NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 30 May 2014 09:00am – 12:00pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (i) *and Part* (ii)*, which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i)*. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) For a timelike geodesic in the equatorial plane ($\theta = \pi/2$) of the Schwarzschild space-time with line element

$$ds^2 = -(1 - r_s/r)c^2dt^2 + (1 - r_s/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

where r_s is the Schwarzschild radius, derive the equation

$$\tfrac{1}{2}\dot{r}^2 + V(r) = \tfrac{1}{2}(E/c)^2\,,$$

where $2V(r)/c^2 = 1 - r_s r^{-1} + (h/c)^2 r^{-2} - (h/c)^2 r_s r^{-3}$ and h and E are constants. The dot denotes the derivative with respect to an affine parameter τ satisfying $c^2 d\tau^2 = -ds^2$.

(ii) Using the results from part **(i)** show that if $h^2 > 3r_s^2c^2$, there is a stable circular orbit at $r = R$, where R is the smaller root of the quadratic equation

$$r^2 = (h/c)^2(2r/r_s - 3)\,,$$

and that Ω, its orbital angular frequency with respect to τ is given by

$$\frac{\Omega^2R^2}{c^2} = \frac{\epsilon}{(2 - 3\epsilon)}\,,$$

where $\epsilon = r_s/R$.

Show also that the angular frequency ω of small radial perturbations is given by

$$\frac{\omega^2R^2}{c^2} = \frac{\epsilon(1 - 3\epsilon)}{(2 - 3\epsilon)} = \frac{\Omega^2R^2}{c^2}(1 - 3\epsilon)\,.$$

Deduce that the rate of precession, with respect to coordinate time t, of the perihelion of the nearly circular Earth orbit is approximately $3\widetilde{\Omega}^3T^2$, where T is the time taken for light to travel from the Sun to the Earth and $\widetilde{\Omega}$ is the orbital angular frequency of the Earth with respect to coordinate time.

Question 2Y - Astrophysical Fluid Dynamics

(i) Explain under which physical conditions sound waves and shocks are generated.

Give at least three astrophysical examples where shocks occur and describe their astrophysical significance.

Assume that the properties of a fluid change discontinuously within a small layer dx along the x-axis. On the left side of this layer fluid has pressure p_1, density ρ_1 and velocity u_1 along the x-axis and on the right side p_2, ρ_2 and u_2 respectively. From the continuity and momentum equations derive the first two Rankine-Hugoniot relations.

In the case of an adiabatic shock derive the third Rankine-Hugoniot relation from the energy equation.

(ii) Derive an expression for the compression factor ρ_2/ρ_1 in the case of a strong isothermal shock, where ρ_1 and ρ_2 are the densities either side of the shock.

Derive the corresponding expression for a strong adiabatic shock and give a physical explanation for the different results.

Now consider a fluid that undergoes an adiabatic strong shock but then gradually cools downstream of the shock. Show that the ratio of ram pressure to thermal pressure is always $\leq \frac{1}{2}(\gamma - 1)$, where γ is the adiabatic index.

Consider a fluid that is separated by an infinitesimally thin boundary. On the left of this boundary the fluid density is ρ_1 and the fluid pressure is p_1, while on the right of the boundary the fluid density is $\rho_2 < \rho_1$ and the pressure is $p_2 < p_1$. On both sides of the boundary the fluid is at rest. At time $t = 0$ the boundary is removed. Describe qualitatively what you expect to happen and also what happens in the limit $p_2 = p_1$.

TURN OVER...

Question 3X - Physical Cosmology

(i) Consider two Friedmann-Robertson-Walker cosmological models.

Model A: $\Lambda \neq 0,\ \rho = \rho_{mat}$ and $p = p_{mat} = 0$.

Model B: $\Lambda = 0,\ \rho = \rho_{mat} + \rho_{vac}$ and $p = p_{mat} + p_{vac} = -\rho_{vac}\, c^2$.

Show that, with an appropriate relation between Λ in model A and ρ_{vac} in model B, the two models are the same.

(ii) In Big Bang nucleosynthesis the following reactions take place

$$n \leftrightarrow p^+ + e^- + \overline{\nu}_e\,,$$

$$n + e^+ \leftrightarrow p^+ + \overline{\nu}_e\,,$$

$$n + \nu_e \leftrightarrow p^+ + e^-\,.$$

Assuming that the particles are non-relativistic and the reactions are in equilibrium show that the neutron to proton ratio is given by

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[-\frac{(m_n - m_p)c^2}{k_{\mathrm{B}}T}\right]\,,$$

where T is the temperature, k_{B} is Boltzmann's constant, m_n is the mass of the neutron and m_p is the mass of the proton. What is n_n/n_p when $k_{\mathrm{B}}T \gg (m_n - m_p)c^2$?

As the Universe cools these reactions stop creating neutrons. Explain why this happens.

If neutron creation stops at the freeze-out temperature T_f, estimate n_n/n_p at this time using $(m_n - m_p)c^2 = 1.3\mathrm{MeV}$ and $k_{\mathrm{B}}T_f = 0.8\mathrm{MeV}$.

Describe the nucleosynthesis process after freeze-out and explain why this process ends.

Assume that at freeze-out the age of the Universe is 10 s and at the end of nucleosynthesis the age of the Universe is 300 s. If the half-life of a neutron is 615 s estimate the final value of n_n/n_p.

Show that at the end of nucleosynthesis the abundance of helium by mass is

$$Y = 2\left(1 + \frac{n_p}{n_n}\right)^{-1}$$

and find its value.

Question 4Z - Structure and Evolution of Stars

(i) The luminosity of the Sun is produced by the conversion of hydrogen to helium. In the process 0.7% of the hydrogen rest mass energy is released. Estimate the number of hydrogen nuclei which are consumed each second.

The nuclear reactions involved in the conversion release three neutrinos per hydrogen nucleus consumed. Given that the neutrino collision cross-section is negligibly small, derive an expression for the total number of neutrinos which pass per day through a 30 meter diameter neutrino detector on Earth.

Only about five neutrinos are detected per day by the detector, roughly one third of the expected number. What could be the reason?

(ii) A star of total mass M has a fully convective core of mass $M_{\rm core}$ with adiabatic index γ where all the nuclear energy generation occurs. The core is surrounded by a radiative envelope of constant opacity κ and where pressure support $P(r)$ at radius r is given by

$$P(r) = P_{rad}(r) + P_{gas}(r) = P_{rad}(r) + \beta\, P(r)\,,$$

with $P_{rad}(r)$ the radiation pressure, $P_{gas}(r)$ the gas pressure and β is a constant satisfying $\beta < 1$. By considering the continuity of the temperature gradient at the boundary between the convective core and the radiative envelope, show that

$$\frac{\gamma - 1}{\gamma}\, G M_{\rm core}\,(1-\beta) \;=\; \frac{\kappa\, L}{16\pi c}\,,$$

where G is the gravitational constant, L is the luminosity of the star and c is the velocity of light.

Show that in the radiative envelope

$$\frac{dP_{rad}}{dP} \;=\; \frac{\kappa L}{4\pi G M c} = \frac{L}{L_{\rm Edd}}\,,$$

where $L_{\rm Edd}$ is the Eddington luminosity of the star.

Hence deduce that the fractional mass in the core is given by

$$\frac{M_{\rm core}}{M} \;=\; \frac{\gamma}{4(\gamma - 1)}$$

and estimate this fraction for plausible values of γ.

TURN OVER...

Question 5Y - Statistical Physics

(i) Give expressions for the equation of state and the internal energy of a monatomic ideal gas.

Briefly describe the Carnot process and sketch the cycle in the P, V (pressure, volume) plane and in the T, S (temperature, entropy) plane.

(ii) The Diesel cycle is an idealised version of the process realised in a Diesel engine. It consists of the following four reversible steps applied to a gas with fixed number of particles N: $A{\to}B$ adiabatic compression; $B{\to}C$ expansion at constant pressure; $C{\to}D$ adiabatic expansion; $D{\to}A$ cooling of the gas at constant volume. The efficiency of the cycle is defined as

$$\eta = \frac{Q_{\rm in} - Q_{\rm out}}{Q_{\rm in}},$$

where $Q_{\rm in}$ is the heat entering the gas in step $B{\to}C$ and $Q_{\rm out}$ is the heat leaving the gas in step $D{\to}A$. Consider the Diesel cycle for the case of a monatomic ideal gas. Sketch the cycle in the P, V (pressure, volume) plane quantifying the shape of the lines in each step.

Calculate the efficiency η as a function of the temperatures at points A, B, C and D.

Express the efficiency in terms of the so-called compression ratio r and cut-off ratio α defined in terms of the volume at the various points as

$$r = \frac{V_A}{V_B}, \qquad \alpha = \frac{V_C}{V_B}.$$

Question 6Y - Principles of Quantum Mechanics

(i) Let $\hat{x}, \hat{p}$ and $H(\hat{x}, \hat{p}) = \hat{p}^2/2m + V(\hat{x})$ be the position operator, momentum operator and Hamiltonian for a particle of mass m moving in one-dimension. Let $|\psi\rangle$ be a state vector for the particle. The position and momentum eigenstates are connected by

$$\langle x|p\rangle \;=\; \frac{1}{\sqrt{2\pi\hbar}}\, e^{\,ipx/\hbar}\,,$$

where $\langle x|x'\rangle = \delta(x - x')$ and $\langle p|p'\rangle = \delta(p - p')$. Show that

$$\begin{aligned}\langle x|\hat{p}|\psi\rangle \;&=\; -i\hbar\frac{\partial}{\partial x}\psi(x)\,,\\ \langle p|\hat{x}|\psi\rangle \;&=\; i\hbar\frac{\partial}{\partial p}\tilde{\psi}(p)\,,\end{aligned}$$

where $\psi(x)$ and $\tilde{\psi}(p)$ are the wavefunctions in the position representation and momentum representation respectively.

(ii) Using the definitions of part **(i)**, show how $\psi(x)$ and $\tilde{\psi}(p)$ may be expressed in terms of each other.

Hence for general $V(\hat{x})$, express $\langle p|V(\hat{x})|\psi\rangle$ in terms of $\tilde{\psi}(p)$ and so write down the time-independent Schrödinger equation in the momentum representation that is satisfied by $\tilde{\psi}(p)$.

Consider now the case $V(x) = -(\hbar^2\lambda/m)\delta(x),\ \lambda > 0$. If there is a bound state with energy $E = -\varepsilon,\ \varepsilon > 0$, show that the wavefunction $\tilde{\psi}(p)$ satisfies

$$\tilde{\psi}(p) = \frac{\hbar\lambda}{\pi}\frac{1}{2m\varepsilon + p^2}\int_{-\infty}^{\infty}\tilde{\psi}(p')dp'\,.$$

Hence show that there is a unique value for ε and determine what it is.

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Two compact objects of mass m_1 and m_2 orbit about their common centre-of-mass in elliptical orbits. Show that the equation of motion for their relative separation r is identical to the equation of motion for a single test particle in an elliptical orbit about a body of mass $M = m_1 + m_2$. Hence show that for the Earth-Moon system, where $m_1/m_2 \approx 1/80$, naive use of Kepler's third law would yield a period about 4 hours too long.

Find the amplitude of the angular oscillation in the apparent direction of the Sun as viewed from the Earth due to the Earth-Moon orbit.

(ii) Approximate the Local Group of galaxies as an isolated binary system of two point masses corresponding to the Milky Way and M31. Use the result from part **(i)** to analyse their relative separation r as a function of time t and the total mass M of the system. Show that for a Keplerian orbit the energy per unit mass E and orbital angular momentum per unit mass L are given by

$$E = -GM/2a \quad , \quad L = \left[GMa(1-e^2)\right]^{1/2} ,$$

where a is the semi-major axis and e the orbital eccentricity. Hence show that T_{12}, the time taken for their separation to change from r_1 to r_2 satisfies

$$T_{12} = \frac{1}{(GMa)^{1/2}} \int_{r_1}^{r_2} \frac{r\,dr}{\left(e^2 - (r/a-1)^2\right)^{1/2}} .$$

Using the substitution $r = a(1 - e\cos\chi)$, or otherwise, show that the time t since periastron (at $\chi = 0$) is given by

$$t = \left(\frac{a^3}{GM}\right)^{1/2} (\chi - e\sin\chi) .$$

Assuming that both galaxies are on radial orbits ($e = 1$) and that they started off together at time $t = 0$, show that the current orbital phase angle χ is defined by the observables r, t, v, through

$$\frac{t\,v}{r} = \frac{(\chi - \sin\chi)\,(1+\cos\chi)^{1/2}}{(1-\cos\chi)^{3/2}} ,$$

where t is the current age of the Universe, r is their current separation and v is the observed radial velocity of M31 towards the Milky Way.

For $r = 785\,\mathrm{kpc}$, $v = 110\ \mathrm{km\,s^{-1}}$ and $t = 13.7\,\mathrm{Gyr}$, show that a value of $\chi \approx 240°$ is consistent with these observables and hence determine the total mass of the Local Group.

Question 8Z - Topics in Astrophysics

(i) Charon is a satellite of Pluto with an orbital period of 6.4 days. The maximum angular distance of Charon from Pluto was observed to be 0.8 arcsec when Pluto was at a distance of 30 au from the Earth. If the mass of Charon is much less than that of Pluto and Charon is on a circular orbit, determine the distance of the satellite from Pluto and the mass of Pluto in units of Earth mass.

As observed from the Earth, the orbit of Charon about Pluto traces out an ellipse with axial ratio $b/a = 0.5$. Estimate the amplitude of the line-of-sight velocity variation of Charon.

(ii) A planet is in a circular orbit of radius R about a Solar-type star. Assume the star acts like a perfect black body of effective temperature $T_{\rm eff} = 5800\,$K. If the planet also acts as a perfect black body, is in thermodynamic equilibrium and has no other source of internal energy generation, find an expression for the effective surface temperature of the planet.

Estimate these surface temperatures for an Earth-like planet at distance of 1 au from the star and for a Pluto-like planet at a distance of 30 au from the star. What are the approximate wavelengths at which the emission is a maximum? You may assume that for a Solar-type star the maximum emission occurs in the visible part of the spectrum at $\lambda = 0.55\,\mu$m.

Sketch the relative black body curves for a Pluto-like planet compared to a Solar-type star and use these to help argue why in practice most of the visible part of the spectrum from such a planet is due to reflected starlight.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 2 June 2014 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 2

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Question 1X - Relativity

(i) Use the expression for the Christoffel connection $\Gamma^a{}_{bc}$ in terms of the metric tensor for the Friedmann-Lemaître-Roberston-Walker space-time in units with $c = 1$ and with line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,,$$

to show that $\Gamma^1{}_{01} = \dot{a}/a$ and $\Gamma^0{}_{01} = 0$.

(ii) For the Friedmann-Lemaître-Roberston-Walker metric from part **(i)**

$$G_0{}^0 = -3\dot{a}^2/a^2 \quad \text{and} \quad G_1{}^1 = -2\ddot{a}/a - \dot{a}^2/a^2\,,$$

where $G_a{}^b$ is the Einstein tensor. Verify by direct calculation that $\nabla_b G_a{}^b = 0$.

Solve the vacuum Einstein equations in the presence of a cosmological constant to determine the form of $a(t)$.

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a vertical patch of the Earth's atmosphere where the gas is isothermal, in hydrostatic equilibrium and characterized by a mean molecular weight μ. What forces are acting on the gas?

From the balance of these forces deduce how gas density $\rho(z)$ varies with the vertical height z.

Does this equation always hold as you consider vertical heights further and further away from the Earth's surface? If it does hold, explain the physical reasons why this is the case. If it does not hold, explain why and deduce what sets the vertical height where the equation breaks down.

(ii) A galaxy cluster contains dark matter and a hot ionized gas component. Assuming the gas is well described by the ideal fluid approximation, derive how the total enclosed mass $M(< r)$ within radius r depends on the gas density and temperature profiles $\rho(r)$ and $T(r)$. You may assume that hydrostatic equilibrium holds and that the system is spherically symmetric.

Now consider that the dark matter density profile $\rho_{\mathrm{DM}}(r)$ is described by a Navarro, Frenk and White (NFW) profile such that

$$\rho_{\mathrm{DM}}(r) \;=\; \rho_{\mathrm{crit}} \frac{\delta_{\mathrm{c}}}{(r/r_{\mathrm{s}})(1 + r/r_{\mathrm{s}})^2} \,,$$

where ρ_{crit} is the critical density of the Universe today, δ_{c} is a characteristic overdensity and r_{s} is a scale radius. Under the simplifying assumptions that the gas is isothermal and that its self-gravity can be neglected, derive an expression for the gas density profile $\rho(r)$ as a function of radius r and gas temperature T.

Compare the functional form of $\rho_{\mathrm{DM}}(r)$ and $\rho(r)$ as r goes to zero and discuss at a qualitative level why the central gas density profile does not follow the dark matter distribution.

TURN OVER...

Question 3X - Physical Cosmology

(i) A volume V of a universe contains material with an internal energy U and pressure p. Assume that as the universe expands the energy of a comoving volume is conserved. Derive the fluid equation

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left[\rho + \frac{p}{c^2}\right] = 0\,,$$

where R is the scale factor, ρ is the density and dots denote differentiation with respect to cosmic time.

(ii) A population of sources has a redshift-independent comoving number density n_0 and luminosities as a function of frequency ν given by

$$L(\nu) \;=\; L(\nu_0)\left(\frac{\nu}{\nu_0}\right)^{-\alpha}\,,$$

where $L(\nu_0)$ is a fiducial luminosity at frequency ν_0. Show that the observed flux density $S(\nu_0)$ of sources at frequency ν_0 is given by

$$S(\nu_0) \;=\; \frac{L(\nu_0)(1+z)^{1-\alpha}}{4\pi {d_L}^2}\,.$$

Now show that for $\alpha > -3/2$ in an Einstein-de Sitter ($\Lambda = 0,\ \Omega = 1$) universe the integrated background intensity at frequency ν_0 from this population of sources distributed out to very high redshift is

$$I(\nu_0) \;\approx\; \frac{2\,c\,n_0 L(\nu_0)}{H_0(2\alpha+3)}\,,$$

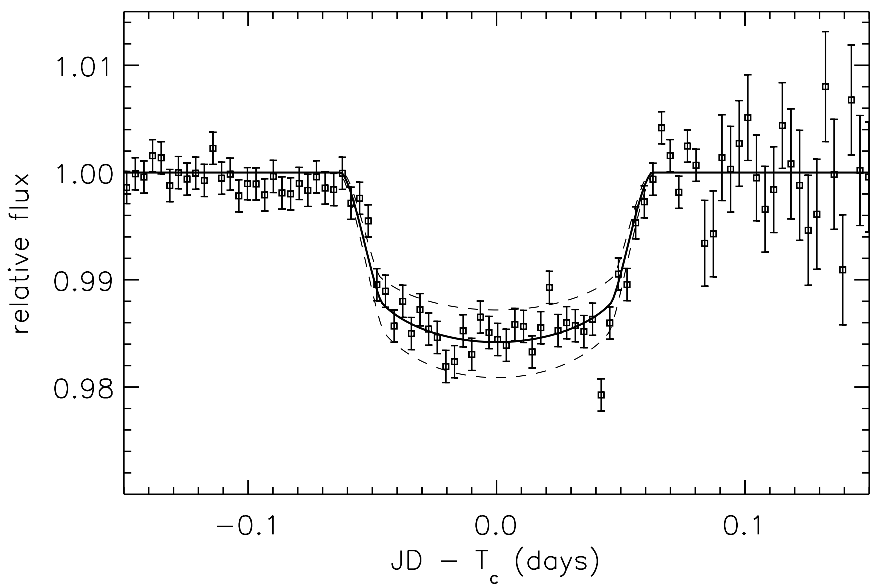
where H_0 is the Hubble constant and $I(\nu_0)$ has units of flux per unit solid angle.

Explain what happens when $\alpha \leq -3/2$.

[You may assume that the comoving distance d_C is related to the luminosity distance d_L by $d_C = d_L/(1+z)$ and that a comoving volume element for an observed solid angle of $d\psi$ is $dV = {d_C}^2\, d\psi\, c\, dz/H(z)$.]

Question 4Z - Structure and Evolution of Stars

(i) A gas giant planet orbits a $1M_{\odot}$ Solar-type star and transits in front of the star once every 3.5 days. The figure shows the light curve of the transit. From the depth of the eclipse calculate the radius of the planet in units of



Earth radii, assuming that radiation from the planet is negligible.

Explain briefly what other information might be inferred from the shape of the light curve.

If the mass of the planet is $0.001M_{\odot}$, calculate the amplitude of perturbations to the radial velocity of the star due to the orbiting planet.

(ii) A star of mass M and radius R is in hydrostatic equilibrium. Derive a general expression for the gravitational potential energy U of the star.

For the special case of stars with density profile $\rho(r) \propto r^{-\alpha}$, find U in terms of the mass and radius of the star and comment on the allowed ranges of α.

TURN OVER...

For an ideal gas show that the total thermal energy of a star is given by

$$K = \int_0^R \frac{3}{2} P(r)\, 4\pi r^2\, dr\,,$$

where $P(r)$ is the pressure at radius r. Hence derive a relation between U and K and explain the physical meaning of the result.

Use this to derive an expression for the average temperature $\overline{T}$ of the star and estimate $\overline{T}$ for a star with mass $\mathrm{M}_\odot$, radius $\mathrm{R}_\odot$, mean molecular weight $\mu = 0.6$ and where $\rho(r) \propto r^{-2}$.

Question 5Y - Statistical Physics

(i) Briefly describe what is meant by the canonical ensemble and define the probability distribution of the state of the system. What is the partition function Z of the ensemble?

Consider a harmonic oscillator in one spatial dimension with energy eigenvalues E_n

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega \, ,$$

in contact with a heat reservoir of temperature T, where ω is the oscillation frequency. Show that the partition function is

$$Z = \exp\left(-\beta\hbar\omega/2\right)[1 - \exp\left(-\beta\hbar\omega\right)]^{-1}, \qquad (*)$$

where $\beta = (k_{\rm B}T)^{-1}$ and $k_{\rm B}$ is Boltzmann's constant.

[You may assume that $\sum_{i=0}^{\infty} x^i = (1-x)^{-1}$ for $x < 1$.]

(ii) Consider the harmonic oscillator of part **(i)**, the Hamiltonian H of which is

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \, ,$$

where p and x are the momentum and position of the oscillator and m is the mass. Calculate the partition function Z, the mean energy $\langle E \rangle$ and the energy fluctuation $\Delta E^2 = \langle \, [\, E - \langle E \rangle \,]^2 \rangle$ using classical statistics.

Using equation $(*)$, or otherwise, calculate $\langle E \rangle$ and ΔE^2 using quantum statistics.

Discuss the classical and quantum mechanical results for $\langle E \rangle$ and ΔE^2 in the low- and the high-temperature limits.

[You may assume the following integral $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$ for $a > 0$.]

TURN OVER...

Question 6Y - Principles of Quantum Mechanics

(i) Let the Hamiltonian of a simple harmonic oscillator be

$$H_0 \;=\; \omega\,\left(a^\dagger a + \tfrac{1}{2}\right)\,,$$

where units are such that $\hbar = 1$, and a and $a^\dagger$ are respectively the annihilation and creation operators with $[a, a^\dagger] = 1$. Explain what properties the eigenstates $|n\rangle$ of H_0 must satisfy and deduce the eigenenergies for these states.

Show that

$$a|n\rangle = \sqrt{n}|n-1\rangle\,, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,.$$

(ii) Consider a system whose unperturbed Hamiltonian is the sum of those for two independent harmonic oscillators given by

$$H_0 = \left(a^\dagger a\,+\tfrac{1}{2}\right)\,+\,2\left(b^\dagger b + \tfrac{1}{2}\right),$$

where $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$ and all other commutators are zero. Find the degeneracies of the eigenvalues of H_0 with energies $E_0 = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$ and $\frac{11}{2}$.

The system is perturbed so that it is now described by the Hamiltonian

$$H = H_0 + \lambda H',$$

where $H' = (a^\dagger)^2 b + a^2 b^\dagger$. Using degenerate perturbation theory calculate the energies of the levels associated with $E_0 = \frac{9}{2}$ to lowest order in λ.

Write down the eigenstates correct to $O(\lambda)$ associated with these perturbed energies.

By explicit evaluation show that they are in fact exact eigenstates of H with these energies as eigenvalues.

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) What are the main components of the Milky Way? Describe briefly how the masses of the individual components are determined and why this indicates the need for dark matter.

(ii) In the halo of the Milky Way several so-called hypervelocity stars have been discovered moving with speeds close to, or above, the estimated Galactic escape velocity. It is conjectured that these were ejected from the centre of the Galaxy where they interacted with the central supermassive black hole of mass $M_{BH} = 4 \times 10^6 \mathrm{M}_\odot$. Estimate the initial speed such a hypervelocity star can attain if it originated in the tidal disruption of a stellar binary system with equal mass stars of mass $1\mathrm{M}_\odot$ separated by 2 au falling in on a parabolic trajectory to within 1 au of the central black hole.

Assume the Galactic rotation curve $v(r)$ is well described by

$$v^2(r) \;=\; \frac{v_c^2}{(1 + r^2/r_h^2)^{1/2}} \, ,$$

where $v_c = 200 \ \mathrm{km\,s^{-1}}$ is the asymptotic circular velocity at small radii and $r_h = 100\,\mathrm{kpc}$ is the outer scale radius of the Halo. Show that the Galactic potential $\phi(r)$ is given by

$$\phi(r) \;=\; -v_c^2 \, \ln\left[\frac{(1 + r^2/r_h^2)^{1/2} + 1}{r/r_h}\right] \, ,$$

and hence find an expression for the Galactic escape velocity as a function of radius r.

A star is observed in the Halo at a distance $r = 10\,\mathrm{kpc}$ moving with a radial velocity of $700 \ \mathrm{km\,s^{-1}}$. Can the star escape from the Galaxy?

TURN OVER...

Question 8Z - Topics in Astrophysics

(i) The Swift satellite has monitored the centre of the Milky Way for an average of 1100 seconds on 715 separate occasions over the five years from 2006-2011. The satellite detected a total of 6 X-ray flares over this period. Assume that these flares are due to accretion of gas clouds, each 1/1000th of an Earth-mass, onto the $M_{BH} = 4 \times 10^6 \mathrm{M}_\odot$ black hole at the centre of the Milky Way and that the radiative efficiency is 10%. Estimate how much mass has been accreted over this period and make a rough estimate of the average luminosity of each event assuming the gas is in free-fall from ≈10 Schwarzschild radii.

If the overall accretion rate is $\propto M_{BH}$ estimate how long would it take the central black hole to double in mass assuming a sufficiently steady supply of similar gas clouds. Comment on your answer.

(ii) Active Galactic Nucleii (AGN) are powered by accretion of matter onto a supermassive black hole. The non-thermal continuum observable from the central region can be characterised by a power law of the form

$$f(\nu)\, d\nu \;\propto\; \nu^{-\alpha} d\nu\,,$$

where $f(\nu)d\nu$ is the luminosity radiated between frequencies ν to $\nu + d\nu$ and $\alpha \approx 0.5$ is the power law index. Neglecting the flux from any AGN emission lines, find an expression for the $g - r$ colour in magnitudes as a function of redshift if the photon detection efficiencies in the g passband (400nm $< \lambda <$ 550nm) and in the r passband (550nm $< \lambda <$ 700nm) are the same, constant as a function of wavelength across each band, and the magnitude system in each band has the same zeropoint.

Sketch how the $g - r$ colour varies as a function of redshift z over the range $0 < z < 5$ commenting on other factors that could influence the observed AGN colour.

The number $\Phi(L)dL$ of AGN in the luminosity range L to $L + dL$ is well-described by a Schechter function such that

$$\Phi(L)dL = \frac{\Phi_*}{L_*}\left(\frac{L}{L_*}\right)^{-\beta} e^{-L/L_*}\, dL\,,$$

where Φ_* is the volume number density of sources at the AGN characteristic luminosity L_* and β characterises the faint-end slope. If the far ultra-violet of the power law continua of AGN is a significant contributor to the cosmological background ionising flux, by what factor would this contribution alter if the faint-end slope changed from $\beta = 0.5$ to $\beta = 1.5$?

In these two cases which AGN luminosities contribute most to the total AGN ionising flux?

What happens if the faint-end slope steepens to $\beta = 2$?

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Wednesday 4 June 2014 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (**i**) *and Part* (**ii**), *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (**ii**) *of each question carrying twice as many marks as Part* (**i**). *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) The vector field V^a is the normalised $V_a V^a = -c^2$ tangent to a congruence of timelike geodesics and $B_{ab} = \nabla_b V_a$. Show that

$$V^a B_{ab} = V^b B_{ab} = 0$$

and that

$$V^c \nabla_c B_{ab} = -B^c{}_b B_{ac} - R^d{}_{acb} V^c V_d \,.$$

[Hint: the relevant Ricci identity is $\nabla_c \nabla_b X_a = \nabla_b \nabla_c X_a - R^d{}_{acb} X_d$.]

(ii) Use the results from part **(i)** and assume that B_{ab} is symmetric. Now let $\theta = B_a{}^a$ and show by writing $B_{ab} = \widetilde{B}_{ab} + \frac{1}{4}\theta g_{ab}$, or otherwise, that

$$\frac{d\theta}{d\tau} \leq -\tfrac{1}{4}\theta^2 - R_{00} \,,$$

where $R_{00} = R_{ab} V^a V^b$ and $\dfrac{d\theta}{d\tau} \equiv V^a \nabla_a \theta$.

Assume in addition that the stress-energy tensor T_{ab} takes the perfect fluid form $(\rho + p/c^2) V_a V_b + p g_{ab}$ and that $\rho c^2 + 3p > 0$. Show that

$$\frac{d\theta^{-1}}{d\tau} > \frac{1}{4}$$

and deduce that, if $\theta(0) < 0$, then $|\theta(\tau)|$ will become unbounded for some value of τ less than $4/|\theta(0)|$.

[You may use without proof the result that $\widetilde{B}_{ab}\, \widetilde{B}^{ab} \geq 0$.]

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a star of mass M subject to a spherically symmetric inflow of isothermal gas with a sound speed $c_{\rm s}$. If the radial velocity of the flow is u show from the continuity and momentum equations that the Mach number $\mathcal{M} = u/c_{\rm s}$ satisfies

$$\left(\frac{1}{\mathcal{M}} - \mathcal{M}\right)\frac{d\mathcal{M}}{dr} = \frac{GM}{c_{\rm s}^2 r^2} - \frac{2}{r},$$

where r is radial distance from the star and G is the gravitational constant.

The general solution to this equation is given by

$$\mathcal{M}^2 - \ln\mathcal{M}^2 = 4\ln\frac{r}{r_{\rm s}} + 4\frac{r_{\rm s}}{r} + C, \qquad (*)$$

where C is a constant and $r_{\rm s} = GM/2c_{\rm s}^2$ is the sonic radius. Consider solutions that correspond to the spherical inflow to a star (Bondi accretion) and subsonic wind outflow from a star. Sketch these solutions in the $(r/r_{\rm s}, \mathcal{M})$ plane and discuss their physical meaning.

Calculate the value of the constant C for these two cases.

(ii) Using the results from part **(i)**, consider now a steady subsonic wind emanating from the surface of the Sun at radius $\mathrm{R}_\odot$. Assume the gas is fully ionized, has mean molecular weight $\mu = 0.6$, temperature $T = 10^6$K and proton number density $n_{\rm p} = 10^8\,\mathrm{cm}^{-3}$ at the Solar surface. With the aid of equation $(*)$, or otherwise, roughly estimate the mass flow per unit time $\dot{M}$ of the Solar wind as it leaves the Solar surface.

Assuming that the wind stays steady as it reaches the Earth, estimate the proton number density of the Solar wind at the Earth's location.

A small cloud G2 of density $\rho_{\rm c} = 6\times10^{-19}\mathrm{g\,cm}^{-3}$ has been detected moving with a velocity v on a highly eccentric orbit towards the supermassive black hole at the centre of our Galaxy. Derive an expression for the binding energy of this cloud per unit mass.

Deduce if G2 is gravitationally bound to the black hole, or not, assuming a central black hole mass $M_{\rm BH} = 4\times10^6\,\mathrm{M}_\odot$, a pericentre distance of 4×10^{15}cm and velocity at pericentre of $5250\,\mathrm{km\,s}^{-1}$.

TURN OVER...

The supermassive black hole is surrounded by a tenuous and hot gaseous atmosphere, where the gas density profile is given by

$$\rho_{\rm hot}(r) = \rho_0 \left(\frac{r_0}{r}\right) = 1.7 \times 10^{-21} \left(\frac{10^{16}{\rm cm}}{r}\right) {\rm g\,cm^{-3}}\,.$$

Assuming that the gas is roughly in hydrostatic equilibrium within the potential of the black hole, derive an expression for the gas temperature profile $T_{\rm hot}(r)$.

Under adiabatic conditions calculate the gas sound speed as a function of distance from the black hole and determine if the G2 cloud is moving sub- or super-sonically through this medium at pericentre. You may assume a mean molecular weight $\mu = 0.6$ and an adiabatic index $\gamma = 5/3$.

If the G2 cloud is not losing mass as it orbits through the hot atmosphere it adjusts to be in rough pressure equilibrium with the surroundings. Given that the cloud temperature $T_{\rm c} = 10^4$K is roughly constant due to photo-ionization equilibrium, deduce how the cloud density $\rho_{\rm c}$ and radius $R_{\rm c}$ change as the cloud moves inwards.

List at least three fluid dynamic processes that can lead to mass loss from the cloud as it moves towards the black hole.

Question 3X - Physical Cosmology

(i) Explain what is meant by the term particle horizon.

Show that in a Friedmann-Robertson-Walker cosmology a flat universe which started to expand at time $t = 0$ has a proper horizon given by

$$d_H(t) = R(t) \int_0^t \frac{c dt'}{R(t')} ,$$

where $R(t)$ is the scale factor.

Find d_H for a matter-dominated universe.

Assume that at last scattering the age of a matter-dominated universe is $t = 4 \times 10^5$ years and $z = 1000$. Estimate the physical horizon size at the epoch of last scattering and its physical size today. Hence calculate the angular size on the sky subtended by this distance and use your result to explain the so-called horizon problem.

(ii) For a Friedmann-Robertson-Walker cosmology show that

$$|\Omega - 1| = \frac{|k| \, c^2}{H^2 R^2} ,$$

where Ω is the density parameter, k is the curvature, H is the Hubble parameter and R is the scale factor.

What is the time dependence of $H^2 R^2$ for a universe dominated by either radiation or matter? Use these results to explain the so-called flatness problem.

During inflationary expansion what is the time dependence of the scale factor? Show how this time dependence solves the flatness problem.

Using

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) ,$$

where G is the gravitational constant, show that the equation of state $p = -\rho c^2$, where p is the pressure and ρ the density, results in inflationary expansion.

Explain briefly how an early period of inflation can also resolve the horizon problem.

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) The radiation emitted by a black body at temperature T is given by

$$f_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/k_{\rm B}T} - 1} ,$$

where $f_\nu d\nu$ is the energy density of the radiation for frequencies from ν to $\nu + d\nu$, $k_{\rm B}$ is Boltzmann's constant, h is Planck's constant and c the speed of light. Find the equivalent black body expression in terms of $f_\lambda d\lambda$, the energy density at wavelengths from λ to $\lambda + d\lambda$.

Hence derive an approximation for the blackbody spectrum valid at long wavelengths ($hc \ll \lambda\, k_{\rm B}T$) and use this to explain why limb darkening in stars is less pronounced at longer wavelengths.

(ii) Explain what is meant by radiative opacity and briefly discuss the main sources of radiative opacity in main sequence stars, distinguishing between line and continuum opacity.

In the centre of cool white dwarfs composed of pure carbon, detailed calculations show that the conduction opacity can be described as

$$\kappa_{\rm cond} \simeq 5 \times 10^{-7} \left(\frac{T}{\rho}\right)^2 \,{\rm cm}^2\ {\rm g}^{-1} ,$$

where T is the temperature and ρ is the density. By assuming that radiative opacity is dominated by Thompson scattering, show that in the centre of such stars, where $T = 10^7\,{\rm K}$ and $\rho = 10^6\,{\rm g\,cm^{-3}}$, conduction completely dominates energy transport.

Question 5Y - Statistical Physics

(i) The free energy F is defined in terms of the energy E, temperature T and entropy S by $F = E - TS$. Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V ,$$

where P and V respectively denote the pressure and volume of the system.

Show that for a function $f(x, y)$ with $(\partial f/\partial x)_y \neq 0$

$$\left(\frac{\partial x}{\partial y}\right)_f = -\frac{(\partial f/\partial y)_x}{(\partial f/\partial x)_y} ,$$

and use this to show that

$$\left(\frac{\partial V}{\partial T}\right)_P = -\frac{(\partial P/\partial T)_V}{(\partial P/\partial V)_T} .$$

(ii) Consider a thermodynamic system with fixed particle number N with the system characterised uniquely by two variables from temperature T, pressure P, volume V, energy E and entropy S. Using the results from part **(i)**, verify that the specific heat at constant volume C_V and the specific heat at constant pressure C_P are given by

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V , \qquad C_P = T\left(\frac{\partial S}{\partial T}\right)_P ,$$

and show that

$$C_P - C_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P .$$

Hence for an ideal gas show that

$$C_P - C_V = Nk_{\mathrm{B}} ,$$

where k_{B} is Boltzmann's constant and briefly interpret this result.

The Van der Waals' equation of state can be written as

$$P = \frac{Nk_{\mathrm{B}}T}{V - bN} - \frac{aN^2}{V^2} ,$$

where a and b are positive constant parameters. Show that for a

TURN OVER...

Van der Waals' gas

$$C_P - C_V = \; Nk_\mathrm{B}\left[1 - \frac{2(V-bN)^2 aN}{V^3 k_\mathrm{B} T}\right]^{-1} .$$

For $a, b \ll 1$ find the fractional correction, linear in N, compared to the ideal gas case.

Many gases are well described by the Van der Waals' equation with

$$a \;=\; \frac{27}{64}\,\frac{(k_B T_\mathrm{cr})^2}{P_\mathrm{cr}} ,$$

with the critical temperature and pressure

$$P_\mathrm{cr} = 71.5 \text{ bar}, \qquad T_\mathrm{cr} = 304.2 \text{ K}.$$

Using these values, give an order-of-magnitude estimate for the percentage deviation of the Van der Waals' result at room temperature and pressure for $C_P - C_V$ from the value obtained for an ideal gas.

Question 6Y - Principles of Quantum Mechanics

(i) Let $\boldsymbol{J} = (J_1, J_2, J_3)$ and $|j, m\rangle$ denote the standard angular momentum operators and states with units such that $\hbar = 1$. Show that $U(\theta) = e^{-i\theta J_2}$ is unitary.

Defining $J_k(\theta) \;=\; U(\theta)\, J_k\, U^{-1}(\theta)$ for $k = 1, 2, 3$, show that

$$J_3(\theta) \;=\; J_3 \cos\theta + J_1 \sin\theta$$

and find the corresponding expressions for $J_1(\theta)$ and $J_2(\theta)$ as linear combinations of J_1, J_2, J_3.

Briefly explain why $U(\theta)$ represents a rotation of $\boldsymbol{J}$ through angle θ about the 2-axis.

(ii) Defining $|j, m\rangle_\theta \;=\; U(\theta)|j, m\rangle$ and using the results and definitions from part **(i)** and the relation

$$J_3|j, m\rangle = m|j, m\rangle\,,$$

show that

$$J_3(\theta)|j, m\rangle_\theta \;=\; m|j, m\rangle_\theta\,. \qquad (*)$$

Hence express $|1, 0\rangle_\theta$ as a linear combination of the states $|1, m\rangle$, where $m = 1, 0, -1$ and make use of equation $(*)$ to determine the coefficients in this expansion. You may assume that for $J_\pm = J_1 \pm iJ_2$ the following relation applies

$$J_\pm\,|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle\,.$$

A particle of spin 1 subject to the Hamiltonian

$$H = -\mu \boldsymbol{B} \cdot \boldsymbol{J}\,,$$

with $\boldsymbol{B} = (0, B, 0)$, is in the state $|1, 0\rangle$ at time $t = 0$. At time t the value of J_3 is measured and found to be $J_3 = 0$. At time $2t$ the value of J_3 is measured again and found to be $J_3 = 1$. Show that the joint probability for these two values to be measured is

$$\frac{1}{8}\,(\sin 2\mu Bt)^2\,.$$

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Explain how dynamical friction works and discuss its importance in theories of structure formation. What observational evidence is there to support this picture?

(ii) A self-gravitating spherically symmetric dark matter halo of density $\rho(r)$, made up of discrete particles of equal mass m, satisifies the isotropic distribution function

$$f(r, v_m) \;=\; \frac{n_0}{(2\pi\sigma)^{3/2}}\,\exp\!\left(\frac{\psi(r)}{\sigma^2} - \frac{v_m^2}{2\sigma^2}\right),$$

where n_0 is a constant, $\psi(r)$ is the gravitational potential at radius r, $v_m = |\mathbf{v_m}|$ is the velocity of dark matter particles and σ their isotropic velocity dispersion. Use Poisson's equation to solve for $\rho(r)$ and hence show that a singular isothermal sphere is a solution of this system.

A body of mass M, where $M \gg m$, moves with velocity $\mathbf{v_M}$ through this dark matter halo. The total deceleration of the body along the direction of $\mathbf{v_M}$ due to dynamical drag is

$$\frac{d\mathbf{v_M}}{dt} = -8\pi^2 \ln(1+\Lambda^2) G^2 M m \left(\int_0^{v_M} f(r, v_m) v_m^2 dv_m\right)\frac{\mathbf{v_M}}{v_M^3},$$

where $v_M = |\mathbf{v_M}|$, G is the gravitational constant and $\Lambda \gg 1$ and is constant throughout the halo. If the body is on a circular orbit write down an expression for the drag force acting on the body at radius r.

A satellite galaxy of mass $M = 5 \times 10^9 \mathrm{M}_\odot$ is on a circular orbit at radius $r = 50\,\mathrm{kpc}$ in this dark matter halo. Find a general expression for the time it will take such a satellite to fall to the centre of the halo and evaluate the time taken for the specific case given if the Coloumb logarithm $\ln\Lambda = 5$.

[You may assume that $\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}dx = \sqrt{2\,\pi\sigma^2}$, $\mathrm{erf}(X) = \frac{2}{\sqrt{\pi}}\int_0^X e^{-x^2}dx$ and that $\mathrm{erf}(1) = 0.8427$]

Question 8Z - Topics in Astrophysics

(i) The effective potential for a satellite orbiting in the plane of the rotating Earth-Sun coordinate system can be approximated as

$$\phi(r,\theta) \;=\; -\frac{G\mathrm{M}_{\odot}}{r} - \frac{G\mathrm{M}_{\oplus}}{(r^2+D^2-2rD\cos\theta)^{1/2}} - \frac{1}{2}r^2\omega^2\,,$$

where r is the distance of the satellite from the Sun, θ the angle of the satellite relative to the Sun-Earth vector, ω the angular orbital velocity of the Earth and D the distance of the Earth from the Sun, assumed to be constant. The Lagrange points in this system are defined as the locations of the stationary values of $\phi(r,\theta)$, with the L1 and L2 Lagrange points lying along the vector defined by $\theta = 0$. Using the above formalism, or otherwise, show that the distance D_{L1} of the near-side Lagrange point and D_{L2} the distance of the far-side Lagrange point from the Earth satisfy

$$D_{\mathrm{L1}} \;\simeq\; D_{\mathrm{L2}} \;\simeq\; D\left(\frac{\mathrm{M}_{\oplus}}{3\,\mathrm{M}_{\odot}}\right)^{1/3}. \qquad (*)$$

(ii) The Gaia satellite was launched on 19th December 2013 and is now at the L2 Lagrange point of the Earth-Sun system. Estimate the distance of the satellite from the Earth using the expression $(*)$.

What is the angular motion of the satellite as seen from the Earth in arcsec per hour with respect to the distant background stars?

Assume that whilst the Gaia satellite is at L2 the sunshield of the spacecraft acts as a perfect diffuse reflective surface with an effective area A_{G} of 15m^2 towards the Earth. Derive an expression relating the apparent magnitude of the satellite m_V to the distance D_{L2} of L2 and the effective reflective area A_{G} and estimate the value of m_V.

Observations show the satellite to be ten times fainter than predicted. What is the observed value of m_V and suggest why the satellite might appear fainter than expected.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 6 June 2014 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

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Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) A plane wave space-time has line element

$$ds^2 = H du^2 + 2du\, dv + dx^2 + dy^2\,,$$

where $H = x^2 - y^2$. Show that the line element is unchanged by the coordinate transformation

$$u = \bar{u}, \qquad v = \bar{v} + \bar{x}e^{\bar{u}} - \tfrac{1}{2}e^{2\bar{u}}, \quad x = \bar{x} - e^{\bar{u}}, \quad y = \bar{y}\,. \qquad (*)$$

(ii) For the plane wave from part **(i)** show more generally that the line element is unchanged by coordinate transformations of the form

$$u = \bar{u} + a, \qquad v = \bar{v} + b\bar{x} + c, \quad x = \bar{x} + p, \quad y = \bar{y}\,,$$

where a, b, c and p are suitably chosen functions of $\bar{u}$ which depend in total on four parameters (arbitrary constants of integration).

Deduce (without further calculation) that the line element is unchanged by a 6-parameter family of coordinate transformations, of which a 5-parameter family act in the wave surfaces $u =$ constant.

For a general coordinate transformation $x^a = x^a(\bar{x}^b)$, give an expression for the transformed Ricci tensor $\bar{R}_{ab}$ in terms of the Ricci tensor R_{cd} and the transformation matrices $\dfrac{\partial x^a}{\partial \bar{x}^c}$.

Calculate $\bar{R}_{\bar{x}\bar{x}}$ when the transformation is given by $(*)$ and deduce that $R_{vv} = R_{vx}$.

Question 2Y - Astrophysical Fluid Dynamics

(i) A steady thin axisymmetric accretion disc surrounds a star of mass M and radius R_*. By considering a portion of the disc in vertical hydrostatic equilibrium at a distance R from the central star and approximating the vertical pressure gradient as $\partial p/\partial z \approx -p/h$, where h is the disc scale height, derive an approximate relation between h, R and the sound speed c_s.

The disc is optically thick and radiates as a black body with a temperature $T_{\rm eff}(R)$ such that

$$2\sigma T_{\rm eff}^4(R) = \frac{3GM\dot{m}}{4\pi R^3}\left[1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right],$$

where σ is the Stefan-Boltzmann constant, G is the gravitational constant and $\dot{m}$ is the mass accretion rate through the disc. If the gas temperature is close to $T_{\rm eff}(R)$, derive how h/R scales with R at large distances ($R \gg R_*$) from the star and sketch an edge-on view of the disc.

By considering the mass in the annulus at R of radial width ΔR, or otherwise, derive a relation between $\dot{m}$, the mass surface density Σ and the radial velocity $u_{\rm R}$.

Given that

$$\nu\Sigma = \frac{\dot{m}}{3\pi}\left[1-\left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right],$$

where $\nu = \alpha c_s h$ is the kinematic viscosity and the constant α lies in the range $0 < \alpha < 1$, comment on how the radial and azimuthal velocities in the disc compare to $c_{\rm s}$.

(ii) By considering a linear shear flow explain at a qualitative level why viscosity arises in fluids and comment on its temperature dependence.

Describe what is meant by the stress tensor σ_{ij} and use this to write the continuity and momentum equations in component form.

If the viscous stress tensor is given by

$$\sigma'_{ij} = -\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)-\zeta\delta_{ij}\frac{\partial u_k}{\partial x_k},$$

derive the general form of the Navier-Stokes equation.

TURN OVER...

What is the physical meaning of η and ζ and which thermodynamical variables do they depend on?

An incompressible non-gravitating viscous fluid with a constant shear viscosity η is enclosed between two planes parallel to the x-z plane. One of these planes is stationary at $y = 0$ and the other at $y = h$ is moving at a constant velocity u along the x-axis. Assuming the flow is steady what are the boundary conditions for the fluid velocity adjacent to the two planes?

Use the Navier-Stokes equation to derive how fluid pressure and velocity depend on spatial coordinates.

How does velocity vary across the fluid in the presence of a pressure gradient along the x-axis?

Question 3X - Physical Cosmology

(i) The number density of fermions in thermal equilibrium at temperature T is given by

$$n = g_s \frac{4\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\left[\exp\left(\frac{E(p)-\mu}{k_{\rm B}T}\right) + 1\right]},$$

where g_s is the number of degrees of freedom for the particles with energy $E(p)$, p is the momentum of a particle, h is Planck's constant, μ is the chemical potential and $k_{\rm B}$ is Boltzmann's constant. Give a brief justification of this equation.

Explain the physical significance of the chemical potential μ, and for $\mu = 0$ derive an expression for the number density n of non-relativistic fermions as a function of temperature.

[You may use the integral $\int_0^\infty x^2 e^{-x^2} dx = \sqrt{\pi}/4$.]

(ii) Using the appropriate form of the Friedmann equations and stating clearly all your assumptions show that at early times the age of the Universe t is related to the photon temperature T by

$$t = \left(\frac{3c^2}{32\pi a G}\right)^{\frac{1}{2}} \frac{1}{T^2},$$

where a is the radiation constant and G is the gravitational constant. When did e^-/e^+ annihilation occur? Why should this equation not be used to calculate when inflation started?

Calculate when inflation ends assuming inflation starts at $t = 10^{-36}$ s and that during inflation the vacuum energy $\Lambda = 1.2 \times 10^{51}\ {\rm s^2 m^{-2}}$ and the Universe expands by 60 e-foldings.

The cross-section for neutrino interactions is $\sigma = 3.5 \times 10^{-67}\, T^2\ {\rm m}^2$ and the number density of neutrinos is given by

$$n_\nu = \frac{3aT^3}{8k_{\rm B}},$$

where $k_{\rm B}$ is Boltzmann's constant. Show that the interaction timescale for neutrinos is

$$t_\nu = 1.4 \times 10^{51}\, T^{-5}\, {\rm s}.$$

At what temperature and at what time did the neutrinos decouple?

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) Assume that the light curve of a supernova is dominated by the energy released in the radioactive decay of an isotope with decay constant $\lambda = \tau^{-1} \ln 2$, where τ is the isotope half-life. Show that the slope of the light curve is

$$\frac{d \log_{10} L}{dt} = -0.434\lambda$$

and derive an analogous expression for the change in bolometric magnitude with time.

In a core collapse supernova explosion $0.075M_\odot$ of ${}^{56}_{27}\mathrm{Co}$ are produced. The half-life of ${}^{56}_{27}\mathrm{Co}$ is $\tau = 77.7\,\mathrm{days}$ and the energy released by the decay of one ${}^{56}_{27}\mathrm{Co}$ atom is $3.72\,\mathrm{MeV}$. If ${}^{56}_{27}\mathrm{Co}$ is the dominant isotope responsible for the supernova light curve, estimate the supernova luminosity both immediately after the formation of cobalt and one year after the explosion.

(ii) Explain in a few sentences what is meant by the term homology when applied to the equations of stellar structure including a brief outline of the mathematical basis of the method.

Consider a set of fully radiative stars with constant opacity κ and uniform mean molecular weight μ. The energy generation rate per unit mass is given by $\epsilon = \epsilon_0 \rho T^{16}$ where ρ is the density, T the temperature and ϵ_0 is a constant. Neglecting radiation pressure and assuming an ideal gas, use an homology argument to show that for this set of stars the following scaling applies

$$L \propto \mu^4 M^3 \,,$$

where M is the stellar mass and L the stellar luminosity.

Find the equivalent scalings for R the stellar radius and T_c the central temperature as a function of μ and M. Hence show that the slope of the theoretical main sequence for such a set of stars is

$$\frac{d \log_{10} L}{d \log_{10} T_\mathrm{eff}} = \frac{76}{9} \,,$$

where T_eff is the effective temperature.

Question 5Y - Statistical Physics

(i) In the grand canonical ensemble the entropy of a system is given by

$$S = -k_\mathrm{B} \sum_i p_i \ln p_i \,,$$

where k_B is Boltzmann's constant and p_i is the probability that the system is in the state i. Show that the entropy can be calculated from the partition function Z through

$$S = k_\mathrm{B} \frac{\partial}{\partial T} \left(T \ln Z \right) \,,$$

where T is the temperature.

(ii) The Dietrici equation of state of a gas is given by

$$P = \frac{k_\mathrm{B} T}{v - b} \exp\left(-\frac{a}{k_\mathrm{B} T v} \right) \,,$$

where P is the pressue, $v = V/N$ is the volume divided by the number of particles, T is the temperature and k_B is Boltzmann's constant. Provide a physical explanation for the constants a and b. Sketch the locus of lines of constant temperature in the (P, V) plane and use this to briefly explain how the Dietrici equation captures the liquid-gas phase transition. Find the maximum temperature at which such a phase transition can occur.

The Gibbs free energy is given by $G = E + PV - TS$, where E is the energy and S is the entropy. Explain why the Gibbs free energy is proportional to the number of particles in the system.

If on either side of a first-order phase transition the Gibbs free energies are equal, derive the Clausius-Clapeyron equation for a line of first-order liquid-gas phase transitions

$$\frac{dP}{dT} = \frac{L}{T(V_\mathrm{gas} - V_\mathrm{liquid})} \,,$$

where L is the latent heat, which you should define, and V_gas and V_liquid are respectively the gas and liquid volumes.

Assume that the volume of liquid is negligible compared to the volume of the gas and that the latent heat is constant. If the gas can be well-approximated by the ideal gas law, derive an equation for the line of first-order phase transitions in the (P, T) plane.

TURN OVER...

Question 6Y - Principles of Quantum Mechanics

(i) Define the interaction picture for a quantum mechanical system with Schrödinger picture Hamiltonian $H_0 + V(t)$ and explain why both pictures give the same physical predictions for transition rates between eigenstates of H_0.

Derive the equation of motion for the interaction picture states $|\overline{\psi(t)}\rangle$

$$i\hbar\,\partial|\overline{\psi(t)}\rangle/\partial t = \overline{V(t)}\,|\overline{\psi(t)}\rangle\,, \qquad (*)$$

where $\overline{V(t)}$ is the time dependent part of the interaction picture Hamiltonian.

(ii) A system from part **(i)** consists of just two states $|1\rangle$ and $|2\rangle$, with respect to which

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\,, \quad V(t) = \hbar\,\lambda \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}\,.$$

Writing the interaction picture state as $|\overline{\psi(t)}\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle$, show that the interaction picture equation of motion $(*)$ can be written as

$$\begin{aligned} i\,\dot{a}_1(t) &= \lambda\,e^{i\mu t}\,a_2(t)\,, \\ i\,\dot{a}_2(t) &= \lambda\,e^{-i\mu t}\,a_1(t)\,, \end{aligned}$$

where $\mu = \omega - \omega_{21}$ and $\omega_{21} = (E_2 - E_1)/\hbar$. Hence show that $a_2(t)$ satisfies

$$\ddot{a}_2 + i\mu\,\dot{a}_2 + \lambda^2 a_2 = 0\,.$$

Given that $a_2(0) = 0$ show that the solution takes the form

$$a_2(t) = \alpha\,\sin\Omega t\;e^{-i\mu t/2}\,,$$

where Ω is a frequency to be determined and α is a complex constant of integration.

Use this solution for $a_2(t)$ to determine $a_1(t)$ and by imposing the normalization condition $\||\overline{\psi(t)}\rangle\|^2 = 1$ at $t = 0$, show that $|\alpha|^2 = \lambda^2/\Omega^2$.

At time $t = 0$ the system is in the state $|1\rangle$. Write down the probability to find the system in the state $|2\rangle$ at time t.

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) The Jeans' equation for a steady-state spherically symmetric system is given by

$$\frac{d(\nu\,\overline{v_r^2})}{dr} + \frac{\nu}{r}\left[2\overline{v_r^2} - (\overline{v_\theta^2} + \overline{v_\phi^2})\right] \;=\; -\nu\frac{d\Phi}{dr}\,.$$

Explain the meaning of the various terms and show how this simplifies for the case of a non-rotating isotropic system.

If the mass of this system is everywhere dark matter dominated and a tracer population with constant velocity dispersion has the following density profile

$$\rho(r) \;=\; \rho_0\, e^{-r^2/2a^2}\,,$$

where ρ_0 is the central density and a is a constant scale length, find the enclosed mass $M(<r)$ as a function of radius r and hence deduce the density profile $\rho_{DM}(r)$ of the dark matter.

(ii) The density $\rho(r)$ of a spherically symmetric globular cluster follows

$$\rho(r) = \begin{cases} \rho_0\,(r/r_0)^{-\alpha} & r \le r_0 \\ 0 & \text{otherwise} \end{cases}$$

where ρ_0 is a constant density scale factor, r is the radial distance from the centre, r_0 is the limiting radius of the system and $\alpha \ge 0$. What is the gravitational potential of the system for all physically possible values of α?

Assume further that the velocity dispersion is isotropic and that the system is in a stable dynamical equilibrium. Using the Jeans' equation from part **(i)**, or otherwise, find the corresponding velocity dispersion profiles $\sigma(r)$ of the cluster.

Comment on any special cases and the range of α that corresponds to physically realisable solutions.

TURN OVER...

Question 8Z - Topics in Astrophysics

(i) A primordial gas cloud composed only of H and He has radius R, uniform proton number density n and mass M typical of a large galaxy. The cloud is virialised and has a gas temperature $T > 10^6$K where the radiative cooling of the primordial gas is dominated by bremsstrahlung (free-free) emission with a cooling time t_{cool} given by

$$t_{cool} = 2 \times 10^5 \frac{T^{1/2}}{n} \text{ s}.$$

Equate the dynamical free-fall timescale for such a cloud with the radiative cooling time and find the radius at which the two timescales are equal.

Comment on the value you obtain.

(ii) Most of the baryons in clusters of galaxies reside in an intra-cluster medium of hot gas. The temperature of the gas is close to that given by the virial theorem. If a typical cluster has a virial radius of 1 Mpc and observed velocity dispersion of $1000\,\mathrm{km\,s^{-1}}$ estimate the temperature of the gas at the virial radius assuming the gas is all hydrogen.

X-ray observations show that the hot gas follows a power-law density profile $\rho(r) \propto r^{-0.8}$ and has a temperature profile $T(r) \propto r^{0.3}$. If the hot gas is in hydrostatic equilibrium find the mass profile $M(< r)$ of the cluster and hence deduce the total mass of the cluster within the virial radius. Compare this with the mass estimated directly from application of the virial theorem.

If the gas mass fraction is 10%, estimate the gas mass within 1 Mpc and the number density of protons at a radius of 100 kpc.

If the combined absolute magnitude of the cluster galaxies is $M_v = -23$ estimate the mass-to-light ratio of the cluster out to the virial radius.

END OF PAPER