NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 3 June 2013 09:00am – 12:00pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS
Script Paper
Blue Cover Sheets
Yellow Master Cover Sheets
1 Rough Work Pad
Tags

SPECIAL REQUIREMENTS
Formulae Booklet
Approved Calculators Allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) Using the Lorentz transformations derive the equation for the change of the velocity direction for a particle moving in the xy-plane from one reference frame to another, where the latter is moving with a relative velocity V along the x-axis. What form does this equation take in the case of a photon? Hence, derive from this equation the classical light aberration equation in the limit $V \ll c$.

(ii) A particle of rest mass m collides elastically with a stationary particle of equal mass. The incident particle has kinetic energy T_0. What is its kinetic energy after the collision if the scattering angle is θ? Assume $c = 1$.

Now consider the elastic collision of a particle of mass m_1 with a stationary particle of mass $m_2 < m_1$. Let $\theta_{\max}$ be the maximum scattering angle of m_1. In the non-relativistic limit $\sin\theta_{\max} = m_2/m_1$. Prove that this result also holds relativistically.

Question 2Y - Astrophysical Fluid Dynamics

(i) A sound wave propagates from a medium in which the speed of sound is c_1 and the density is ρ_1 to a medium in which the speed of sound is c_2 and the density is ρ_2. The direction of propagation is perpendicular to the interface between the two media. State and explain the boundary conditions that must be satisfied by perturbations at the interface.

Show that the ratio t of the amplitude of the velocity perturbations of the transmitted wave to those of the incident wave is

$$t = \frac{2\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} \, .$$

(ii) An isothermal infinite self-gravitating ideal-gas slab of temperature T_s is in a state of hydrostatic equilibrium over the range $-z_\mathrm{s} < z < z_\mathrm{s}$ where it has a density profile given by

$$\rho = \rho_0 \, \mathrm{sech}^2 \left(\frac{z}{H} \right) ,$$

where ρ_0 is the mid-plane density,

$$H = \frac{c_\mathrm{ss}}{(2\pi G \rho_0)^{1/2}}$$

and $c_\mathrm{ss}^2 = \mathcal{R}^* T_\mathrm{s}/\mu$ with $\mathcal{R}^*$ the gas constant and μ the mean molecular weight of the gas. For $|z| > z_\mathrm{s}$ from the mid-plane the same ideal gas is isothermal at temperature T_a (with $T_\mathrm{a} \gg T_\mathrm{s}$). Show that, when the self-gravity of this hot atmosphere can be neglected, the hydrostatic density profile for $|z| > z_\mathrm{s}$ is

$$\rho = \rho_0 \frac{T_\mathrm{s}}{T_\mathrm{a}} \, \mathrm{sech}^2 \left(\frac{z_\mathrm{s}}{H} \right) \exp \left(- \frac{4\pi G \rho_0 H \tanh\left(z_\mathrm{s}/H \right) \left(|z| - z_\mathrm{s} \right)}{c_\mathrm{sa}^2} \right) ,$$

where $c_\mathrm{sa}^2 = \mathcal{R}^* T_\mathrm{a}/\mu$. Explain the boundary condition at $|z| = z_\mathrm{s}$.

A sound wave is launched in the hot atmosphere at $z > z_\mathrm{s}$ and propagates downwards into the slab. What condition on the wavelength of the disturbance would mean that its propagation is almost unaffected by the density stratification within the atmosphere and within the slab? Write down the propagation speeds of the disturbances in the atmosphere and the slab in this limit.

Using the result given in part **(i)**, or otherwise, comment on whether disturbances in the hot atmosphere of a disc are likely to heat efficiently cold material near the mid-plane.

TURN OVER...

Now consider a case in which the slab and atmosphere described above are supported by magnetic, rather than thermal, pressure. Explain why a purely vertical magnetic field cannot support the disc.

When the field is purely horizontal with magnitude $B(z)$, find expressions for $B(z)$ in a slab and atmosphere of the same structure.

[You may assume without proof that magnetic pressure p_B is related to field strength by $p_B = B^2/2\mu_0$.]

Question 3X - Physical Cosmology

(i) In a static infinite Euclidian universe populated with sources of luminosity L and space density n, show that the total flux S from the sky for sources up to a distance $x_{\rm max}$ is

$$S = nLx_{\rm max}\,.$$

Now consider an expanding universe. A small co-moving space element at redshift z has volume

$$dV = R_0^2\, r_{\rm e}^2\, \Omega \frac{cdz}{H(z)}\,,$$

where Ω is a solid angle element, R_0 is the scale factor at $z = 0$, $r_{\rm e}$ is the coordinate distance and $H(z)$ is the Hubble parameter at redshift z. Assuming an Einstein-de Sitter universe populated with sources of luminosity L and co-moving space density n, show that the total observed flux S seen from the whole sky for sources with $0 < z < \infty$ is

$$S = \frac{2}{5} nL \frac{c}{H_0}\,,$$

where H_0 is the Hubble parameter at $z = 0$. Compare and discuss the two results.

(ii) In the early Universe the number density n of particle species in equilibrium with $mc^2 \ll k_{\rm B}T$ at temperature T is given by

$$n(T) = \frac{g}{(2\pi\hbar)^3} \int_0^\infty f(E) 4\pi \tilde{p}^2 d\tilde{p}\,,$$

where $f(E)$ is given by

$$f(E) = \frac{1}{e^{E/k_{\rm B}T} \pm 1} \qquad (+ \text{ for fermions, } - \text{ for bosons}),$$

with E the particle energy, $\tilde{p}$ the momentum, g the number of degrees of freedom and $k_{\rm B}$ Boltzmann's constant. Show that

$$\rho c^2 = \frac{1}{2}\, gaT^4 \quad \text{for bosons and}$$

$$\rho c^2 = \frac{7}{16}\, gaT^4 \quad \text{for fermions,}$$

where ρ is the density and a is the radiation constant.

TURN OVER...

Derive similar expressions for the entropy density $s(T)$.

Let T_b and T_a be the temperature just before and just after electron-positron annihilation respectively. What is the value of the ratio $T_\mathrm{a}/T_\mathrm{b}$?

[You may assume that

$$\int_0^\infty \frac{y^3}{e^y - 1} dy = \frac{\pi^4}{15}$$

$$\int_0^\infty \frac{y^3}{e^y + 1} dy = \frac{7\pi^4}{120}$$

and that pressure p and density ρ are related by $p = \rho c^2/3$.]

Question 4Z - Structure and Evolution of Stars

(i) Explain in a few words what is meant by a P-Cygni line profile.

Use a sketch to illustrate how such a profile arises and explain the necessary conditions for producing a P-Cygni profile.

What physical parameters can be deduced from the analysis of P-Cygni profiles?

Consider two stars, both of spectral type O7V. Star 1 has a solar metallicity, while Star 2 has metallicity 1/100th of solar. Which of the two stars would you expect to have stronger P-Cygni lines? Give a physical reason for your answer.

(ii) Explain what is meant by the Kelvin-Helmholtz timescale $\tau_{\rm KH}$. Use the virial theorem to deduce the dependence of $\tau_{\rm KH}$ on the stellar mass M, radius R and luminosity L.

The mass-loss rate in massive stars can be approximated by the expression

$$\dot{M} = \left(\epsilon \frac{v_{\rm esc}}{c}\right) \frac{LR}{GM},$$

where $v_{\rm esc}$ is the escape velocity at the surface of the star, c is the speed of light, G is the gravitational constant and ϵ is an efficiency factor ($\epsilon \leq 1$). How does the mass-loss timescale $\tau_{\rm ml}$ compare with $\tau_{\rm KH}$?

Show that the rate of energy supply required to sustain a mass-loss rate $\dot{M}$ is much smaller than L.

Find a relation between $\tau_{\rm ml}$ and the nuclear timescale $\tau_{\rm nuc}$ of the star and show that for massive stars $\tau_{\rm ml} < \tau_{\rm nuc}$.

TURN OVER...

Question 5Y - Statistical Physics

(i) By considering the number of ways N fermions of total energy E can be distributed over states i of energy ϵ_i and degeneracy g_i, show that the mean occupancy $\bar{n}_i$ is given by the Fermi-Dirac distribution

$$\bar{n}_i = \frac{g_i}{e^{\beta(\epsilon_i - \mu)} + 1} \, ,$$

where $\beta = 1/k_{\rm B}T$, $k_{\rm B}$ is Boltzmann's constant, T is the temperature and μ is the chemical potential.

(ii) A semi-infinite thin metal slab occupies $z \leq 0$. The space $z > 0$ is a vacuum. An electron with momentum (p_x, p_y, p_z) inside the slab escapes from the metal in the $+z$ direction if it has a sufficiently large momentum p_z to overcome a potential barrier $V_0 > 0$ relative to the Fermi energy $\epsilon_{\rm F}$ so that

$$\frac{p_z^2}{2m} \geq \epsilon_{\rm F} + V_0 \, ,$$

where m is the electron mass. At temperature T some fraction of electrons satisfy this and so give rise to a current density j_z in the $+z$ direction. Each electron escaping provides a contribution $\delta j_z = -ev_z$ to this current density, where v_z is its velocity and e is the elementary charge. Sketch the Fermi-Dirac distribution as a function of energy in the limit $k_{\rm B}T \ll \epsilon_{\rm F}$. What does this mean for the chemical potential μ?

Assume that the electrons behave as an ideal non-relativistic Fermi gas, that $k_{\rm B}T \ll V_0$ and that $k_{\rm B}T \ll \epsilon_{\rm F}$. Calculate the current density j_z associated with the electrons escaping from the metal in the $+z$ direction.

How could we easily increase the magnitude of the current?

Question 6Y - Principles of Quantum Mechanics

(i) Consider a composite system of several identical, non-interacting particles. Describe how the multi-particle state is constructed from single particle states. For the special case of two identical particles, describe how the interchange symmetry leads to the emergence of bosons and fermions.

(ii) A quantum mechanical system consists of two identical particles, each of which has spin 1. Each single particle i has a spin-independent Hamiltonian $\hat{H}_i = \hat{H}(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{p}}_i)$ with non-degenerate eigenvalues E_j ($E_0 < E_1 < E_2 < \ldots$) with corresponding wavefunctions $\psi_j(\boldsymbol{x})$. In terms of these single particle wavefunctions and single particle spin states $|1\rangle$, $|0\rangle$ and $|-1\rangle$, write down all of the multi-particle states and energies for both the ground state and the first excited state.

For a particular two-particle system E_i is a linear function of i. Show that the degeneracy of the nth excited state can be written as a polynomial in n and find the polynomials explicitly when n is even and when n is odd.

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Show that for a particle on a circular orbit at radius R_0 in the $z = 0$ plane of an axisymmetric potential $\Phi(R, z)$ the circular velocity V_c is given by

$$V_c^2 = R_0 \frac{\partial \Phi}{\partial R}\bigg|_{R=R_0, z=0} .$$

Describe how the rotation curves of spiral galaxies are determined and why their shapes indicate the need for dark matter.

(ii) For the Kuzmin disk potential

$$\Phi(R, z) = \frac{GM}{\sqrt{R^2 + (a + |z|)^2}} ,$$

show that $\nabla^2 \Phi = 0$ for $z \neq 0$, where M and a are constants, G is the gravitational constant and R and $|z|$ are the radial and perpendicular-to-plane distances respectively.

Determine the mass surface density $\Sigma(R)$ and hence show that the constant M is the total mass of the disk.

Derive the circular velocity $V_c(R)$ of a particle in the plane of the disk and sketch the variation of this velocity as a function of radius. At what radius is the circular velocity a maximum?

Question 8Z - Topics in Astrophysics

(i) A distant Solar-like star has an Earth-like planet orbiting at a constant radius of 1 AU with a period of 1 year. Estimate the fractional change in the brightness of the star when the planet passes directly in front of the star and the duration of the transit of the planet.

Derive the annual change in the radial velocity of the star due to the orbital motion of the planet and estimate the relative orbital angular momenta of the two bodies about their common centre-of-mass. Comment on your results.

(ii) The recently discovered Kuiper Belt dwarf planet Makemake is in a circular orbit at a distance of 52 AU from the Sun and has a diameter of 1450 km. A Solar-like star at a distance of 10 pc from the Earth is predicted to be occulted by Makemake when it is in opposition at 00:00 UT on 21st September 2013. This occultation will be optimally visible in Cambridge (latitude 52°), weather permitting. At this point in time the star will have a tangential velocity of $10\,\mathrm{km\,s^{-1}}$ relative to the Earth. Estimate the physical size and shape of the projected shadow of Makemake on the surface of the Earth and the maximum duration of the occultation.

What is the precision in arcseconds required for the predicted relative positions of the star and Makemake such that an occultation could be observed in Cambridge at the anticipated time.

If Makemake has an albedo of 4% at optical wavelengths estimate its apparent visual magnitude m_{v}.

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Wednesday 5 June 2013 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

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Tags	

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Question 1X - Relativity

(i) A mirror moves perpendicular to its plane with a speed v. With what angle to the normal is a ray of light reflected if it is incident at an angle θ? What is the change in the frequency of the light?

(ii) Show by explicit examination of components that the equations

$$\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}} = 0 \quad \text{and}$$

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\beta}} = 4\pi J^{\alpha}\,,$$

where $F_{\alpha\beta}$ is the electromagnetic tensor, J^{α} is the four-current and α, β and $\gamma = 0, 1, 2$ or 3, contain Maxwell's equations in Minkowski space.

[Recall that

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{bmatrix} \quad .]$$

Question 2Y - Astrophysical Fluid Dynamics

(i) Explain what is meant by a barotropic flow. For a steady spherically symmetric barotropic wind in the presence of an inwardly directed radial gravitational field of acceleration $|g_r|$, show that the flow velocity u is given by

$$\left(u^2 - c_\mathrm{s}^2\right) \frac{\partial \ln u}{\partial r} = \frac{2c_\mathrm{s}^2}{r} - |g_r| \, ,$$

where c_s is the local sound speed and r is the distance from the origin.

(ii) A barotropic outflow is everywhere driven by pressure in the atmosphere of a planet composed purely of ionised hydrogen. Its density ρ and temperature T are related by

$$T = T_0 \left(\frac{\rho}{\rho_0}\right)^a ,$$

where ρ_0 and T_0 are fiducial values and pressure, density and temperature obey the ideal gas law. For what values of a does the temperature of the flow increase with radius from the planet? Explain why there is no steady outflow solution if $a < -1$.

A planet of roughly Jupiter mass (2×10^{27} kg) orbits a star which gives rise to an X-ray flux at the planet of magnitude $F_\mathrm{X} = 1\,\mathrm{W\,m^{-2}}$. The stellar X-rays heat the planet's outer layers and drive a spherically symmetric wind from the planet. The temperature of X-ray heated gas of number density n is given by

$$T = 1000 \left(\frac{F_\mathrm{X}/n}{5 \times 10^{-15}\,\mathrm{W\,m}}\right)^{0.9} \mathrm{K} \, .$$

When any variation of F_X with distance from the planet can be neglected show that, in a steady state, the temperature of the wind must increase with increasing distance from the planet.

The sonic surface of the X-ray heated flow is located at a radius $R_\mathrm{s} = 8 \times 10^9$ m from the centre of the planet. Using the relation given in part **(i)**, or otherwise, calculate the temperature and density of the wind at R_s. Hence determine the mass-loss rate in the wind and the timescale on which the planet loses 10% of its mass in this way.

When the radius of the sonic surface is independent of X-ray luminosity also determine how the mass-loss rate changes when the X-ray luminosity of the star is increased by a factor of 10.

TURN OVER...

Question 3X - Physical Cosmology

(i) For a matter-dominated universe

$$\ddot{\delta} + 2\frac{\dot{R}}{R}\dot{\delta} - 4\pi G\bar{\rho}\delta = 0\,,$$

where $\delta = (\rho - \bar{\rho})/\bar{\rho}$, $\bar{\rho}$ is the mean density and R is the scale factor. Find the growing mode solutions $\delta(t)$, $\delta(R)$ and $\delta(z)$ for an Einstein-de Sitter universe.

What sets the upper and lower length scales for growth to actually occur?

(ii) Show that the sound speed c_s in the early Universe is given by

$$c_\mathrm{s}^2 = \frac{dp}{d\rho} = \frac{1}{3}\left(\frac{4\rho_\mathrm{r}}{3\rho_\mathrm{m} + 4\rho_\mathrm{r}}\right)c^2\,,$$

where ρ_r is the radiation density, ρ_m is the matter density and c is the velocity of light.

How does c_s depend on the scale factor R before the time of equivalence t_eq, between t_eq and the time of recombination t_rec and after t_rec?

The sound speed just before recombination is $10^{11/3}$ greater than it is just after recombination. Give a descriptive explanation for this large difference.

The Jeans length is given by

$$\lambda_\mathrm{J} = c_\mathrm{s}\sqrt{\frac{\pi}{G\bar{\rho}}}\,,$$

where $\bar{\rho}$ is the mean density and the Jeans mass M_J just before recombination is $10^7\,\mathrm{M}_\odot$. Sketch $M_\mathrm{J}(R)$ from a time before t_eq to a time after t_rec. What does this imply for the growth of structure in a universe where the matter is purely baryonic?

For non-baryonic dark matter during the period $t_\mathrm{eq}/4 < t < 4t_\mathrm{eq}$ perturbations grow according to

$$\delta \propto 3\eta + 2\,,$$

where $\eta = \rho_\mathrm{DM}/\rho_\mathrm{r}$ and ρ_DM is the dark matter density. Find the growing modes $\delta(t)$, $\delta(R)$ and $\delta(z)$ for an Einstein-de Sitter universe before and after t_eq for the non-baryonic dark matter. What value of δ is needed at t_eq to explain the structures we see in the present day Universe?

[You may assume that $\Delta T/T \approx 10^{-5}$ for the anisotropies in the cosmic microwave background.]

Question 4Z - Structure and Evolution of Stars

(i) Assume that the density $\rho(r)$ of a star varies linearly from the centre of the star $r = 0$ to the surface $r = R$ such that

$$\rho(r) = \rho_{\rm c}\left(1 - \frac{r}{R}\right) ,$$

where $\rho_{\rm c}$ is the central density. Derive expressions for the total mass of the star $M(R)$ and for $\rho_{\rm c}/\bar{\rho}$, where $\bar{\rho}$ is the mean density of the star.

(ii) For the star from part **(i)** derive an expression for the pressure $P(r)$ as a function of radius. Use this result to deduce the dependence of the central pressure $P_{\rm c}$ on stellar radius and mass.

Assuming an ideal gas law, derive an expression for the temperature $T(r)$ as a function of radius r and show that $T(R) = 0$.

The star is composed only of completely ionized hydrogen. Derive an expression for the central temperature $T_{\rm c}$ as a function of the stellar radius R.

Compare the central temperature computed in this way to that of the Sun, 1.6×10^7 K. What conclusions do you draw from this comparison?

Question 5Y - Statistical Physics

(i) Give the first law of thermodynamics for a closed system containing a fixed number of particles. Explain carefully the meaning of each term.

What is equipartition of energy? Under which conditions is it valid? Use it to show that the heat capacity at constant volume C_V for a monatomic ideal gas of N particles is given by

$$C_V = \frac{3}{2} N k_{\mathrm{B}} \,,$$

where k_{B} is Boltzmann's constant.

(ii) Use the first law of thermodynamics and the fact that for an ideal gas $(\partial E/\partial V)_T = 0$ to show that the entropy S of an ideal gas containing N particles can be written as

$$S(T,V) = \int \frac{C_V(T)}{T}\, dT + N k_{\mathrm{B}} \ln \frac{V}{N} + \mathrm{const}\,,$$

where T and V are the temperature and volume of the gas, k_{B} is Boltzmann's constant and C_V is the heat capacity at constant volume.

Now consider a gas in which the number of particles N can vary. The first law of thermodynamics becomes

$$dE = T dS - p dV + \mu dN \,,$$

where μ is the chemical potential. The Gibbs free energy G is the thermodynamic potential for which temperature T, pressure p and particle number N are regarded as independent variables. How is it obtained from the energy E?

Calculate the chemical potential μ for an ideal gas with heat capacity per particle $c_V(T) = C_V/N$.

Thence show that for an ideal monatomic gas

$$\mu(T,p) = k_{\mathrm{B}} T \left(\frac{\mu(T_0,p_0)}{k_{\mathrm{B}} T} - \frac{5}{2} \ln \frac{T}{T_0} + \ln \frac{p}{p_0} \right),$$

where T_0 and p_0 are a fiducial temperature and pressure.

Question 6Y - Principles of Quantum Mechanics

(i) A particle has normalized angular momentum states $|j\ m\rangle$ such that $\hat{J}^2|j\ m\rangle = j(j+1)\hbar^2|j\ m\rangle$ and $\hat{J}_3|j\ m\rangle = m\hbar|j\ m\rangle$, where $\hat{\boldsymbol{J}}$ is the angular momentum operator. Write down the general form for the commutator $[\hat{J}_i, \hat{J}_j]$.

Ladder operators $\hat{J}_+$ and $\hat{J}_-$ are defined by $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$. Find the commutators $[\hat{J}^2, \hat{J}_i]$, $[\hat{J}_3, \hat{J}_\pm]$ and $[\hat{J}^2, \hat{J}_\pm]$ explicitly in terms of $\hat{J}^2$, $\hat{J}_3$ and $\hat{J}_\pm$.

Show that $\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_3^2 + \hbar\hat{J}_3$.

(ii) Show that $\hat{J}_+|j\ m\rangle$ and $\hat{J}_-|j\ m\rangle$ are eigenstates of $\hat{J}^2$ and $\hat{J}_3$ and find their eigenvalues.

Find real coefficients $a_\pm$ such that the states $a_+\hat{J}_+|j\ m\rangle$ an $a_-\hat{J}_-|j\ m\rangle$ are normalized. Thence evaluate

$$\langle j\ j{-}1|\hat{J}_+^{j-1}\hat{J}_-^{j}|j\ j\rangle$$

explicitly in terms of j.

Consider the combination of a spinless nucleus with an electron of spin $\frac{1}{2}\hbar$ and orbital angular momentum $\hbar$. Calculate the probability that the electron has a spin of $+\frac{1}{2}\hbar$ in the z direction when the combined system has an angular momentum of $+\frac{1}{2}\hbar$ in the z direction and a total angular momentum of (a) $+\frac{3}{2}\hbar$ and (b) $+\frac{1}{2}\hbar$.

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Give a brief account of the dynamical structure and evolution of globular clusters in the Halo of the Milky Way. Include an explanation of the physical processes and timescales that might drive cluster development. What makes globular clusters useful as dynamical probes of the Halo?

(ii) A globular cluster on an orbit with eccentricity e falls to its certain death in a host galaxy with a flat circular velocity curve with $V_{\rm c} = 200\,{\rm km\,s^{-1}}$. The stars tidally stripped from the cluster arrange themselves into two tails, leading and trailing. Assume that all tearing happens at pericentre where the stars leave the cluster through two Lagrange points separated by $2r_{\rm J}$. At pericentre the debris has the systemic velocity of the cluster. Find the difference in apocentre distances for stars in the leading and trailing tails.

A globular cluster in this host galaxy is observed to have a spread in apocentre distances of its tidal tails of 3 kpc. The apocentric distance of the main body of the cluster is at 100 kpc and the orbital eccentricity $e = 0.8$. Estimate the total mass of the cluster.

Question 8Z - Topics in Astrophysics

(i) An Active Galactic Nucleus (AGN) is powered by accretion of matter on to a supermassive black hole of mass M. Energy is generated by gravitational infall of material that is heated to a high temperature. Derive the Eddington limit L_{E} due to outward radiation pressure on the infalling material and show that for radially accreting ionized hydrogen the AGN has a maximum luminosity given by

$$L_{\mathrm{E}} = \frac{4\pi G M m_{\mathrm{p}} c}{\sigma_{\mathrm{T}}},$$

where G is the gravitational constant, c is the velocity of light, m_{p} is the proton mass and σ_{T} is the Thomson scattering cross-section of an electron.

Assuming a 100% conversion efficiency, what mass of AGN radiating at the Eddington limit would be required to power a typical household?

(ii) There is dynamical evidence that the nucleus of the Milky Way harbours a supermassive black hole with mass $4 \times 10^{6}\,\mathrm{M}_{\odot}$. If this nuclear region radiates with a luminosity at 10% of the Eddington limit what is its luminosity in units of $\mathrm{L}_{\odot}$? What is this as an absolute bolometric magnitude M_{bol}, if for the Sun $M_{\mathrm{bol}} = 4.75$? What mass accretion rate would be required to sustain this luminosity?

Starting from a black hole seed mass of $100\,\mathrm{M}_{\odot}$, how long would it take the black hole in the Milky Way accreting at this rate to reach its current mass? What are the implications of the value obtained?

Assume that this central radiation is completely absorbed by a giant opaque spherical dust cloud of radius 10 pc surrounding the nucleus. The cloud is in thermal equilibrium and behaves as if it were a large black body. What is the temperature of the cloud and at what wavelength is the cloud emission a maximum?

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 6 June 2013 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 3

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Question 1X - Relativity

(i) A particle falls radially into a non-rotating black hole from rest at infinity. What is its inward coordinate velocity dr/dt at a coordinate radius r? What is the locally measured velocity relative to a stationary observer at the same radius?

(ii) Derive the equations of motion relating t, r and τ, where τ is the proper time, for the same particle as in **(i)**. Consider two cases: a particle released from rest at infinity; and a particle released from rest at $r = R$.

Question 2Y - Astrophysical Fluid Dynamics

(i) State and explain the conditions on the speeds u_1 and u_2 that are satisfied at an isothermal shock front and hence prove that

$$\frac{u_2}{u_1} = \left(\frac{c_s}{u_1}\right)^2 ,$$

where subscripts 1 and 2 denote conditions upstream and downstream of the shock and c_s is the sound speed.

For an ideal gas state the sign of the specific entropy change between the pre-shock state and the post-shock state calculated above and describe the physical processes that occur in the thin region separating these two regimes and how these affect the specific entropy of the flow.

(ii) Derive the dispersion relation for sound waves in a steady and uniform self-gravitating medium and hence show that there is a critical (Jeans) length scale

$$\lambda_J = c_s\sqrt{\frac{\pi}{G\rho}} ,$$

where c_s is the sound speed, G is the gravitational constant and ρ the gas density, such that scales larger than this are subject to gravitational collapse. Explain why a gravitationally unstable cloud, which remains at constant temperature, should fragment into successively smaller pieces as the cloud collapse proceeds.

The temperature T and density ρ of a gas cloud that is collapsing under gravity is governed by the equation

$$4\kappa\sigma T^4 = \frac{\mathcal{R}^* T}{\mu}\,(16\pi G\rho)^{\frac{1}{2}} ,$$

where κ is the opacity of the gas, σ is the Stefan-Boltzmann constant, $\mathcal{R}^*$ is the gas constant and μ is the mean molecular weight. The left hand side represents the rate of cooling per unit mass owing to optically thin cooling. Suggest, without detailed calculation, what process is represented by the right hand side of this equation.

The optical depth across fragments of length λ_J,

$$\tau_J = \rho\kappa\lambda_J ,$$

increases as the cloud collapses until it eventually attains a value of unity.

TURN OVER...

At this point fragments cannot cool efficiently and further sub-fragmentation is impossible. Use this information to calculate the density and temperature corresponding to fragments with $\tau_{\mathrm{J}} = 1$ when the opacity is given by

$$\kappa = 2 \times 10^{-5} \left(\frac{T}{\mathrm{K}}\right)^2 \mathrm{m}^2\,\mathrm{kg}^{-1}$$

and the gas is neutral hydrogen. Hence calculate the minimum mass for gravitational fragmentation and comment on the implications of your answer for the existence of free-floating planets forming from interstellar gas clouds.

Question 3X - Physical Cosmology

(i) The cross section for neutrino interactions is given by

$$\sigma = 3.5\times10^{-67}\left(\frac{T}{\mathrm{K}}\right)^2 \quad m^2$$

and the number density of a single neutrino type is given by

$$n(T) = 7\times10^6\left(\frac{T}{\mathrm{K}}\right)^3 \quad m^{-3}\,,$$

where T is the temperature. Estimate the typical time between interactions. As the Universe cools, at what temperature do the neutrinos decouple?

When the electrons and the positrons annihilate the temperature of the photons increases by a factor of $(11/4)^{1/3}$. Assuming that neutrinos are massless, what is the temperature of the relic neutrinos present in the Universe today?

(ii) At a time just before recombination estimate the number density of the electrons, the mean free path and the average time between photon-electron collisions. You may assume that the intergalactic gas is fully ionized at the present epoch with an electron density of $0.2\,\mathrm{m}^{-3}$ and that $z = 1000$ just before recombination. What does this imply about the opacity of the Universe at early times?

Assume that the gas remains neutral after recombination until an early generation of stars reionizes it at redshift $z = z_*$ and that from then on the electron fraction $\chi_\mathrm{e} = n_\mathrm{e}/n_\mathrm{H}$ remains constant. Show that in an Einstein-de Sitter universe the optical depth for scattering by the free electrons obeys

$$\tau = C\chi_\mathrm{e}(1 + z_*)^{3/2}\,,$$

where C is a dimensionless constant. Estimate the redshift z_* at which the gas was ionised when $C = 0.001$, $\chi_\mathrm{e} = 0.1$ and $\tau = 0.01$.

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) X-ray observations have shown that the corona of the Sun reaches a temperature of nearly 10^6 K. Why doesn't the Sun appear as a blackbody with $T_{\rm eff} \approx 10^6$ K?

When the temperature gradient of a stellar atmosphere is such that the temperature increases *outwards*, what type of spectral line would you expect to see in the stellar spectrum at wavelengths where the opacity is greatest?

Consider a star surrounded by a large hollow spherical shell of hot gas. Under what circumstances would you see this shell as a ring around the star? If you observed the ring with a spectrograph what type of spectrum would you see?

(ii) A white dwarf star can be modelled as an isothermal degenerate core with temperature $T_{\rm c}$, mass $M_{\rm c}$ and molecular weight $\mu_{\rm c}$. The core cools and loses energy at a rate

$$L = -\frac{3}{2}\frac{\mathcal{R}^* M_{\rm c}}{\mu_{\rm c}}\frac{dT_{\rm c}}{dt}\,,$$

where $\mathcal{R}^*$ is the gas constant and is overlaid by a thin non-degenerate envelope with opacity following Kramers' law

$$\kappa = \frac{A\,\rho}{T^{3.5}}\,,$$

where A is a constant and $\rho(r)$ and $T(r)$ are respectively the density and temperature at radius r. Using equations of stellar structure show that in the envelope pressure and density are related by

$$P(r) \propto \rho(r)^{\frac{n+1}{n}}\,,$$

with $n = 3.25$.

The transition density from core to envelope is given by $\rho_{\rm t} = C T_{\rm c}^{3/2}$, where C is a constant. Show that the luminosity L depends on the core temperature as $L \propto T_{\rm c}^{3.5}$.

Show also that the luminosity decreases with time as $L \propto t^{-7/5}$. Explain how you would verify empirically that this result holds for real white dwarfs.

Suppose that in the Milky Way white dwarfs are formed at a constant rate and that we are able to see all white dwarfs to a given distance from the Earth. How would you expect the number of white dwarfs per luminosity bin to vary with luminosity? Is this what is observed? If not give a plausible interpretation of the discrepancy.

Question 5Y - Statistical Physics

(i) Describe the microcanonical ensemble.

For quantum mechanical systems the energy levels are discrete. How can the number of states be interpreted in this case?

(ii) Consider N independent, linear harmonic oscillators of equal frequency ω. Their total energy is given by

$$E(n_1, n_2, \ldots, n_N) = \sum_{i=1}^{N} \hbar\omega \left(n_i + \frac{1}{2}\right) = M\hbar\omega + \frac{N}{2}\hbar\omega\,,$$

where

$$M = \sum_{i=1}^{N} n_i\,,$$

and $n_i = 0, 1, 2, \ldots$ is the excitation number of oscillator i. Show that for fixed N and M the number $g_N(M)$ of ways to distribute the M excitations over N oscillators is given by

$$g_N(M) = \frac{(M+N-1)!}{M!\,(N-1)!}\,.$$

For a given total energy between E and $E + \Delta E$ calculate the probability distribution $P(E_1)$ for the first oscillator as a function of its energy $E_1 = n_1\hbar\omega + \frac{1}{2}\hbar\omega$.

Show that for $\Delta E = \hbar\omega \ll E$

$$P(E_1) \approx \frac{g_{N-1}(M - n_1)}{g_N(M)}\,.$$

Approximate this in the limit when $N \gg 1$ and $M \gg n_1$ and argue that the result is unchanged when $\Delta E > \hbar\omega$ as long as $\Delta E \ll E$.

TURN OVER...

Question 6Y - Principles of Quantum Mechanics

(i) A Hamiltonian $\hat{H}$ has time independent, non-degenerate, normalised eigenstates $|\psi_n\rangle$ such that $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, with $n = 1, 2, 3, \ldots$. The system is perturbed so that the Hamiltonian becomes $\hat{H}' = \hat{H} + \lambda\hat{H}^{(1)}$, where $\lambda \ll 1$ and is real and positive. Show that, to first order in λ, the energy of the nth excited state is perturbed to $E_n + \lambda E_n^{(1)}$, where $E_n^{(1)} = \langle\psi_n|\hat{H}^{(1)}|\psi_n\rangle$ and find the first order perturbation to the nth state.

(ii) The infinite square-well potential has $V(x) = 0$ for $|x| < a$ and is infinite for $|x| > a$. It is perturbed by the potential $\delta V = \epsilon x/a$ for $|x| < a$. Find the perturbation in the nth energy level E_n to first order in ϵ. Show that the ground-state wavefunction is

$$\psi_1(x) = \frac{1}{\sqrt{a}}\left[\cos\frac{\pi x}{2a} + \frac{32\epsilon}{\pi^2 E_1}\sum_{m=1}^{\infty}(-1)^m\frac{m}{(4m^2-1)^3}\sin\frac{m\pi x}{a} + \mathcal{O}(\epsilon^2)\right].$$

Comment on the conservation of parity in the unperturbed and perturbed systems.

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) Using Kepler's third law, or otherwise, deduce the free-fall time for a sphere of radius R with constant density ρ to collapse to a point under self-gravity.

Estimate this timescale for a constant density spherical cloud of molecular gas with mass $1000\,\mathrm{M}_\odot$ and radius $100\,\mathrm{pc}$.

(ii) Massive elliptical galaxies have at least half of their mass $M_{\rm ini}$ already assembled at high redshift. They then grow further through a series of mergers that bring in a total of $M_{\rm acc}$ in accreted mass. The fractional mass increase due to the accreted material is $\eta = M_{\rm acc}/M_{\rm ini}$, while the total kinetic energy of the material is $K_{\rm acc} = 0.5 M_{\rm acc} < v_{\rm acc}^2 >$ and $\epsilon = < v_{\rm acc}^2 > / < v_{\rm ini}^2 >$. Using the virial theorem deduce the ratio of the final to initial mean square speeds, gravitational radii and densities.

What are the differences between accretion of a single equal mass system compared with the equivalent mass randomly accreted from much smaller sub-units $i = 1, 2, 3, \ldots$ where $M_i \ll M_{\rm ini}$.

TURN OVER...

Question 8Z - Topics in Astrophysics

(i) Sketch the observed optical spectrum ($300 - 1000\,\mathrm{nm}$) of a redshift $z = 4$ quasar noting the main features that would typically be present.

The spectrum of a distant quasar shows two broad emission lines with observed wavelengths of 395.2 nm and 910.0 nm. The strongest lines are most likely to be either Lyman-α with a rest wavelength of 121.6 nm, CIV with a rest wavelength of 154.9 nm, or MgII with a rest wavelength of 280.0 nm. Determine the redshift of the quasar and identify the two lines that have been detected.

(ii) The supermassive black hole powering the quasar in **(i)** has a mass of $10^9\,\mathrm{M}_\odot$. The broad emission lines seen in the spectrum have observed line widths of 10 nm and 23 nm respectively. Assuming that these lines originate in a large ionised gas cloud close to the quasar nucleus estimate the approximate distance of the cloud from the black hole. How does this compare with the Schwarzschild radius of the black hole?

An intervening absorption system along the line-of-sight to this quasar produces two narrow absorption lines at wavelengths of 449.2 nm and 812.0 nm, with intrinsic line widths of 0.016 nm and 0.028 nm respectively, plus one very broad (saturated) absorption line at a wavelength of 352.6 nm. If the lines are due to neutral gas in thermal equilibrium estimate the temperature and bulk random motion of the gas.

What type of object is this intervening system likely to be and at what redshift?

[The atomic weights of hydrogen, carbon and magnesium are 1.674×10^{-27} kg, 1.994×10^{-26} kg and 4.036×10^{-26} kg respectively.]

END OF PAPER

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 7 June 2013 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (i) *and Part* (ii)*, which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i)*. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) In a space of fewer than four dimensions simple expressions can be given for the Riemann tensor. Derive the Riemann tensor in a one-dimensional space and express the Riemann tensor for a two-dimensional space in terms of the metric and the Ricci scalar.

(ii) An infinitesimal circuit in the shape of a parallelogram is specified by four-vector differential displacements $\boldsymbol{u}$ and $\boldsymbol{v}$ representing the sides of the parallelogram. Let a four-vector $\boldsymbol{A}$ be parallel transported around this circuit. Show that the change in $\boldsymbol{A}$ due to the transport is

$$\delta A^{\alpha} = -R^{\alpha}_{\beta\gamma\delta} A^{\beta} u^{\gamma} v^{\delta} \, ,$$

where $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann tensor.

Suppose K is a curvature at a point in a two-dimensional space defined as

$$K = \frac{R_{\alpha\gamma\beta\delta} A^{\alpha} A^{\beta} B^{\gamma} B^{\delta}}{(g_{\kappa\lambda} g_{\mu\nu} - g_{\kappa\nu} g_{\lambda\mu}) A^{\kappa} A^{\lambda} B^{\mu} B^{\nu}} \, ,$$

where $g_{\alpha\beta}$ is the metric tensor and $\boldsymbol{A}$ and $\boldsymbol{B}$ are two four-vectors tangent at a point to the two-surface. If $\boldsymbol{A}$ is parallel transported around a small circuit lying in the two-surface, not necessarily a rectangle, show that the change in the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ is of magnitude

$$\Delta\theta = |K \Delta\Sigma| \, ,$$

where $\Delta\Sigma$ is the area enclosed by the circuit.

Question 2Y - Astrophysical Fluid Dynamics

(i) In order to illustrate its viscous evolution sketch a series of snapshots of the surface density distribution of an accretion disc for which the initial configuration is a narrow ring of material at radius $R = R_0$.

State the sign of the evolution of the radius enclosing half of the disc mass and the radius enclosing half of the disc angular momentum. Comment briefly on the principles that dictate the sign of these evolutionary changes.

Such a disc accretes on to a star through a zero torque inner boundary. Explain why such a boundary is appropriate and explain the sign of the rate of change of the angular momentum of the star and of the disc.

State what boundary condition applies at its outer edge when the disc is truncated by the tidal influence of a companion. How is the global angular momentum conservation of a binary star plus disc system ensured in this case?

(ii) An accretion disc is convectively unstable and contains convection cells which are typically of size H, the vertical scale height of the disc. Explain how convection transports energy from the disc mid-plane to its surface.

If the disc consists of a diatomic gas, state and explain the relationship between pressure and density as a function of height at a fixed cylindrical radius in the disc.

The kinematic viscosity ν can be written in the form $\nu = \tilde{v}\lambda$ where $\tilde{v}$ and λ are the velocity and length scales associated with angular momentum transport in the disc. Use dimensional analysis to determine the coefficients a and b in the equation

$$t_\nu = kR^a\nu^b,$$

where t_ν is the viscous timescale at radius R in the disc and k is a dimensionless constant of order unity.

Accretion discs around young solar mass stars are typically about 100 AU in radius and are observed to evolve on a timescale of a few million years. Use this information to estimate ν at the outer edge of the disc.

The density and temperature of the disc at this radius are $3 \times 10^{-11}\,\mathrm{kg\,m^{-3}}$ and 10 K. Estimate λ when the velocity $\tilde{v}$ associated with angular momentum transport is of order the local sound speed. Compare this with the molecular mean free path and the vertical disc scale height H. What does this suggest about viable viscosity mechanisms in accretion discs?

TURN OVER...

Question 3X - Physical Cosmology

(i) The cosmic microwave background (CMB) is observed today as a perfect black body with

$$u_\nu d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/k_\mathrm{B}T}-1}\,,$$

where $u_\nu d\nu$ is the energy density of the radiation for frequencies from ν to $\nu + d\nu$, T is the temperature and k_B is Boltzmann's constant. Assuming that the temperature of the photons obeys $T_\gamma \propto 1+z$, show that an observer in the past would also see the CMB as a perfect black body.

What does the assumption $T_\gamma \propto 1+z$ imply about the z dependence of the total number of CMB photons? At what point in the past does this assumption break down and what does this imply about the baryon to photon number ratio?

(ii) Sketch the $\Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$ plane for $-2 \le \Omega_{\Lambda,0} \le 3$ and $0 \le \Omega_{\mathrm{m},0} \le 3$, where $\Omega_{\mathrm{m},0}$ is the cosmic density parameter of matter today and $\Omega_{\Lambda,0}$ is the cosmic density parameter of the cosmological constant today. Draw boundaries that divide the plane into open and closed universes, into accelerating and decelerating universes, into universes that will expand forever and those which will eventually collapse. Clearly label the different regions. Indicate on this diagram which universes have an age of 14 Gyr and which are younger.

Using the appropriate Friedmann equation, or otherwise, determine whether a universe with $\Omega_{\mathrm{m},0} = 0.5$ and $\Omega_{\Lambda,0} = 3.5$ is open or closed. Show that these values can correspond to a universe with no big bang. What is the maximum observed redshift in such a universe?

[You may require the factorisation $x^3 - 6x^2y + 7y^3 = (x+y)(x^2 - 7xy + 7y^2)\,.$]

Question 4Z - Structure and Evolution of Stars

(i) Describe in a few sentences the main properties of globular clusters and explain their importance as a window on stellar evolution.

Sketch the Hertzsprung-Russell diagram of a typical globular cluster and discuss its nomenclature in terms of different stages of stellar evolution.

Why are planetary nebulae rarely seen in globular clusters?

(ii) A gas cloud collapses to form a star cluster. The number of stars formed with masses in the range M to $M + dM$ is given by

$$dN = \phi(M)dM\,,$$

where the function $\phi(M)$ is observed to have the form

$$\phi(M) \propto M^{-2.35}\,.$$

The luminosity L of a star on the main sequence follows the relation $L \propto M^{3.5}$. Assume that the total luminosity of the cluster is dominated by stars on the main sequence. Calculate the change in the luminosity of the cluster as the main-sequence turn-off mass decreases from $2\,\mathrm{M}_\odot$ to $1\,\mathrm{M}_\odot$.

The initial mass function of the cluster $\xi(M)$ is defined in terms of the mass in stars in the range M to $M + dM$ such that

$$MdN = \xi(M)dM\,.$$

How does the initial mass function depend on mass?

As stars leave the main sequence, those with mass $M \geq M_{\mathrm{SN}}$ return 90% of their mass to the interstellar medium in core-collapse supernova explosions. Stars with masses $M < M_{\mathrm{SN}}$ produce a white dwarf with mass $M_{\mathrm{WD}} = 0.6\,\mathrm{M}_\odot$ returning the remainder of their main-sequence mass to the interstellar medium. Derive an expression for the fraction of the total mass of the cluster returned to the interstellar medium when the main-sequence turn-off occurs at a mass of $1\,\mathrm{M}_\odot$. Using suitable values for the maximum and minimum stellar mass in the cluster at its birth and for M_{SN}, provide a numerical estimate of this fraction.

TURN OVER...

Question 5Y - Statistical Physics

(i) What is meant by the canonical ensemble in statistical physics? Show that in such an ensemble the probability of finding a system in a particular microstate i of energy E_i is given by the Boltzmann distribution

$$P_i \propto \exp(-E_i/k_{\mathrm{B}}T)\,,$$

where k_{B} is Boltzmann's constant and T is the temperature.

Define the partition function Z for the ensemble.

(ii) A classical particle of mass m moves non-relativistically in a two-dimensional space enclosed within a circle of radius R and is attached through a spring with constant κ to the centre of the circle so that it moves in a potential

$$V(r) = \begin{cases} \frac{1}{2}\kappa r^2 & \text{for } \; r < R\,, \\ \infty & \text{for } \; r \geq R\,, \end{cases}$$

where $r^2 = x^2 + y^2$. The particle is coupled to a heat reservoir of temperature T. Calculate the average energy of the particle.

What is this average energy in the limits of strong coupling, $\frac{1}{2}\kappa R^2 \gg k_{\mathrm{B}}T$, and weak coupling, $\frac{1}{2}\kappa R^2 \ll k_{\mathrm{B}}T$, where k_{B} is Boltzmann's constant?

Compare these two results with what is expected from equipartition of energy.

Question 6Y - Principles of Quantum Mechanics

(i) The Hamiltonian of a harmonic oscillator is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2,$$

where $\hat{p}$ and $\hat{x}$ are momentum and position operators. Describe how to construct creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ so that

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$$

and $[\hat{a}, \hat{a}^\dagger] = 1$.

(ii) Explain how the space of eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$ of $\hat{H}$ of part **(i)** is formed and deduce the energies for these states.

Show that

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

and give the form of the number operator $\hat{N}$ such that $\hat{N}|n\rangle = n|n\rangle$.

The operator $\hat{K}_r$ is defined to be

$$\hat{K}_r = \frac{\hat{a}^{\dagger r}\hat{a}^r}{r!} \qquad r = 0, 1, 2, \ldots.$$

Show that $\hat{K}_r$ commutes with $\hat{N}$ and that

$$\hat{K}_r|n\rangle = \begin{cases} \frac{n!}{(n-r)!r!}|n\rangle & \text{if } r \le n \\ 0 & \text{otherwise.} \end{cases}$$

By considering the action of $\hat{K}_r$ on the state $|n\rangle$, show that

$$\sum_{r=0}^{\infty}(-1)^r\hat{K}_r = |0\rangle\langle 0|\,.$$

TURN OVER...

Question 7Z - Stellar Dynamics and the Structure of Galaxies

(i) A planet is formed as a perfect cube of constant density. Derive expressions for the gravitational field inside the planet and on its surface.

If an astronaut went for a long walk over the surface what would they experience as they moved away from the centre of a face and approached the edges and vertices?

(ii) Assume that the mass budget of the Galactic Halo is dominated by dark matter spherically distributed with density $\rho(r)$ given by the Navarro-Frenk-White (NFW) law

$$\rho(r) = \frac{\rho_0}{(r/r_\mathrm{s})\,(1 + r/r_\mathrm{s})^2}\,,$$

where ρ_0 is a constant density scale factor, r_s is a characteristic scale radius and r is the radial distance from the centre. Derive the Galactic Halo mass within a sphere of radius r and hence deduce the circular velocity at distance r.

An outer Galactic Halo stellar tracer population at $50\,\mathrm{kpc} < r < 150\,\mathrm{kpc}$ is observed to follow power-law distributions in density $\nu(r)$ and line-of-sight velocity dispersion $\sigma(r)$ such that

$$\nu(r) = \nu_0\,(r/r_0)^{-\alpha} \quad \text{and} \quad \sigma^2(r) = \sigma_0^2\,(r/r_0)^{-\gamma}\,,$$

where ν_0 and σ_0 are respectively the density and line-of-sight velocity dispersion at radius $r = r_0 = 100\,\mathrm{kpc}$ and α and γ are the constant power-law indices. Assuming an isotropic velocity dispersion for the tracer population use the Jeans' equation in spherical polars, or otherwise, to estimate the total amount of matter in the Galaxy in this radial range.

When $\alpha = 2.5$ and $\gamma = 0.5$ estimate the scale factor r_s of the NFW profile that reproduces the observed mass dependence on r at $r = r_0$.

[You may assume that the line-of-sight velocity measurement is equivalent to a radial velocity measured with respect to the Galactic centre.]

Question 8Z - Topics in Astrophysics

(i) Galaxies orbiting in the strong gravitational field of a galaxy cluster are susceptible to tidal stripping as they approach the central cluster mass concentration. A galaxy of mass $m_{\rm g}$ and outer radius $r_{\rm g}$ is infalling slowly to a cluster. If the total mass of the cluster within a radius R is M_R, show that the infalling galaxy will be tidally stripped when

$$R \approx r_{\rm g} \left(\frac{2M_R}{m_{\rm g}} \right)^{1/3} .$$

A spherical galaxy cluster has a mass $10^{14}\,{\rm M}_\odot$ within a radius 1 Mpc and a mass of $10^{13}\,{\rm M}_\odot$ within 200 kpc. A proto-galaxy of total mass $10^{9}\,{\rm M}_\odot$, outer radius 20 kpc and density proportional to r^{-2} falls toward the central region of the cluster. Estimate what fraction of the proto-galaxy will survive at these radii.

(ii) A gas-rich disk galaxy is moving through a hot ionized intracluster medium with a velocity component V perpendicular to the disk. The gravitational field of the disk is dominated by stars with mean surface density Σ_*. Show that the ram pressure exerted by the intracluster medium can strip the gas from the disk if

$$\rho_{\rm c} > \frac{2\pi G \Sigma_* \mu}{V^2} ,$$

where $\rho_{\rm c}$ is the density of the intracluster medium, G is the gravitational constant and μ is the mean gas surface density in the disk.

If the stellar mass of the disk is $2 \times 10^{11}{\rm M}_\odot$ and the gas mass in the disk is $10^{9}\,{\rm M}_\odot$, both distributed over a radius of 20 kpc, compute $\rho_{\rm c}$ if the galaxy is moving through the cluster with $V = 1000\,{\rm km\,s^{-1}}$.

What is the total mass of ionised cluster gas if gas of this density is uniformly spread out over a sphere of radius 1 Mpc?

END OF PAPER