NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 1 June 2015 13.30pm – 16.30pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 7Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) State the two necessary conditions for the Lorentz transformations to apply.

Using 4-vector notation determine the equation of motion of a particle moving along the x-axis whose acceleration stays constant in its own reference frame.

(ii) Inertial frame S' moves with a velocity $\vec{v}$ along the x-axis with respect to inertial frame S. A cylinder of length l' and radius r' is at rest in frame S' and makes an angle θ' with respect to the forward direction of motion. Express the cylinder's length l in frame S as a function of l', θ' and β, where $\beta = v/c$, $v = |\vec{v}|$ and c is the speed of light.

Draw the cylinder's projection on the $x - y$ plane in both frames to deduce how its shape is transformed in frame S.

Derive an expression for the area of the cylinder's lid surfaces in frame S as a function of r', θ' and β.

With the aid of the drawing, or otherwise, explicitly calculate the cylinder's volume V and V' in frames S and S', respectively, and find a simple expression to relate the two.

[You may find the following relation useful: $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$.]

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a circumstellar disk with Keplerian orbital frequency Ω and mid-plane density ρ_0 whose gravity is dominated by a central star of mass M. Under the assumption of hydrostatic equilibrium derive how gas density ρ varies as a function of height z above the disk's mid-plane at a radial distance r, and sketch ρ as a function of z. You may assume that gas temperature T and mean molecular weight μ are independent of z and that $z \ll r$.

If a gas atom which is part of this circumstellar disk is launched upwards from the mid-plane of the disk with its typical thermal velocity calculate the maximum height h_{max} above the disk that it can attain, and comment on the result obtained. You may ignore collisions with other gas atoms.

(ii) Consider the circumstellar disk discussed in Part (i) and assume a central star mass $M = 1M_\odot$. A spherical dust grain of radius s_{d} and mass m_{d} in the disk experiences a drag force

$$\vec{F}_{\mathrm{D}} = -4\pi\, s_{\mathrm{d}}^2\, \rho\, c_{\mathrm{s}}\, \Delta\vec{v}_{\mathrm{d}},$$

where ρ is the disk gas density, c_{s} is the sound speed in the disk and $\Delta\vec{v}_{\mathrm{d}}$ is the relative velocity of the dust particle with respect to gas. The dust particle will move together with the gas if its momentum stopping time,

$$t_{\mathrm{stop}} = m_{\mathrm{d}}\, |\Delta\vec{v}_{\mathrm{d}}| \,/\, |\vec{F}_{\mathrm{D}}|,$$

is comparable to the dynamical time of the disk t_{dyn}. Using the results from Part (i) or otherwise, estimate the size of dust particle of density $\rho_{\mathrm{s}} = 1\,\mathrm{g/cm^3}$ for which $t_{\mathrm{stop}} \approx t_{\mathrm{dyn}}$ in the mid-plane of the disk at a radius $r = 1\,\mathrm{au}$, where the gas disk surface density is $\Sigma(1\mathrm{au}) = 10^3\,\mathrm{g/cm^2}$.

A particle of radius $s_{\mathrm{d}} = 1\,\mu\mathrm{m}$ is at a height $z = c_{\mathrm{s}}/\Omega$ above the disk mid-plane at $r = 1\,\mathrm{au}$. Find an expression for the speed at which the particle settles towards the mid-plane of the disk in terms of ρ_{d}, s_{d}, c_{s} and Σ.

Determine whether the dust particle settles towards the mid-plane sub- or super-sonically.

Estimate how many years it takes for the dust particle to fall to half of its original height above the mid-plane.

TURN OVER...

Estimate the velocity difference Δv between the gas orbital velocity v at the mid-plane of the disk and the Keplerian velocity $v_\mathrm{k} = \Omega r$ at the same radius r as a function of c_s and v_k alone. You may assume that $dp/dr \sim -p/r$, where p is the gas pressure.

Hence calculate a numerical value for Δv at $r = 1\,\mathrm{au}$ assuming that the temperature $T(1\mathrm{au}) = 200\,\mathrm{K}$ and that the disk is made of hydrogen atoms.

Question 3X - Physical Cosmology

(i) The two Friedmann equations are

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3}\,,$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}\,,$$

where R is the scale factor, ρ is the mass density, p is pressure, k is a constant that specifies the geometry of the universe, Λ is the cosmological constant and c is the speed of light. Use these equations to derive the fluid equation in the following form

$$\frac{d}{dt}(\rho R^3) + \frac{p}{c^2}\frac{d}{dt}(R^3) = 0.$$

Assume that $p = w\rho c^2$, where w is a constant. Show that

$$\rho R^{3(1+w)} = \text{constant}\,,$$

and from this equation deduce how density depends on redshift in matter-, radiation-, and cosmological constant-dominated universes.

(ii) Explain briefly what Big Bang Nucleosynthesis (BBN) is. What is the dominant component of the Universe during the BBN and what are the main products of BBN?

In which important way does the BBN differ from the nucleosynthesis occurring in the cores of stars?

During nucleosynthesis the effective degeneracy factor g_* is given by

$$g_* = \left[\sum g_i\right]_{\text{bosons}} + \left[\frac{7}{8}\sum g_i\right]_{\text{fermions}}\,,$$

where g_i is the number of possible states of species i. In the standard model of particle physics there are 3 neutrino types and for each type there is a particle and its corresponding anti-particle.

TURN OVER...

The following reactions

$$\nu_e + n \longleftrightarrow p + e^-$$

$$e^+ + n \longleftrightarrow p + \bar{\nu}_e$$

stop at the freeze-out temperature $T_{\rm F}$. Assuming that electron-positron annihilation and neutrino decoupling occur simultaneously at the freeze-out temperature, derive an expression for g_* before freeze-out in terms of the number of neutrino types N_ν and calculate its value for both 3 and 4 neutrino types.

Assuming that the effective degeneracy factor $g_* \propto T_{\rm F}^6$ and $k_{\rm B}T_{\rm F}$ =0.8 MeV for $N_\nu = 3$, where $k_{\rm B}$ is Boltmann's constant, show that $k_{\rm B}T_{\rm F} = 0.82\,{\rm MeV}$ for $N_\nu = 4$.

For both protons and neutrons the number density is given by

$$N = AT^{3/2}\exp(-mc^2/k_{\rm B}T)\,,$$

where m is the particle mass and A is a constant. Use this to derive an expression for the ratio of the number of neutrons to the number of protons $(N_{\rm n}/N_{\rm p})_{\rm F}$ at the freeze-out temperature.

Calculate the values of $(N_{\rm n}/N_{\rm p})_{\rm F}$ at the freeze-out temperature for both $N_\nu = 3$ and $N_\nu = 4$.

After freeze-out free neutrons are destroyed by the reaction

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

until there are no free neutrons and most of the surviving neutrons end up in He nuclei. Assuming that cosmic time $t \propto T^{-2}$ show that

$$\frac{\left[\frac{N_n}{N_p}\right]_{\rm end}}{\left[\frac{N_n}{N_p}\right]_{\rm F}} \approx \exp\left[-\frac{t_3}{\tau_n}\left(1+\frac{7}{43}\Delta N_\nu\right)^{-1/2}\right],$$

where ΔN_ν is the change in the number of neutrino types with respect to $N_\nu = 3$, t_3 is the age of the universe at the end of nucleosynthesis for $N_\nu = 3$, $\tau_n = 887\,{\rm s}$ is the free-neutron lifetime and $[N_{\rm n}/N_{\rm p}]_{\rm end}$ is the neutron to proton ratio at the end of nucleosynthesis.

If $t_3 = 300\,{\rm s}$ show that the change of the primordial helium abundance ΔY due to a change in the number of neutrino types is

$$\Delta Y \approx 0.014\Delta N_\nu.$$

Question 4Z - Structure and Evolution of Stars

(i) Explain briefly the physical basis for the two-dimensional classification of stars into the Morgan-Keenan (M-K) scheme consisting of a spectral type and a luminosity class.

Explain what is meant by the term *opacity*, and briefly describe the different sources of opacity in stellar atmospheres.

Which are the dominant sources of opacity in the atmosphere of an O5 V star? Likewise, what are the dominant opacity sources in the atmosphere of a M2 I star?

(ii) Consider a star that is fully radiative. Suppose that the energy generation rate per unit mass $\mathcal{E}$ is independent of radius r. Show that the temperature gradient in the star is given by

$$\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{3\kappa\mathcal{E}}{16\pi\, ac\, GT^3},$$

where T is the temperature, P is the pressure, κ is the opacity, a is the radiation density constant, and c is the speed of light.

If the opacity κ is also constant, as is the case for electron scattering, show that

$$\frac{T^4}{4} = \frac{T_0^4}{4} + \frac{3\kappa\mathcal{E}}{16\pi\, ac\, G}\,(P - P_0)\;,$$

where T_0 and P_0 are the temperature and pressure at the surface. Hence show that $P \approx \mathcal{C}T^4$ (where $\mathcal{C}$ is a constant) in the interior of the star where $T \gg T_0$.

A system is termed a 'polytrope of index n' if the dependence of pressure on density is of the form $P = K\rho^{(n+1)/n}$, where K is a constant. Use the result above to show that the interior of a star where the total pressure is the sum of gas pressure (assuming an ideal gas) and radiation pressure can be described as a polytrope of index $n = 3$, provided that the energy generation rate satisfies the inequality

$$\mathcal{E} < \frac{4\pi cG}{\kappa}.$$

TURN OVER...

Question 5Y - Statistical Physics

(i) Let $\mathcal{S}$ be a system with states $|n\rangle$, $n = 1, \ldots, N$, with energy E_n and occupied with probabilities $p(n)$. By considering a large number W of identical systems $\mathcal{S}$ coupled to a reservoir $\mathcal{R}$ such that the number of systems in state $|n\rangle$ is $W\,p(n)$, show that the number of possibilities to have $W\,p(n)$ systems in state $|n\rangle$ is

$$\Omega_{\mathcal{S}} = \frac{W!}{\prod_n [W\,p(n)]!}\,.$$

Hence show that the entropy of system $\mathcal{S}$ in the canonical ensemble is

$$S = -k_{\mathrm{B}} \sum_n p(n)\, \ln p(n)\,,$$

where k_{B} is Boltzmann's constant.

What is the entropy of the system in the microcanonical limit where all states $|n\rangle$ have the same energy E_n?

(ii) Consider a system of 3 fermions described by 2 quantum numbers $n \in \{0,\, 1,\, 2\}$ and $s \in \{-\frac{1}{2},\, \frac{1}{2}\}$ such that the energy of fermion i is

$$E_i = n_i \epsilon - s_i B\,,$$

where ϵ is a constant and B is the local magnetic field strength. Let this system be in contact with a reservoir of temperature T. Show that the average spin per fermion is given in terms of the canonical partition function Z by

$$\langle s \rangle = \frac{1}{3\beta} \frac{\partial}{\partial B} \ln Z\,,$$

where $\beta = 1/(k_{\mathrm{B}} T)$ and k_{B} is Boltzmann's constant.

List all possible states of the 3-fermion system and determine the partition function Z.

Calculate $\langle s \rangle$ and its leading-order term in the high-temperature limit.

Question 6Z - Principles of Quantum Mechanics

(i) If A and B are operators which each commute with their commutator $[A,B]$, show that $[A, e^B] = [A,B]e^B$.

By considering

$$F(\alpha) = e^{\alpha A} e^{\alpha B} e^{-\alpha(A+B)} ,$$

and differentiating with respect to the parameter α, show also that

$$e^A e^B = C e^{A+B} = e^{A+B} C,$$

where $C = e^{\frac{1}{2}[A,B]}$.

(ii) The annihilation and creation operators for a harmonic oscillator of mass m and frequency ω are, respectively, given by

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \quad , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) ,$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, respectively. The energy eigenstates for the Hamiltonian H are denoted by $|n\rangle$, $n = 0, 1, 2, \ldots$ with $a|0\rangle = 0$. The oscillator is subject to a small perturbation so that it is now described by the Hamiltonian $H + \lambda V(\hat{x})$ with $V(\hat{x}) = \cos(\mu \hat{x})$, where λ and μ are constants. By expressing $\hat{x}$ as a linear combination of a and $a^\dagger$, and using the results from Part (i) or otherwise, show that

$$V(\hat{x}) = e^{-\frac{\mu^2 \hbar}{4m\omega}} \operatorname{Re} \left(e^{i\mu\sqrt{\frac{\hbar}{2m\omega}}\, a^\dagger} e^{i\mu\sqrt{\frac{\hbar}{2m\omega}}\, a} \right) .$$

Hence calculate the matrix element $\langle n|V(\hat{x})|0\rangle$ for all n.

Show that to $O(\lambda^2)$ the shift in the ground state energy is

$$\lambda e^{-\frac{\mu^2\hbar}{4m\omega}} - \lambda^2 e^{-\frac{\mu^2\hbar}{2m\omega}} \frac{1}{\hbar\omega} \sum_{k=1}^{\infty} \frac{1}{(2k)!2k} \left(\frac{\mu^2 \hbar}{2m\omega} \right)^{2k} .$$

[You may quote standard results for first and second order corrections from perturbation theory.]

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) For a star cluster orbiting in the potential of a host galaxy what is the significance of the tidal radius.

Derive the tidal radius assuming that the galaxy and cluster are both point-like with masses M and m, respectively, and that the cluster is at a distance R from the galaxy.

A cluster of effective radius R_{c} starts orbiting its host galaxy at a distance for which its tidal radius is much larger than R_{c}. Describe the mechanisms which might be responsible for causing some of the cluster's stars to eventually reach beyond the tidal radius.

(ii) A stellar stream is made up of a train of stars orbiting within a host galaxy potential on the same circular orbit with velocity v_{s}. A perturber of mass M flies directly through the stream modifying the orbits of stream stars. The perturber is moving in the stream's orbital plane at a constant velocity v_{p} at an angle α to the stream, where $\cos\alpha = v_{\mathrm{s}}/v_{\mathrm{p}}$. The potential of the perturber is described by a Plummer model

$$\Phi(r) = \frac{-GM}{\sqrt{r^2 + r_{\mathrm{s}}^2}},$$

where r is the distance from the centre of the perturber and r_{s} is a constant scale radius. Use the impulse approximation to deduce how the velocity kick imparted by the perturber in the along-stream direction Δv_x changes as a function of the initial distance x of stars from the impact point in the frame moving with the stream.

Sketch how Δv_x varies with x and hence describe what happens to the stream stars immediately after the impact.

Using conservation of energy for orbits within a Keplerian potential, deduce how changes in the velocities of stream stars determine whether their orbits within the galaxy become larger or smaller in size.

Hence describe how you expect the density of the stream to evolve on long timescales.

Question 8Z - Physics of Astrophysics

(i) It is assumed that a planet cannot exist in a stable orbit at distance $a_{\rm pl}$ from a star of mass M if this star also has a binary companion with separation $a_{\rm bin}$ in the range $0.3a_{\rm pl} < a_{\rm bin} < 3a_{\rm pl}$. Sixty percent of solar mass stars have a binary companion and for these binaries the distribution of their orbital periods P is roughly flat in $\log(P)$ over the range $P_1 = 10$ days to $P_2 = 10^6$ years. What fraction of solar type stars cannot host a planet at 1 au on account of being in an unsuitable binary?

Suggest why the incidence of binaries is low outside the period range P_1 to P_2.

(ii) A rocky object falls radially towards a star, with mass M_*, starting at rest at a large distance r_0 from the star. Using conservation of energy or otherwise, write down an expression for the radial velocity of the object as a function of radial distance r from the star and hence sketch how the time spent in a given interval of log (r) depends on r.

Assuming that the object acts as a black-body and that it is in radiative equilibrium with incident radiation from the star, determine the surface temperature T of the body as a function of r. You may assume that the star radiates as a black-body of radius R_* and temperature T_*.

Derive an expression for the thermal timescale of the object $t_{\rm therm}$ as a function of T if its radius, density and specific heat capacity are R, ρ and C respectively. You may assume that the body has a uniform internal temperature.

Derive an expression for the amount of time that the object spends in equal intervals of log (T) as a function of T.

Hence derive a condition on the radius R of the object that must be satisfied in order that the object always has time to attain its thermal equilibrium surface temperature at all distances from the star.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 2 June 2015 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 2

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Tags	

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Question 1X - Relativity

(i) Determine whether an isolated free electron can absorb or emit a photon.

Consider an inertial frame S with an electron with mass m_e initially at rest. A photon with 4-momentum $\vec{p}_\gamma$ travelling along the x-axis collides with the electron. Derive the photon's frequency after the scattering ν' as a function of its initial frequency ν and its scattering angle θ.

(ii) With the aid of the electro-magnetic field 4-tensor F_{ik}, or otherwise, show that $B^2 - E^2$ and $\vec{E} \cdot \vec{B}$ are invariant under Lorentz transformations. Here $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field and the speed of light is set to 1.

If $\vec{E}$ is at an angle θ with respect to $\vec{B}$, is there an unique value of θ such that it is invariant to all observers?

By constructing the Lorentz transformations relating $\vec{E}$ and $\vec{B}$ between the two inertial frames S and S' show that if $\vec{E} \cdot \vec{B} = 0$, but $E^2 \neq B^2$, then there is a frame S' such that $\vec{E}' \times \vec{B}' = 0$.

Can $\vec{E}' \times \vec{B}' = 0$ if $|\vec{E}| = |\vec{B}|$?

[You may assume that:

$$F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} . \qquad]$$

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a steady and spherically symmetric inflow of gas onto a star of mass M. If the gas is at rest at infinity, qualitatively explain how gas velocity u will change with distance r from the star.

Derive an equation for the spatial dependence of velocity which involves only u, r, M and the gas sound speed $c_{\rm s}$, and use this to get an expression for the sonic radius $r_{\rm s}$.

For an isothermal gas with density ρ_∞ at infinity derive an expression for the mass accretion rate onto the star.

If the gas at infinity is not at rest will the accretion rate increase or decrease?

(ii) Consider a steady flow of gas through a pipe whose radius is r_1 at all locations, except in the middle of the pipe where it is reduced to r_2. In the section of the pipe with radius r_1 the gas density and pressure are ρ_1 and p_1, respectively, while in the middle of the pipe they are ρ_2 and p_2. In the limit of highly sub-sonic motion derive an expression for the rate of mass flow through the pipe.

Consider now the case where gas moving through a De Laval nozzle is initially very sub-sonic with sound speed $c_{\rm s,0}$ but eventually becomes supersonic. Find an expression for the gas velocity u and sound speed $c_{\rm s}$ at the narrowest part of the nozzle in terms of $c_{\rm s,0}$ and the adiabatic index γ, where you may assume $p \propto \rho^\gamma$.

Hence derive an expression for the gas velocity at infinity in terms of $c_{\rm s,0}$ and γ.

If the initial gas pressure is p_0 find an expression for gas pressure at the narrowest part of the nozzle and hence deduce how gas pressure changes along the whole nozzle.

Consider now a steady jet with pressure $p_{\rm jet} \propto \rho_{\rm jet}^\gamma$, where $\gamma = 5/3$ and $\rho_{\rm jet}$ is the density of the jet, moving through an external medium whose pressure $p_{\rm med}$ scales with distance x as x^{-2}. Using the results derived for the De Laval nozzle, determine how the opening angle of the jet scales with x very far along the jet, and explain the physical meaning of this result. You may assume the jet to have a circular cross-section and neglect gravity.

TURN OVER...

Question 3X - Physical Cosmology

(i) Explain the definition of angular diameter distance D_θ and show that it is given by

$$D_\theta = R(t_\mathrm{e}) r_\mathrm{e} = \frac{R(t_0) r_\mathrm{e}}{1+z},$$

where R is the scale factor, t is cosmic time, subscripts 0 and e refer to the present day epoch and the epoch when the observed photons were emitted, respectively, r_e is the comoving distance and z is the observed redshift.

In an Einstein–de Sitter universe the angular diameter distance is given by

$$D_\theta = \frac{2c}{H_0} \frac{1}{1+z} \left[1 - (1+z)^{-1/2}\right] , \qquad (*)$$

where H_0 is the value of the Hubble constant at the present day and c is the speed of light. At what redshift does the angular diameter distance reach a maximum value?

How long does light take to travel from this redshift to us assuming that $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$?

Derive how the angular diameter distance is related to the luminosity distance d_L and comment on what happens in the limit $t_\mathrm{e} \to t_0$?

(ii) In an expanding universe what are the temperature dependencies of matter density ρ_mat and radiation density ρ_rad?

Use these two temperature dependencies to show that the sound speed c_s is given by

$$c_\mathrm{s}^2 = \frac{1}{3} \left(\frac{\frac{4}{3}\rho_\mathrm{rad}}{\rho_\mathrm{mat} + \frac{4}{3}\rho_\mathrm{rad}} \right) c^2 ,$$

where c is the speed of light. What is the sound speed in the radiation-dominated era?

Assume that the universe is radiation-dominated for all epochs up to the time of recombination. Also assume an Einstein–de Sitter universe with $H_0 = 70\,\mathrm{kms^{-1}\,Mpc^{-1}}$ and redshift at recombination $z_\mathrm{r} = 1100$. Using equation $(*)$ or otherwise, derive expressions for the observed angular size and the physical proper size of the sound horizon on the last scattering surface and calculate their values.

Sketch the angular cosmic microwave background power spectrum of temperature anisotropies. On this sketch indicate the observed angular size of the horizon and sound horizon. Briefly explain the physical meaning of this result.

Explain how the measurement of the sound horizon in redshift surveys can be used as a standard ruler and calculate the minimum spatial scale that a survey of local galaxies needs to span to detect the sound horizon. Has this detection been made yet?

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) A massive star undergoes mass loss via a radiation-driven wind. Assuming that all photons transfer their entire momentum to the outflowing wind, show that the maximum mass loss rate that can be driven by radiation is given by

$$\dot{M}_{\max} = \frac{L}{v_\infty c},$$

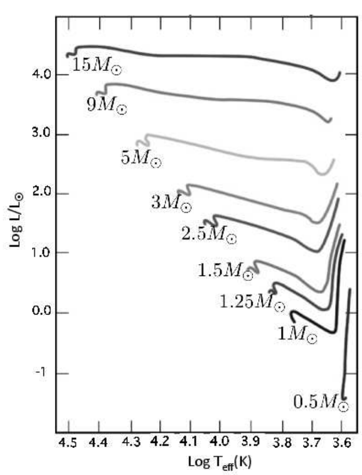
where v_∞ is the wind terminal velocity, L is the luminosity and c is the speed of light.

Show that with this maximum mass loss rate, the kinetic energy of the wind is only a small fraction of the luminosity.

(ii) In the final stage of contraction to the main sequence, most stars evolve to higher effective temperatures until the core temperature is sufficiently high to initiate the thermonuclear reactions that supply the energy radiated from the surface. State under what conditions it is reasonable to assume that the star is in hydrostatic equilibrium during this phase.

Assuming that energy transport is dominated by radiative diffusion and that the opacity follows Kramers' law, use homology arguments to derive the dependence of the star's luminosity L on its effective temperature $T_{\rm eff}$ during the contraction stage.

The figure below shows theoretical evolutionary tracks for pre-main sequence stars. How do they compare with the answer to the above question? What conclusions can be drawn from this comparison?



Question 5Y - Statistical Physics

(i) Give a brief description of the phenomenon of Bose-Einstein condensation.

A system of N bosons in an isotropic harmonic potential can be modelled as a set of harmonic oscillators of frequency ω with energy

$$E_{n_x n_y n_z} = \tilde{E} + E_0 \,,$$

where $\tilde{E} = \hbar\omega(n_x+n_y+n_z)$, n_x, n_y, $n_z \in \mathbb{N}_0$, and $E_0 = 3\hbar\omega/2$ is the zero-point energy. Show that there are $(n+2)(n+1)/2$ possible choices for n_x, n_y, n_z such that $n_x + n_y + n_z = n \in \mathbb{N}_0$.

Show that the density of states for a single particle with energy above the ground state in the range $(n-1)\hbar\omega < \tilde{E} \leq n\hbar\omega$ can be approximated for $\tilde{E} \gg \hbar\omega$ by

$$g(\tilde{E}) \approx \frac{1}{2}\frac{\tilde{E}^2}{(\hbar\omega)^3}\,. \qquad (*)$$

(ii) Consider the system of N bosons discussed in Part (i). Use the density of states derived in equation $(*)$ to show that the number of particles N can be expressed in terms of temperature T as

$$N - N_0 = F(y)G(z)\,,$$

where N_0 is the number of particles in the ground state, F is a function of $y = \frac{k_{\rm B}T}{\hbar\omega}$ that should be determined, $z = \exp(\frac{\mu - E_0}{k_{\rm B}T})$, μ is the chemical potential, $k_{\rm B}$ is the Boltzmann constant, and the polylogarithm

$$G(z) = \frac{1}{\Gamma(3)}\int_0^\infty \frac{x^2}{z^{-1}e^x - 1}dx,$$

where $\Gamma(n)$ is the Gamma function.

Show that $z = N_0/(N_0+1)$, and that for $1 \ll N_0 \ll N$ the critical Bose-Einstein condensation temperature obeys $T_c \approx KN^{1/3}$, where the proportionality factor K should be determined in terms of constants including the Riemann Zeta function $\zeta(n) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{e^x-1}dx$.

Calculate for this case the fraction N_0/N at temperatures T just below T_c as a function of T/T_c.

Briefly compare the behaviour with the result for free bosons for which $N_0/N \approx 1 - (T/T_c)^{3/2}$.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Express the spin operator $\mathbf{S}$ for a particle of spin $\frac{1}{2}$ in terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Show that $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = \mathbb{I}$ for any unit vector $\mathbf{n}$ and deduce that

$$e^{-i\theta\, \mathbf{n}\cdot\mathbf{S}/\hbar} = \mathbb{I} \cos(\theta/2) - i(\mathbf{n} \cdot \boldsymbol{\sigma}) \sin(\theta/2) .$$

(ii) The space of states V for a particle of spin $\frac{1}{2}$ has basis states $|\uparrow\rangle$, $|\downarrow\rangle$ which are eigenstates of S_3 with eigenvalues $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$, respectively, where S_3 is the 3-component of the spin operator $\mathbf{S}$. The Hamiltonian for the particle is $H = \frac{1}{2}\alpha\hbar\sigma_1$, where α is a real constant and σ_1 is the Pauli matrix defined in Part (i). Find explicit expressions for the states

$$e^{-itH/\hbar} |\uparrow\rangle \qquad \text{and} \qquad e^{-itH/\hbar} |\downarrow\rangle$$

as linear combinations of the basis states.

The space of states for a system of two spin $\frac{1}{2}$ particles is $V \otimes V$. Write down explicit expressions for the joint eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J}$ is the sum of the spin operators for the particles, and J_3 is the 3-component of $\mathbf{J}$.

Suppose that the two-particle system has Hamiltonian $H = \frac{1}{2}\lambda\hbar(\sigma_1 \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_1)$, where λ is a constant, and that at time $t = 0$ the system is in the state with J_3 eigenvalue $+\hbar$. Calculate the probability that at time $t > 0$ the system will be measured to be in the state with $\mathbf{J}^2$ eigenvalue zero.

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Describe the epicyclic approximation for nearly circular orbits, and use this to sketch the motion, in a frame centred on the Sun, of stars that are currently in the immediate vicinity of the Sun, but on orbits that lie at larger or smaller Galactocentric radii.

Sketch the distribution of the tangential velocities of stars orbiting in the Galactic disk in the vicinity of the Sun.

Explain what is meant by the *asymmetric drift* of nearby stars and give a qualitative explanation of the asymmetry using epicycles.

(ii) Show that the Laplacian of a spherically symmetric potential $\Phi(r)$ can be written in the form

$$\nabla^2\Phi(r) = \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\Phi).$$

Hence derive an expression for the gravitational potential for a galaxy in which the density of matter as a function of radius r obeys the following law

$$\rho(r) = \frac{\rho_0}{1+(r/a)^2},$$

where ρ_0 and a are constants.

Derive an expression for how the mass interior to a given radius varies with radius for this density distribution.

Deduce how the circular velocity and escape velocity change with radius.

Can real galaxies be described by this law at larger radii?

TURN OVER...

Question 8Z - Physics of Astrophysics

(i) A star cluster of mass m orbits at a radius R within a galaxy of mass M for which the bulk of the mass is located within a radius $\ll R$. Derive an estimate for the distance r_T from the centre of the cluster at which the gravity due to the cluster balances the tidal effect of the galaxy.

A cluster of luminosity $3 \times 10^5 L_\odot$ orbits at a galactocentric radius of $R = 7\,\mathrm{kpc}$ and is associated with a pair of tidal tails which originate at a distance 50 pc from the cluster centre. Assuming that the average mass to light ratio of the cluster's stars is $M/L = 2M_\odot/L_\odot$, estimate the mass of the parent galaxy.

(ii) It has been argued that the presence in meteorites of daughter products of the rare isotope ^{60}Fe implies that the primitive solar nebula was impacted by ejecta from a supernova that exploded in its neighbourhood before the nebula condensed into planets. The nebula would however be stripped away by the blast wave if it intercepted an energy greater than the gravitational energy binding it to the proto-Sun. Assuming that the nebula was of mass $0.01M_\odot$ and radius 10 au, estimate the minimum distance $d_{\min}$ between the Sun and the supernova at the point that it exploded. You may assume that the supernova releases an energy of 10^{44} J.

Using dimensional analysis or otherwise, estimate the time required for the blast wave to propagate a distance $d_{\min}$ calculated above, assuming that the blast wave is spherically symmetric and the intervening medium is of uniform density $10^{-18}\,\mathrm{kg\,m^{-3}}$.

Assume that the maximum distance to which the ejecta can contaminate a nebula is $d_{\max} = 1\,\mathrm{pc}$. Given that the half-life of ^{60}Fe is $t_{0.5} = 1.5 \times 10^6$ years, do you expect significant decay of ^{60}Fe between the explosion of the supernova and its interaction with a nebula at distance $d_{\max}$?

The average stellar density and velocity dispersion in the solar neighbourhood is $0.1\,\mathrm{pc^{-3}}$ and $20\,\mathrm{km\,s^{-1}}$. Following the explosion of a supernova, calculate the expected number of stars that are likely to pass within a distance $d_{\max}$ of the explosion site within a time $t_{0.5}$. You may neglect the effect of gravitational focussing.

If stars possess a protostellar nebula for around 0.1% of their lifetimes, how many supernovae must explode before a protostellar nebula is contaminated in this way?

If the local rate of supernovae per unit volume is $10^{-13}\,\mathrm{pc^{-3}\,yr^{-1}}$, within what volume of the Galaxy is one contamination event expected within 10^{10} years, and how many stars are contained in this volume?

Comment on whether you think that this a likely scenario for contaminating the primordial solar nebula, and explain without calculation how the answer would change if stars are preferentially clustered at birth.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 4 June 2015 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 7Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) The metric of the surface of a sphere in spherical polar coordinates is

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2 \, ,$$

where a is a constant. Write down all metric components g_{ik}.

Given the dimensionality of the problem how many independent Christoffel symbols are there?

Hence calculate all independent Christoffel symbols which are non-zero.

By explicit calculation show how many independent components of the curvature tensor there are.

With the aid of Christoffel symbols calculate all these components and thus derive the Gaussian curvature K of a sphere.

(ii) By considering the metric describing the geometry in the vicinity of a non-rotating black hole of mass M explain the significance of the radius $r_\mathrm{s} = 2GM/c^2$, where c is the speed of light.

Express the Lagrangian of this metric and thus derive the four geodesic equations using the affine parameter σ along the geodesic $x^\mu(\sigma)$ with $\mu = 0, 1, 2, 3$. From these equations derive the geodesic equations describing the motion of both a massive particle and a photon in the equatorial plane, i.e. $\theta = \pi/2$. Instead of the $\mu = 1$ equation use a simpler expression given by the first integral of the geodesic equations.

A spacecraft falls radially from rest at infinity towards a black hole of mass M. Using the geodesic equations from above derive an expression for the spacecraft's coordinate speed dr/dt which depends only on constants and on the coordinate r.

According to an external observer how long does it take for the spacecraft to reach r_s? Discuss the physical meaning of this result.

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a barotropic fluid which is at rest in a steady state with a uniform density ρ_0 and pressure p_0. Assuming a small perturbation to this equilibrium configuration characterized by some density $\Delta\rho$, pressure Δp and velocity $\Delta\vec{u}$, derive the equation governing the time evolution of this perturbation.

By explicit calculation show that the perturbation travels with speed $c_{\rm s} = \sqrt{dp/d\rho}$.

Determine how $|\Delta\vec{u}|$ compares to $c_{\rm s}$ and discuss the physical implications.

(ii) Consider a strong adiabatic shock travelling through the interstellar medium (ISM). The unperturbed ISM has a density $n_1 = 1\,\mathrm{cm}^{-3}$ and temperature $T_1 = 10^4\,\mathrm{K}$. What pre-shock speed u_1 is needed for the post-shock temperature to be $T_2 = 10^7\,\mathrm{K}$? What is the post-shock velocity u_2 for this situation? You may assume $\gamma = 5/3$ and that for a strong shock

$$\frac{T_2}{T_1} = \left(\frac{\gamma - 1}{\gamma + 1}\right)\left(\frac{2\gamma M_1^2}{\gamma + 1}\right),$$

where M_1 is the Mach number of pre-shocked gas.

A star emits a wind of constant velocity $v_{\rm w}$ and mass-loss rate $\dot{M}$. The wind propagates into the surrounding interstellar medium of density ρ_0 and terminates in a shock. Estimate how the shock radius R scales as a function of time assuming that all kinetic wind energy is used to drive the shocked interstellar medium outwards (i.e. there are no radiative losses).

Explain the physical reason why R scales with time differently than is the case for the Sedov-Taylor solution.

Calculate the velocity of the flow just behind the shock in the star's rest frame, as a function of the rate of change of the shock radius $\dot{R}$. You may assume that the shock is strong and that the adiabatic index $\gamma = 5/3$.

Comment on why this velocity is different from $v_{\rm w}$.

After some time the star that was blowing the wind explodes as a supernova. Assuming that all the kinetic energy of the supernova is used to sweep up the shocked stellar wind, estimate how the supernova blast wave radius $R_{\rm s}$ scales with time.

TURN OVER...

Question 3X - Physical Cosmology

(i) Rich clusters of galaxies are observed embedded in hot gas which emits X-rays. What physical process gives rise to this radiation?

Explain why the observed X-ray surface brightness $B_{\rm X}$ is related to the electron density $n_{\rm e}$ in the gas by

$$B_{\rm X} \propto \int n_{\rm e}^2 dl \,,$$

where l is distance measured along the line of sight through the cluster.

Sketch how and explain why the cosmic microwave background (CMB) black-body spectrum changes as seen through the cluster. Hence explain why, at radio frequencies, the CMB as seen through the cluster is observed to have a lower temperature than it would otherwise have?

Given that the observed fractional temperature decrement in the CMB is given by

$$\frac{\Delta T}{T_{\rm CMB}} \propto \int n_{\rm e} dl \,,$$

explain how the observables $\Delta T/T_{\rm CMB}$ and $B_{\rm X}$ in conjuction with the angular size of the cluster can be used to estimate the angular diameter distance to the cluster. Discuss whether this method can be used to arbitrarily high redshifts, explaining your reasoning.

(ii) The comoving radius of the cosmological horizon at time t is given by

$$r_{\rm H}(t) \propto [H(t)R(t)]^{-1} \,,$$

where $H(t)$ is the Hubble constant and $R(t)$ is the scale factor. Explain what the horizon problem is and why inflation is invoked to solve it.

Sketch $r_{\rm H}$ as a function of time for times from before the inflation until the present day.

Assume that the energy scale of the GUT phase transition is $10^{14}\,$GeV, the temperature of the cosmic microwave background is $T_0 = 2.73\,$K and the temperature at the end of the radiation-dominated era is $T_{\rm eq} = 13900\,$K. Estimate the minimum number of e-foldings of expansion that are required during inflation to solve the horizon problem for a flat universe. Does one need to invoke the anthropic principle with this number of e-foldings?

If the present age of the Universe is 13.7 billion years show that inflation ends at a cosmic time of order $10^{-34}\,$s.

Question 4Z - Structure and Evolution of Stars

(i) Two stars begin their lives on the Zero Age Main Sequence with the same initial mass, $M_{\rm i} = 50\,M_{\odot}$. Star A has solar composition, while in star B elements heavier than He are 100 times less abundant than in the Sun. Which of the two stars would you expect to spend longer on the Main Sequence and why?

Explain in a few sentences the main difference between H II regions and planetary nebulae.

Observationally what differences might you expect in chemical composition, degree of ionisation, and kinematics between H II regions and planetary nebulae?

(ii) A $1 M_{\odot}$ white dwarf accretes matter of solar composition from a companion star for 10^5 yr at a rate of $10^{-10} M_{\odot}$ yr^{-1}. At the end of this accretion period, the accreted hydrogen ignites and burns at a constant luminosity for 100 days. What is the composition that results from the hydrogen burning?

Is mass lost from the system? You can assume that the main source of opacity is electron scattering, $\kappa_{\rm es} = 0.02\,(1 + X)$ m^2 kg^{-1}, where X is the hydrogen mass fraction.

Explain briefly what is meant by the term *thermonuclear supernova.*

Suggest observational tests that may help distinguish between different proposals for the origin of thermonuclear supernovae.

TURN OVER...

Question 5Y - Statistical Physics

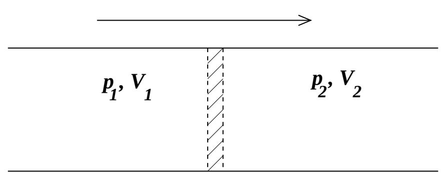
(i) A sample of gas has pressure p, volume V, temperature T and entropy S. Use the first law of thermodynamics to derive the Maxwell relation

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$

Let x, y and u represent three related thermodynamic variables. Show that along the curve in the (x, y) plane defined by the condition that $u(x, y)$ is constant

$$\left(\frac{\partial x}{\partial u}\right)_y \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_u = -1 .$$

(ii) Consider a perfectly insulated pipe with a *throttle valve* (see figure). Gas initially occupying volume V_1 on the left is forced slowly through the valve at constant pressure p_1. A constant pressure p_2 is maintained on the right and the final volume occupied by the gas after passing through the valve is V_2. Show that the enthalpy H of the gas is conserved during this process.



Show that the specific heat at constant pressure $C_p = T\,(\partial S/\partial T)_p$ is related to the enthalpy by $C_p = (\partial H/\partial T)_p$.

Using the results from Part (i), or otherwise, show that the Joule–Thomson coefficient $(\partial T/\partial p)_H$ is

$$\frac{V}{C_p}\left[\frac{T}{V}\left(\frac{\partial V}{\partial T}\right)_p - 1\right] .$$

Calculate this coefficient for an ideal gas.

Suppose that the gas obeys an equation of state

$$p = k_{\mathrm{B}} T \left[\frac{N}{V} + B_2(T)\frac{N^2}{V^2}\right] ,$$

where N is the number of particles, k_{B} is Boltzmann's constant and B_2 is a function only of temperature T. Derive a condition in terms of $\dfrac{d}{dT}\left(\dfrac{B_2(T)}{T}\right)$ for obtaining a positive Joule–Thomson coefficient.

Question 6Z - Principles of Quantum Mechanics

(i) The angular momentum operator $\boldsymbol{J}$ obeys commutation relations which may be written in the form

$$[J_3, J_\pm] = \pm\hbar J_\pm, \qquad [J_+, J_-] = 2\hbar J_3, \qquad [\mathbf{J}^2, J_i] = 0,$$

where $J_\pm = J_1 \pm iJ_2$. The normalized angular momentum eigenstates are denoted $|j, m\rangle$, where $m\hbar$ are the eigenvalues of J_3 and $j(j+1)\hbar^2$ are the eigenvalues of $\mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2$. State the allowed values that the quantum numbers j and m may take.

Show that the states $|j, m\rangle$ satisfy

$$J_-|j, m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle\ .$$

(ii) Consider two quantum systems with angular momentum states $|\frac{1}{2}, r\rangle$ and $|j, m\rangle$, following the notation defined in Part (i). The eigenstates corresponding to their combined angular momentum can be written as

$$|\, J,\, M\,\rangle \;=\; \sum_{r\,m} C^{J\,M}_{r\,m}\, |\, \frac{1}{2},\, r\,\rangle\, |\, j,\, m\,\rangle\ ,$$

where $C^{J\,M}_{r\,m}$ are Clebsch-Gordan coefficients for addition of angular momenta $\frac{1}{2}$ and j. What are the possible values of J and what is a necessary condition relating r, m and M in order that $C^{J\,M}_{r\,m} \neq 0$?

Using the results from Part (i) or otherwise, calculate the values of $C^{J\,M}_{r\,m}$ for $j = 2$ and for all $M > \frac{1}{2}$. Use the sign convention that $C^{J\,J}_{r\,m} > 0$ when m takes its maximum value.

A particle X with spin $\frac{3}{2}$ and intrinsic parity η_X is at rest. It decays into two particles A and B with spin $\frac{1}{2}$ and spin 0, respectively. Both A and B have intrinsic parity -1. The relative orbital angular momentum quantum number for the two particle system is ℓ. What are the possible values of ℓ for the cases $\eta_X = +1$ and $\eta_X = -1$?

Suppose particle X is prepared in the state $|\,\frac{3}{2},\,\frac{3}{2}\,\rangle$ before it decays. Calculate the probability P for particle A to be found in the state $|\,\frac{1}{2},\,\frac{1}{2}\,\rangle$, given that $\eta_X = +1$.

What is the probability P if instead $\eta_X = -1$?

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Explain what is meant by the *distribution function* of stars in a galaxy.

How is the distribution function connected to the volume density of stars, giving consideration to how this is normalised?

Describe the meaning of the Collisionless Boltzmann Equation and discuss its limitations.

State Jeans Theorem and comment briefly on its use.

(ii) For a distribution function that depends solely on energy, derive an expression that relates the volume matter density ρ to the distribution function $f(\mathcal{E})$, where the relative energy $\mathcal{E} = \Psi - \frac{1}{2}v^2$, the relative potential $\Psi = -\Phi + \Phi_0$, Φ is the potential, Φ_0 is a constant and v is the velocity. Assume stars on average have a mass of m.

Consider a galaxy of mass M that is described by a Plummer model for which

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + a^2}},$$

$$\rho(r) = \frac{3M}{4\pi} \frac{a^2}{(r^2 + a^2)^{5/2}},$$

where a is a constant. Show that the distribution function $f(\mathcal{E}) = -b\,\mathcal{E}^{7/2}$ is a viable solution for this galaxy, and find the value of the constant b.

Derive the behaviour of the velocity dispersion as a function of radius.

[You might find the substitution $\mathcal{E} = \Psi \cos^2\theta$ as well as the following integrals useful:

$$\int_0^{\pi/2} \sin^2\theta \cos^8\theta d\theta = \frac{7\pi}{512}, \qquad \int_0^{\pi/2} \sin^4\theta \cos^8\theta d\theta = \frac{7\pi}{2048}. \quad]$$

Question 8Z - Physics of Astrophysics

(i) Dust grains are exposed to radiation from a starburst in a galactic nucleus. Assuming the dust acts like a black-body and has a density 3000 kg m^{-3}, determine the size range of grains that are blown out by radiation pressure if the mass to light ratio for the starburst region is $M/L = 0.1 M_\odot/L_\odot$.

Comment briefly on how this calculation would have changed if the dust had not been assumed to act like a black-body.

(ii) A spherical nebula of ionised gas located in Orion (at a distance 450 pc) is resolved at a wavelength of 6 cm and found to have a radius of 330 milliarcseconds and a total flux at this wavelength of 5 mJy. Calculate the size of the object.

Assuming that the gas is at a temperature of 10^4 K, what can you deduce about the nature of the emission from the fact that the observed flux is much less than that emitted by a black-body of this size and temperature?

The flux produced at frequency ν by thermal bremsstrahlung from an optically thin gas of temperature 10^4 K, is given by

$$\left[\frac{S_\nu}{\text{mJy}}\right] = 3.4 \left[\frac{\nu}{\text{GHz}}\right]^{-0.1} \left[\frac{\text{VEM}}{10^{63}\text{m}^{-3}}\right] \left[\frac{D}{\text{kpc}}\right]^{-2},$$

where D is the distance to the source and VEM is the volume emission measure, i.e. the integral of $n_{\text{ion}} n_{\text{e}}$ over volume, where n_{ion} and n_{e} are the number densities of ions and electrons respectively (in m^{-3}). Deduce the number density in the nebula assuming it to be of uniform density and composed of pure hydrogen.

The rate of recombinations to excited electronic states per unit volume is given by $\alpha_B n_{\text{ion}} n_{\text{e}}$, where $\alpha_B = 2 \times 10^{-19}\,\text{m}^3\ \text{s}^{-1}$ is the Case B recombination coefficient. Explain why, when calculating ionisation equilibrium in a dense gas, it is this Case B coefficient (which does not include recombinations to the ground state) that is appropriate.

Calculate the output of ionising photons per second that must be provided by a luminosity source that maintains the nebula's ionisation equilibrium.

TURN OVER...

The pressure in the nebula exceeds that of its surroundings and so it expands, initially at the sound speed of the ionised gas. Estimate the expansion timescale and compare it with the recombination timescale. Is it reasonable to assume that the nebula remains in ionisation equilibrium as it expands?

Estimate the size and internal density of the nebula when it has expanded to the point that it is in pressure equilibrium with its surroundings, which have a number density $10^9\,\mathrm{m}^{-3}$ and temperature $100\,\mathrm{K}$, assuming that the nebula remains at temperature $10^4\,\mathrm{K}$ as it expands.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 5 June 2015 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

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STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) The energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - pg^{\mu\nu}\,,$$

where ρ is density, p is isotropic pressure in the instantaneous rest-frame, u^μ are the components of 4-velocity, $g^{\mu\nu}$ is the metric and c is the speed of light. Show that for any fluid

$$u_\nu \nabla_\mu u^\nu = 0\,.$$

Hence show that a perfect fluid in a gravitational field must satisfy

$$\nabla_\mu(\rho u^\mu) + p/c^2 \nabla_\mu u^\mu = 0\,, \qquad (*)$$

$$(\rho + p/c^2)u^\mu \nabla_\mu u^\nu = (g^{\mu\nu} - u^\mu u^\nu/c^2)\nabla_\mu p\,.$$

If the fluid is dust show that the worldline of each particle is a geodesic.

(ii) Using equation $(*)$, or otherwise, show that for a homogeneous and isotropic universe described by the Friedmann-Robertson-Walker geometry, for which $\Gamma^\mu_{0\mu} = c\dot{R}/R$ for $\mu = 1, 2, 3$, the following equation holds

$$d(\rho R^3)/dR = -3pR^2/c^2\,,$$

where R is the scale factor and ρ, p and c are defined as in Part (i).

Outline the physical significance of this equation.

Assuming comoving coordinates $x^\mu = (t, r, \theta, \phi)$ such that the Friedmann-Robertson-Walker metric yields

$$ds^2 = c^2dt^2 - R(t)^2\left[dr^2/(1 - kr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

derive the Friedmann equations from the gravitational field equations $R_{\mu\nu} = -k(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) + \Lambda g_{\mu\nu}$. Here $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid, $g_{\mu\nu}$ is the metric, Λ is the cosmological constant and $k = 8\pi G/c^4$. Due to symmetry consider only 00 and 22 components and recall that $R_{00} = 3\ddot{R}/R$ and $R_{22} = -(R\ddot{R} + 2\dot{R}^2 + 2c^2k)r^2/c^2$.

Consider a massive particle in a flat expanding Friedmann-Robertson-Walker universe. Show that its physical 3-momentum decreases as $p(t) \propto 1/R(t)$, where $R(t)$ is the scale factor.

Question 2Y - Astrophysical Fluid Dynamics

(i) Explain why the viscous stress tensor σ_{ij} needs to be symmetric.

Consider two infinitely long concentric cylinders with radius r_1 and $r_2 > r_1$. The cylinders rotate with angular speeds Ω_1 and Ω_2, respectively. The space between the cylinders is filled with a viscous fluid. With the aid of the Navier-Stokes equation in cylindrical polar coordinates show that in steady state the fluid rotates with tangential velocity

$$v_\phi = Ar + B/r\,,$$

where A and B are two constants, the dependence of which on r_1, r_2, Ω_1 and Ω_2 should be determined.

Assuming the flow is incompressible and of density ρ, find how the pressure difference Δp between the inner and outer cylinder depends on Ω_1 and Ω_2 in the limit where $r_2 - r_1 \ll r_1$ or r_2. You may assume that $\Delta p \approx (r_2 - r_1)\partial p/\partial r$.

(ii) Consider two incompressible stratified fluids in a uniform gravitational field $\vec{g}$ that is normal to the boundary between the fluids. The lower fluid has density ρ and velocity U along the x-axis, while the upper fluid has density ρ' and velocity U' along the x-axis. It can be shown that the instabilities that may develop along the interface between these two fluids have phase velocity given by

$$\omega/k = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left[\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}\right]^{1/2}, \quad (*)$$

where $g = |\vec{g}|$. From equation $(*)$ derive the condition on the wave number k such that the fluids are subject to Kelvin-Helmholtz (KH) instability.

Explain what happens if $g = 0$.

Consider now a cold dense cloud with radius $R_{\rm cl}$ injected into a hot uniform wind tunnel with density ρ_1 moving at a large relative velocity U along the x-axis. Initially the cloud is in pressure equilibrium with the surrounding medium and the cloud density is ρ_2. A shock will form ahead of the cloud and KH instabilities will be important in its wake. Sketch the shock front and the cloud immersed in this wind tunnel indicating the direction of the x-axis and where the flow is super- and sub-sonic.

TURN OVER...

Neglecting gravity for the moment, estimate the characteristic growth time of the KH instability by assuming that instabilities of the order of cloud size are responsible for cloud destruction. You may assume that the density contrast between the cloud and the wind tunnel $D = \rho_2/\rho_1 \gg 1$.

In the case that the self-gravity of the cloud is not negligible, estimate the critical mass of the cloud as a function of wind tunnel properties and D such that it remains stable against the KH instability. Provide a physical interpretation of the equation obtained.

Question 3X - Physical Cosmology

(i) A density perturbation $\delta_{\rm m} = (\rho_{\rm m} - \bar{\rho}_{\rm m})/\bar{\rho}_{\rm m}$ in a pressureless matter-dominated expanding universe with mean density $\bar{\rho}_{\rm m} \propto R^{-3}$, where $\rho_{\rm m}$ is matter density, obeys the equation

$$\ddot{\delta}_{\rm m} + 2\frac{\dot{R}}{R}\dot{\delta}_{\rm m} - 4\pi G\bar{\rho}_{\rm m}\delta_{\rm m} = 0\,,$$

where R is the scale factor. Find the growing and the decaying solutions in an Einstein–de Sitter model as a function of R.

Show that for models with low matter densities today, $\bar{\rho}_{\rm m} = \beta\rho_{\rm crit}$, where $\rho_{\rm crit}$ is the critical density of the universe and $0 < \beta \ll 1$, the growing mode solution at sufficiently early cosmic times is $\delta_{\rm m} \propto t^{2\beta}$. What is the physical reason for the different growth rate with respect to that for the Einstein–de Sitter model?

(ii) Show that in a Friedmann–Robertson–Walker universe, the observed redshift of a distant comoving source changes with proper time t measured by a comoving observer according to

$$\frac{dz}{dt} = H_0(1+z) - H(z),$$

where $H(z)$ is the Hubble parameter with value H_0 today.

The spectrum of a very high-redshift quasar shows an absorption line due to a foreground gas cloud at $z = 4.0$. If the quasar is monitored for 10 years what is the expected change in wavelength of the absorption line $\delta\lambda/\lambda$ for an Einstein–de Sitter universe with $H_0 = 70\,{\rm kms^{-1}Mpc^{-1}}$?

Estimate how this fractional wavelength change compares to the line width if the gas cloud has a velocity dispersion of $100\,{\rm kms^{-1}}$ and comment on your result.

How does the predicted wavelength change for our current best model of the universe differ from the Einstein–de Sitter case? Sketch dz/dt as a function of z for these two model universes to corroborate your answer.

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) Describe briefly at least three different observational techniques used to detect binary stars.

Two stars, X and Y, are separated on the sky by one second of arc. The apparent magnitudes of star X through the V and B filters are measured to be $m_{\rm V} = 6.0$ and $m_{\rm B} = 6.0$, respectively. For star Y, the measured values are $m_{\rm V} = 6.0$ and $m_{\rm B} = 7.0$. Explaining your reasoning, is it likely that the two stars are members of a binary system? What additional information would be required to obtain a definitive answer?

Which of the two stars is more likely to consume its nuclear fuel first?

(ii) In the outer parts of a star, the luminosity and enclosed mass are approximately constant with radius, and the opacity can be assumed to be of the form

$$\kappa = \kappa_0 \, \frac{P^{\alpha-1}}{T^{\beta-4}},$$

where P is the pressure, T is the temperature, and κ_0, $\alpha(>0)$ and $\beta(>0)$ are constants. Radiation pressure may be neglected. With the aid of the equations of stellar structure, and assuming that energy is transported by radiative diffusion, write down a differential equation for $T(P)$, and show that P can be written as

$$P = \left[B + \frac{\alpha}{\beta} A T^{\beta} \right]^{\frac{1}{\alpha}},$$

where A and B are constants.

Hence show that

$$\frac{d \ln P}{d \ln T} = \frac{A T^{\beta}}{B + \frac{\alpha}{\beta} A T^{\beta}}.$$

If the star obeys the surface boundary condition $P = 0$, $T = T_{\rm s}$, determine the limiting value of $d \ln P / d \ln T$ deep in the atmosphere, where $T \gg T_{\rm s}$.

Show that if the opacity follows the Kramers' law, $\kappa \propto \rho T^{-3.5}$, where ρ is the density, the atmosphere is stable against convection.

[You may assume that the Schwarzschild criterion for convective instability is $\frac{d \ln P}{d \ln T} < \frac{\gamma}{\gamma - 1}$, where $\gamma = 5/3$ for a monoatomic gas.]

Question 5Y - Statistical Physics

(i) The Ising model consists of N particles arranged on a D-dimensional periodic Euclidean lattice. Each particle has spin *up*, $s_i = +1$, or *down*, $s_i = -1$, and the energy in the presence of a magnetic field B is

$$E = -B \sum_{i=1}^{N} s_i - J \sum_{\langle ij \rangle} s_i \, s_j \, ,$$

where $J > 0$ is the coupling strength between two particles, and the second summation covers all pairs of interacting particles assumed here to include for every particle only its two nearest neighbours in each spatial direction. Briefly discuss the mean-field theory approximation for this model and derive that in this approximation the canonical partition function has the form

$$Z = C \left(\sum_{s=\pm 1} e^{\beta B_{\text{eff}} s} \right)^{N} ,$$

where $\beta = 1/(k_{\text{B}} T)$, k_{B} is Boltzmann's constant, T is temperature, $B_{\text{eff}} = B + 2DJm$, m is the average spin per particle. The pre-factor C, which is independent of the individual spins s_i and the magnetic field B, should be determined.

(ii) Consider the mean-field theory approximation of the Ising model of Part (i). Show that the average spin $m = \frac{1}{N} \sum_i \langle s_i \rangle$ per particle in this model obeys

$$m = \tanh(\beta B + 2\beta DJm) \, .$$

Consider a variation of the Ising model where now each particle can have a spin value $s_i \in \{-\sigma, -\sigma + 1, \ldots, +\sigma\}$ for a positive integer or half-odd integer value of σ. Show that in the mean-field theory approximation, the average spin per particle obeys

$$m = \left(\sigma + \frac{1}{2} \right) \coth \left[\left(\sigma + \frac{1}{2} \right) \beta (B + 2DJm) \right] - \frac{1}{2} \coth \left[\frac{\beta (B + 2DJm)}{2} \right] .$$

Calculate the critical temperature T_{c} as a function of D, J and σ, and explain its significance.

TURN OVER...

[You may find useful that for $z \in \mathbb{R}$ and positive or half-odd integer n

$$\left(\sum_{s=-n}^{n} e^{sz}\right)(e^{\frac{z}{2}} - e^{-\frac{z}{2}}) = e^{(n+\frac{1}{2})z} - e^{-(n+\frac{1}{2})z},$$

$$\left(n+\frac{1}{2}\right)\coth\left[\left(n+\frac{1}{2}\right)z\right] - \frac{1}{2}\coth\left(\frac{z}{2}\right) = \frac{n}{3}(n+1)z + \mathcal{O}(z^3). \quad]$$

Question 6Z - Principles of Quantum Mechanics

(i) The Hamiltonian for a quantum system in the Schrödinger picture is $H_0 + \lambda V(t)$, where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Suppose that $|\chi\rangle$ and $|\phi\rangle$ are eigenstates of H_0 with distinct eigenvalues E and E', respectively. Show that if the system is in state $|\chi\rangle$ at time zero then the probability of measuring it to be in state $|\phi\rangle$ at time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt'\,\langle\phi|V(t')|\chi\rangle\, e^{i(E'-E)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\,.$$

(ii) Let H_0 be the Hamiltonian for an isotropic three-dimensional harmonic oscillator of mass m and frequency ω, with $\chi(r)$ being the ground state wavefunction (where $r = |\mathbf{x}|$) and $\phi_i(\mathbf{x}) \;=\; (2m\omega/\hbar)^{1/2}\, x_i\, \chi(r)$ being wavefunctions for the states at the first excited energy level ($i = 1, 2, 3$). The oscillator is in its ground state at $t = 0$ when a perturbation

$$\lambda V(t) \;=\; \lambda\, \hat{x}_3\, e^{-\mu t}$$

is applied, where μ is a positive constant, and H_0 is then measured after a large time has elapsed. Show that to first order in perturbation theory the oscillator will be found in one particular state at the first excited energy level with probability

$$\frac{\lambda^2}{2\hbar m\omega\,(\mu^2 + \omega^2)}\,,$$

but that the probability that it will be found in either of the other excited states is zero (to this order).

[You may assume that $4\pi \int_0^\infty r^4\, |\chi(r)|^2\, dr = \frac{3\hbar}{2m\omega}$.]

TURN OVER...

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Discuss how the Virial Theorem can be used to estimate masses of astrophysical objects from observable quantities.

The Jeans equation for spherical systems states that

$$\frac{d(\nu\overline{v_r^2})}{dr} + \nu\left(\frac{d\Phi}{dr} + \frac{2\overline{v_r^2} - \overline{v_\theta^2} - \overline{v_\phi^2}}{r}\right) = 0. \qquad (*)$$

Define the quantities entering this equation and discuss the ease with which each of them can be measured using observations.

With regards to the Jeans equation, what might the *mass-density-anisotropy degeneracy* refer to?

(ii) A handful of stars is orbiting with constant velocity dispersion σ and constant velocity anisotropy β within a spherically symmetric galaxy dominated by dark matter with a density distribution given by

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^3},$$

where ρ_0 and a are constants. Show that the potential of the system can be written as

$$\Phi(r) = \frac{A}{r + a},$$

where the constant A should be determined.

The number density of stars at a radius $r = a$ is ν_a. Using equation $(*)$, or otherwise, deduce how the density of stars changes as a function of radius.

Dark matter halos are rarely spherical. Consider a dark matter halo that is flattened along the z-axis by a constant factor q, so that its potential is

$$\Phi(r) = \ln\left[R^2 + (z/q)^2\right],$$

where R is the radial distance in the plane normal to the z-axis. Deduce the flattening of the iso-density contours.

Comment on the range of q giving rise to a physical density distribution.

Question 8Z - Physics of Astrophysics

(i) Consider an axisymmetric disc with constant circular velocity independent of distance from the star R, i.e. $V(R) = V_0$, viewed at inclination angle i. Sketch contours of constant projected velocity along the line-of-sight.

Show that the flux in a spectral line of rest frequency ν_0 observed in the frequency interval ν to $\nu + d\nu$ is of the form

$$\left[1 - \left(\frac{\nu - \nu_0}{\nu_{\max} - \nu_0}\right)^2\right]^{-1/2} d\nu,$$

where $\nu_{\max}$ is the highest frequency present in the observed spectrum of this line. Sketch this spectrum and state the relationship between $\nu_{\max}$ and V_0.

State without detailed calculation how the line shape differs in the case that $V(R) = V_0(R_{\rm out}/R)$ for a disc that extends over $R_{\rm in} < R < R_{\rm out}$.

(ii) A short burst in the continuum of an Active Galactic Nucleus lasting a few hours triggers a response in the broad emission lines from material surrounding the nucleus which begins at time $T = 3.73$ days later. The width of the Hβ line at 4861 Å detected by the observer at a later time is 64 Å. Use this information to estimate the mass of the black hole, explaining clearly the assumptions made.

The light curve of the response in the emission lines is used to constrain the geometry of the line emitting clouds. Assuming that the clouds are distributed in a narrow axisymmetric ring perpendicular to the plane of the sky, show that the form of the expected response at time t after the burst is first detected is given by

$$I(t) \propto \left[t(2T - t)\right]^{-0.5},$$

and sketch this form.

Derive the expected response if instead the clouds are distributed in a thin spherical shell.

In the case of the ring of clouds, explain without detailed calculation how changes in the *shape* of the line over the period $0 < t < 2T$ could be used to distinguish tangential cloud motions from radial inflow or outflow.

END OF PAPER