 + \gamma^2\hat{x}_3^2\,),$$

where the dimensionless constants α, β, γ are real and positive. Assuming a unique ground state, construct the general normalised eigenstate of H and give a formula for its energy eigenvalue. Results for a one-dimensional oscillator need not be proved if they are stated clearly, with precise definitions.

List all states in the four lowest energy levels in the cases:
(a) $\alpha < \beta < \gamma < 2\alpha$; (b) $\alpha = \beta = 1$ and $\gamma = 1 + \epsilon$, where ϵ is small and positive.

(ii) Consider the Hamiltonian defined in Part (i) with $\alpha = \beta = \gamma = 1$ subject to a perturbation

$$\delta V = \lambda m\omega^2(\,\hat{x}_1\hat{x}_2 \,+\, \hat{x}_2\hat{x}_3 \,+\, \hat{x}_3\hat{x}_1\,),$$

where λ is small. Compute the changes in energies for the ground state and the states at the first excited level of the original Hamiltonian, working to the leading order at which non-zero corrections occur. General results from perturbation theory may be used without proof.

Explain in outline how the energy levels of the perturbed Hamiltonian could be found exactly by an appropriate change of axes.

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Starting with a Maxwellian distribution function $f(\mathcal{E})$ with constant velocity dispersion σ, i.e.,

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{\mathcal{E}}{\sigma^2}\right),$$

where $\mathcal{E} = -E$, E is energy and ρ_1 is a constant, derive the following differential equation for the radial dependence of density $\rho(r)$ of a singular isothermal sphere

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\left(\frac{4\pi G}{\sigma^2}\right) r^2 \rho,$$

where G is the gravitational constant.

Show that

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

is a solution to this equation.

Which astrophysical objects appear to be well described as an isothermal sphere?

(ii) A self-consistent isotropic stellar system has an ergodic distribution function and a power-law density profile

$$\rho = \rho_0 \left(\frac{r_0}{r}\right)^\alpha,$$

where $1 < \alpha < 3$. Derive the radial velocity dispersion $\overline{v_r^2}$.

What does $\overline{v_r^2}$ become in the case $\alpha = 2$ of a singular isothermal sphere?

Consider a galaxy with a spherically symmetric distribution of mass, which is dominated by its dark matter component and has a density $\rho \propto r^{-2.5}$. The mass enclosed within $r_{\rm max} = 100\,{\rm kpc}$ is $10^{12} M_\odot$, and the radial velocity dispersion of stars in the halo of the galaxy at that radius is $100\,{\rm km\,s^{-1}}$. If the density of stars in the halo follows $\nu \propto r^{-\gamma}$, evaluate the power law index γ.

[You might find the following integral useful $\int_0^\infty x^2 e^{-\frac{x^2}{2a^2}} dx = \sqrt{\frac{\pi}{2}} a^3$.]

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) Molecules of carbon monoxide (CO) are desorbed from the surface of a grain of CO ice of temperature T at a rate in s^{-1} per molecule of

$$R_{\rm des} = 1.6 \times 10^{11} \sqrt{\frac{E_{\rm CO}}{\mu_{\rm CO}}} \exp\left(-\frac{E_{\rm CO}}{k_{\rm B}T}\right),$$

where $\mu_{\rm CO}$ is the relative molecular weight of CO, and $k_{\rm B}$ is Boltzmann's constant. Suggest a physical interpretation of $E_{\rm CO}$.

Show that if the ice grain has radius s and is closely packed with density ρ_s, the timescale for the grain to be destroyed by progressive desorption from its surface is given by

$$t_{\rm des} = f \frac{s}{R_{\rm des}} \left(\frac{\rho_s}{\mu_{\rm CO} m_{\rm p}}\right)^{1/3},$$

where $m_{\rm p}$ is the mass of a proton and f is a factor of order unity that you need not evaluate.

Show that for $T = 20\,{\rm K}$, $t_{\rm des}$ is a few Myr for a grain of size 1 cm, given that $E_{\rm CO}/k_{\rm B} = 850\,{\rm K}$, $\rho_{\rm s} = 2000\,{\rm kg\,m^{-3}}$, and the atomic masses of carbon and oxygen are 12 and 16 respectively.

(ii) Gas of density ρ orbiting in a disc at radius R from the central star is in a state of radial force balance between gravity, the centrifugal force and the effect of the radial pressure gradient dP/dR. Show that if the tangential velocity of the gas $v_{\rm g}$ is nearly Keplerian, it can be approximated by

$$v_{\rm g} \approx v_{\rm K} + \frac{1}{2} \frac{R}{\rho v_{\rm K}} \frac{dP}{dR},$$

where $v_{\rm K}$ is the Keplerian velocity.

An ice grain of density $\rho_{\rm s}$ and radius s in a nearly Keplerian orbit in a circumstellar gas disc experiences a tangential acceleration due to gas drag given by $(v_{\rm g} - v_{\rm K})/t_{\rm s}$, where $t_{\rm s}$ is the drag timescale. Write down an expression for the torque about the central star acting on the grain.

Assuming that the grain remains in a nearly Keplerian orbit and that the effect of drag on the radial motion of the grain can be ignored, show that the timescale of in-spiral of the grain as a consequence of gas drag is

$$t_{\rm sp} = \frac{\rho t_{\rm s} v_{\rm K}^2}{R|dP/dR|}.$$

An ice grain is orbiting a star of mass $1M_\odot$ at a radius of $100\,\mathrm{au}$ where the temperature is $20\,\mathrm{K}$ and where the drag time t_s is 300 years. Estimate t_sp asssuming that the disc is isothermal and that the density in the disc varies as $\rho \propto R^{-1}$.

Use the data given in Part (i) to discuss the extent to which the grain is able to spiral inwards before it is desorbed.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 3 June 2016 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

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Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

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Question 1X - Relativity

(i) The Roberston–Walker metric for a spatially-flat cosmology in spherical polar coordinates is

$$ds^2 = c^2dt^2 - a^2(t)\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right), \qquad (*)$$

where $a(t)$ is the scale factor and c is the speed of light. Show that $(*)$ satisfies the Einstein field equations in the absence of matter, but with a cosmological constant Λ,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0,$$

if

$$a \propto e^{Ht} \qquad \text{with} \qquad H = \sqrt{\frac{\Lambda c^2}{3}}.$$

You may assume that the non-zero components of the Ricci tensor of the metric $(*)$ are $R_{tt} = 3\ddot{a}/a$, $R_{rr} = R_{\theta\theta}/r^2 = R_{\phi\phi}/(r^2\sin^2\theta) = -(a\ddot{a} + 2\dot{a}^2)/c^2$, where overdots denote differentiation with respect to t.

Radiation is emitted from a point P at time t. Show that this will never be received by a comoving observer whose proper spatial distance from P at time t exceeds c/H.

(ii) Consider a hyperboloid

$$x^2 + y^2 - z^2 = 1/H^2, \qquad (**)$$

where H is a positive constant, embedded in (2+1)-dimensional Minkowski space with metric

$$ds^2 = dz^2 - dx^2 - dy^2.$$

Show that by introducing coordinates t and ξ, such that

$$\begin{aligned} z &= \frac{1}{H}\sinh(Ht) + \frac{H}{2}\xi^2 e^{Ht}, \\ x &= \frac{1}{H}\cosh(Ht) - \frac{H}{2}\xi^2 e^{Ht}, \\ y &= e^{Ht}\xi, \end{aligned}$$

the constraint $(**)$ is satisfied.

Show that the metric of the hyperboloid in the (t, ξ) coordinates is

$$ds^2 = dt^2 - e^{2Ht}d\xi^2,$$

i.e., the two-dimensional analogue of the spatially-flat cosmology considered in Part (i).

By considering $x + z$, or otherwise, determine what part of the hyperboloid is covered by the (t, ξ) coordinates. Illustrate your answer with a sketch.

By appropriate choice of the time coordinate t, the spatial sections of the hyperboloid can be spatially flat (as here) or have positive or negative curvature, irrespective of the value of the constant H. Why is this possible?

TURN OVER...

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a supernova explosion producing a strong spherically symmetric and thin blastwave which propagates into a uniform medium of density ρ_0 and pressure p_0. Assuming that the pressure within the blastwave cavity is half that of the shocked material p_1, and that $p_1 \gg p_0$, calculate the rate of expansion of the blastwave radius R with time t. You may seek solutions of the form $R \propto t^b$, where b is a constant.

By considering the time dependence of p_1, discuss qualitatively what happens on very long timescales.

How would the solution derived above be affected had the blastwave propagated into a medium with a patchy density distribution?

(ii) Consider a dark matter halo with an isothermal density profile $\rho_{\rm DM} = \sigma^2/(2\pi G r^2)$, where G is the gravitational constant, σ is a constant and r is the radial distance. This halo is filled with gas with a density profile $\rho(r) = f_{\rm g}\rho_{\rm DM}(r)$, where $f_{\rm g}$ is the universal baryon fraction. At the centre of this halo a supermassive black hole of mass M is emitting at its Eddington luminosity $L_{\rm Edd} = 4\pi GMc/\kappa$, where κ is the gas opacity and c is the speed of light. The radiation pressure exerted by these photons sweeps up the gas and drives a forward propagating shell whose thermal pressure is negligible due to the radiative gas cooling. Assuming that all photons are absorbed by the shell, that the mass of the black hole stays constant, and considering the gravitational force exerted on the shell, derive the rate of change of momentum of this shell.

Thus show that there is a critical black hole mass of $M_\sigma = \kappa f_{\rm g}\sigma^4/(\pi G^2)$ such that the radial expansion of the shell always decelerates if $M < M_\sigma$.

In the limit that the shell launching speed is much larger than the escape velocity from the black hole, derive that the maximum shell radius for $M < M_\sigma$ is

$$R_{\rm max} = R_0\sqrt{1 + \frac{\dot{R}_0^2}{2\sigma^2(1 - M/M_\sigma)}}\,,$$

where R_0 and $\dot{R}_0$ are the initial radius and expansion velocity of the shell, respectively.

Discuss qualitatively whether you would expect the shell to expand faster or slower had its thermal pressure not been negligible.

Question 3X - Physical Cosmology

(i) Show that the primordial abundance of helium by mass is

$$Y_{\rm P} \approx 2\left(1+\frac{n_{\rm p}}{n_{\rm n}}\right)^{-1},$$

where $n_{\rm p}/n_{\rm n}$ is the ratio (by number) of protons to neutrons, if all the baryons are in H and ^{4}He.

Consider an alternative universe, described by the same cosmological parameters as our own, but with the one difference that a force of unknown origin further decelerated the universal expansion between times $t_1 = 1\,\rm s$ and $t_2 = 300\,\rm s$. Discuss whether the mass fraction of helium in this alternative universe would be larger or smaller than in our own Universe.

(ii) The Lyα absorption cross-section for radiation of (angular) frequency ω is approximately

$$\sigma_{{\rm Ly}\alpha}(\omega) = \frac{3}{4}\Lambda\lambda_\alpha^2\delta(\omega-\omega_\alpha),$$

where $\Lambda = 6.25\times10^8\,\rm s^{-1}$ is the $2p \to 1s$ decay rate, $\lambda_\alpha = 1.22\times10^{-7}\,\rm m$ is the wavelength of Lyα radiation, $\omega_\alpha = 2\pi c/\lambda_\alpha$, c is the speed of light, and $\delta(\omega)$ is the Dirac delta function. Consider Lyα absorption by a uniform intergalactic medium with (proper) number density of hydrogen atoms $n_{\rm H\,I}(z)$ at redshift z. Show that the optical depth for absorption of radiation with observed frequency $\omega_\alpha/(1+z)$ is

$$\tau_{{\rm Ly}\alpha}(z) = \frac{3\Lambda\lambda_\alpha^3 n_{\rm H\,I}(z)}{8\pi H(z)}, \qquad (*)$$

where $H(z)$ is the Hubble parameter at redshift z.

Explain, with the aid of a sketch, the effect that an intergalactic medium with uniform $n_{\rm H\,I}$ would have on the spectrum of a distant quasar. How does this prediction compare with observed quasar spectra? Give a physical interpretation of any difference.

At redshift $z = 3$, the average Lyα optical depth is $\langle\tau_{{\rm Ly}\alpha}\rangle \simeq 0.5$. Assuming that $H(z) = 300\,\rm km\,s^{-1}\,Mpc^{-1}$ at $z = 3$, use $(*)$ to estimate the average $n_{\rm H\,I}$ in the intergalactic medium at that redshift.

Given that the critical density for a Hubble constant $H_0 = 100\,\rm km\,s^{-1}\,Mpc^{-1}$ corresponds to a density of hydrogen atoms of $10\,\rm m^{-3}$, estimate the fraction of hydrogen atoms in the intergalactic medium that are not ionized at $z = 3$.

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) Five stars have effective temperatures $T_{\rm eff}$, luminosities L and masses M as given in the table below.

Star	$T_{\rm eff}$(K)	$L(\mathrm{L_\odot})$	$M(\mathrm{M_\odot})$
Spica A	25400	13400	10.9
Vega	10000	51	2.6
Sun	5680	1	1
Aldebaran	4000	150	2.5
Sirius B	24800	0.024	0.98

Assume that the radiation produced by the stars can be approximated by a blackbody. Use Wien's Law to estimate the wavelengths of the peak emission for each of the stars and state in which wavelength range (e.g., X-ray, optical) they occur.

Determine the radii, R, for each of the stars and sketch where they lie in a Hertzsprung-Russell diagram.

State the corresponding phase of stellar evolution for each star.

(ii) Stars on the zero-age main sequence formed in a cluster are all homogeneous and have the same chemical composition. If the stars are fully radiative, using homology arguments calculate the form of the luminosity-mass relation for intermediate mass stars. You may assume that the opacity dependence on temperature T is of the form $\kappa \propto \rho^n T^{-\alpha}$, and is given by Thomson opacity, $n = 0$ and $\alpha = 0$, for the higher mass stars and by Kramers opacity, $n = 1$ and $\alpha = 3.5$, for the lower mass stars.

Assuming that the stellar masses are constant, how will the shape of the luminosity-mass relation change as the cluster ages?

The nuclear energy generation rate per unit mass is $\epsilon \propto \rho T^{\beta}$, with $\beta = 4$ for the p-p chain and $\beta = 18$ for the CNO cycle. Determine the radius-mass relation for the stars and hence quantify how the density of the stars changes with mass.

Question 5Z - Statistical Physics

(i) Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V ,$$

where S is entropy, V is volume, p is pressure, and T is temperature.

Consider a thermodynamic system whose energy E at constant temperature T is volume independent, i.e.,

$$\left(\frac{\partial E}{\partial V}\right)_T = 0 .$$

Show that this implies that the pressure is of the form $p(T,V) = T\,f(V)$ for some function f.

(ii) For a photon gas inside a cavity of volume V, the energy E and pressure p are given in terms of the energy density U, which only depends on the temperature T by

$$\frac{E(T,V)}{V} = U(T) , \qquad p(T,V) = \frac{1}{3}U(T) .$$

Show that this implies $U(T) = aT^4$ where $a = \text{const}$.

Show that the entropy is given by

$$S = \frac{4}{3}aVT^3 ,$$

and calculate the thermodynamic potentials, namely, the internal energy $E(S,V)$, the Helmholtz free energy $F(T,V)$, the Gibbs free energy $G(T,p)$, and the Enthalpy $H(S,p)$, each in terms of its respective fundamental variables.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Consider a quantum system with Hamiltonian $H = H_0 + V$, where H_0 is independent of time and the potential V may be time-dependent. Define the interaction picture corresponding to this Hamiltonian and derive an expression for the time derivative of an operator in the interaction picture, assuming it is independent of time in the Schrödinger picture.

The Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ satisfy

$$\sigma_i \sigma_j \;=\; \delta_{ij} \;+\; i\epsilon_{ijk}\,\sigma_k \;.$$

Explain briefly how these properties allow $\boldsymbol{\sigma}$ to be used to describe a quantum system with spin 1/2.

(ii) A particle of spin 1/2 has position and momentum operators $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$. What is the unitary operator corresponding to a rotation through an angle θ about an axis $\boldsymbol{n}$? Check your answer by considering the effect of an infinitesimal rotation on $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{p}}$ and $\boldsymbol{\sigma}$.

Suppose now that this particle has Hamiltonian $H = H_0 + V$ with

$$H_0 = \frac{1}{2m}\hat{\boldsymbol{p}}^2 \;+\; \alpha\,\boldsymbol{L}\cdot\boldsymbol{\sigma} \qquad \text{and} \qquad V = B\,\sigma_3,$$

where $\boldsymbol{L}$ is the orbital angular momentum and α, B are constants. Using results from Part (i) or otherwise, show that all components of the total angular momentum $\boldsymbol{J}$ are independent of time in the interaction picture. Is this true in the Heisenberg picture? You may quote commutation relations of $\boldsymbol{L}$ with $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$.

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Non-gravitating gas can naturally exist at constant density and temperature. Explaining your reasoning, discuss whether a stellar system can survive in an equilibrium, isothermal, homogeneous state.

Describe the situations in which it might be useful to employ Eddington's formula for spherically symmetric density distributions

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{d\mathcal{E}} \int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\mathcal{E} - \Psi}} \frac{d\nu}{d\Psi},$$

where $\Psi = -\Phi + \Phi_0$ is the relative potential, $\mathcal{E} = -E + \Phi_0$ is the relative energy and ν is density.

Eddington's inversion relies on monotonicity of the gravitational potential in spherical systems. Prove that the potential $\Phi(r)$ is indeed a non-decreasing function of r in any spherical system.

(ii) Consider a particle whose offset from the origin is defined in the spherical polar coordinate system (r, θ, ϕ). The particle's velocity in this coordinate system has components v_r, v_θ and v_ϕ. Show that these components can be written as

$$\begin{aligned} v_r &= v \cos\alpha, \\ v_\theta &= v \sin\alpha \cos\delta, \\ v_\phi &= v \sin\alpha \sin\delta, \end{aligned}$$

where v is the magnitude of the particle's velocity, and the geometrical significance of the angles α and δ should be described with the aid of a sketch or otherwise.

The distribution function of a spherical system is proportional to $L^\gamma f(E)$, where L is the angular momentum, E is the energy, and γ is a constant. Derive expressions for the components of the velocity dispersion tensor $\overline{v_r^2}, \overline{v_\theta^2}$ and $\overline{v_\phi^2}$.

The anisotropy parameter is defined as

$$\beta = 1 - \frac{\overline{v_\theta^2} + \overline{v_\phi^2}}{2\overline{v_r^2}}.$$

Show that at all radii $\beta = -\gamma/2$.

[You might find the following reduction equation useful: $\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$.]

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) The age of a white dwarf is related to its luminosity L by

$$\ln(t) = C - (5/7)\ln(L),$$

where t is measured from the point of white dwarf formation and C is a constant that is independent of the mass of the progenitor star. An observer measures that within a fixed volume the number of white dwarfs with luminosity in the range L to $L + dL$ is given by $n(L)dL$ where $n(L) \propto L^{-\beta}$ and $\beta = 12/7$. Show that this is consistent with a steady state population with constant star formation rate.

Explain without detailed calculation what could be inferred if the observer had measured $\beta < 12/7$, and comment on how this inference would have been affected had the sample been magnitude-limited.

(ii) Write down an expression for the rate of pdV work done per unit mass in an isothermal spherical wind of sound speed c_s at a radius r where the local gas velocity u is independent of radius. Hence or otherwise show that if the wind is in a steady state with mass loss rate $\dot{M}$, the total rate of pdV work done by the gas between radii $r_{\rm in}$ and $r_{\rm out}$ is

$$\dot{W} = 2\dot{M}c_s^2 \ln\left(\frac{r_{\rm out}}{r_{\rm in}}\right).$$

An isothermal spherical wind surrounding an O star has temperature $T = 10^4$ K and is observed to extend between radii $r_1 = 100$ au and $r_2 = 1000$ au. At a radius of 1000 au the density and expansion velocity are estimated to be $10^{-22}\,{\rm kg\,m^{-3}}$ and $20\,{\rm km\,s^{-1}}$, respectively. Estimate a lower bound to the luminosity of the O star if it produces 10^{49} ionizing photons per second, and determine if the stellar luminosity is sufficient to offset pdV energy losses in the wind.

If the wind has roughly constant velocity between r_1 and r_2, write down an expression for the total recombination rate between r_1 and r_2, stating whether this quantity is dominated by conditions at large or small radii. Is the stellar luminosity sufficient to keep the wind ionized out to infinite radius?

[You may assume that the rate of recombination per unit volume in a fully ionized plasma of number density n is given by $\alpha_{\rm B} n^2$ where $\alpha_{\rm B} = 2.6 \times 10^{-7}\,{\rm m^3\,s^{-1}}$ and n is in ${\rm m^{-3}}$.]

END OF PAPER