NST Part II Astrophysics

Astrophysical Fluid Dynamics

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Fluid dynamics studies the motions of fluids, i.e., liquids and gases. Solids, in contrast, are composed of atoms which form a lattice structure.

Phenomena considered in fluid dynamics are macroscopic. We describe a fluid as a continuous medium with well-defined macroscopic quantities (e.g., density $\rho$, pressure $p$...) even though, at a microscopic level, the fluid is composed of particles.

In the astrophysical context, the liquid state is not very common (examples are high-pressure environments of planetary surfaces and interiors), so our focus will be on the gas phase. A key difference is that gases are more compressible than liquids.

Examples of fluids in the Universe:
- stars, white dwarfs, neutron stars
- interstellar medium (ISM), intergalactic medium (IGM)
- intracluster medium (ICM)
- stellar winds, jets, accretion disks
- giant planets.

The microphysical complexity of fluids is (usually) contained in the equation of state (e.g., ideal gas, degenerate gas).

In our discussion, we shall use the concept of a fluid element. This is a region of fluid that is...
1. Small enough that there are no significant variations of any property q that interests us...

\[ \text{region} \ll \text{scale} \sim \sqrt[3]{\text{Vol}} \]

2. Large enough to contain sufficient particles to be considered in the continuum limit...

\[ n \text{ region} \gg 1 \]

Where \( n \) is the number density of particles.

2: Collisional and collisionless fluids

In a collisional fluid, any relevant fluid element is large enough such that the constituent particles know about local conditions through interactions with each other.

\[ \text{region} \gg \lambda, \text{the mean free path} \]

Particles will then attain a distribution of velocities that maximizes the entropy of the system at a given temperature. Thus, a collisional fluid at a given \( p \) and temperature \( T \) will have a well defined distribution of particle speeds and hence a well defined pressure, \( p \)

\[ \Rightarrow \text{can relate } p, T \text{ and } \rho \text{ (equation of state)} \]

In a collisionless fluid, particles do not interact frequently enough to satisfy \( \text{region} \gg \lambda \). So distribution of particle speeds locally does not correspond to maximum entropy solution, instead depending on initial conditions and non-local conditions. Examples:

- Stars in a galaxy
- Grains in Saturn's rings
- Dark matter
- ICM (transitional from collisional to collisionless)
Expand example of ICM: treat as fully ionized plasma of H (e-, p). The mean free path is set by Coulomb collisions and an analysis gives

\[ \lambda_e = \frac{3^{3/2} (k_B T_e)^{3/2}}{4 \pi T_e^{1/2} \rho_e^2 \ln \Lambda} \]

\( \rho_e = \text{e- number density} \)
\( \Delta = \text{ratio of largest to smallest impact parameter} \)

and for \( T \gtrsim 4 \times 10^5 \text{K} \) we have \( \ln \Lambda \approx 40 \). So, if \( T_i = T_e \) we have

\[ \lambda_e \approx 23 \text{kpc} \left( \frac{T_e}{10^8 \text{K}} \right)^{3/2} \left( \frac{n_e}{10^{-3} \text{cm}^{-3}} \right)^{-1} \]

So we have

\( R_{\text{Galaxy}} \sim \lambda_e \ll R_{\text{cluster}} \sim 1 \text{Mpc} \)

Collisionless \hspace{1cm} Collisional
B: FORMULATION OF THE FLUID EQUATIONS

1: Eulerian vs. Lagrangian

Two frameworks for understanding fluid flow:

(i) Eulerian description - one considers the properties of the fluid measured in a frame of reference that is fixed in space. So we consider quantities like

\[ P(r, t) \quad p(r, t) \]
\[ T(r, t) \quad \mathbf{v}(r, t) \]

(ii) Lagrangian description - one considers a particular fluid element and examines the change in the properties of that element. So, the spatial reference frame is moving with the fluid flow.

The Eulerian approach is more useful if the motion of particular fluid elements is not of interest.

The Lagrangian approach is useful if we do care about the passage of given fluid elements (e.g., gas parcels that are enriched by metals).

These two different pictures lead to very different computational approaches to fluid dynamics which we will discuss later.

Mathematically, it is straightforward to relate these two pictures. Consider a quantity \( Q \) in a fluid element at position \( r \) and time \( t \). At time \( t + dt \), the element will be at position \( r + \mathbf{dr} \). The change in quantity \( Q \) of
the fluid element is

\[
\frac{DQ}{Dt} = \lim_{St \to 0} \left[ \frac{Q(r + Sr, t + St) - Q(r, t)}{St} \right]
\]

but

\[
Q(r + Sr, t + St) = Q(r, t) + \frac{\partial Q}{\partial t} St + Sr \cdot \nabla Q + O(St^2, Sr^2, StSr)
\]

So,

\[
\frac{DQ}{Dt} = \lim_{St \to 0} \left[ \frac{\partial Q}{\partial t} + \frac{Sr}{St} \cdot \nabla Q + O(St^2, Sr^2, StSr) \right]
\]

\[
\Rightarrow \frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + u \cdot \nabla Q
\]

Lagrangian time derivative

Eulerian time derivative

"Convection" derivative

2: Kinematics

Kinematics is the study of particle (and fluid element) trajectories.

Streamlines, streaklines and particle paths are field lines resulting from the velocity vector fields. If the flow is steady with time, they all coincide.

Definitions:

1. Streamlines: families of curves that are instantaneously tangent to the velocity vector of the flow \( u(r, t) \). They show the direction of the fluid element.
parametrize streamline by label $s \ s.t.$

$$\frac{dr}{ds} = \left( \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right)$$

and demand $dr/ds \parallel u$

$$\Rightarrow \frac{dr}{ds} \times u = 0$$

$$\Rightarrow \frac{dx}{Ux} = \frac{dy}{Uy} = \frac{dz}{Uz}$$

(1) Particle paths: trajectories of individual fluid elements

given by

$$\frac{dr}{dt} = U(r,t)$$

For small time intervals, particle paths follow streamlines since $U$ can be treated as approximately steady.

(2) Streaklines: locus of points of all fluid that have passed through a given spatial point in the past.

$$\Gamma(t) = \Gamma_0 \text{ for some } \Gamma_0 \text{ in the past}$$

We now proceed to discuss the equations that describe the dynamics of a fluid. These are essentially expressions of the conservation of mass, momentum and energy.
Consider a fixed volume $V$ bounded by a surface $S$. If there are no sources or sinks of mass within the volume, we can say

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

This is true for all volumes $V$. So we must have

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

**EULERIAN CONTINUITY EQ.**

The Lagrangian expression of mass conservation is easily found:

\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\nabla \cdot (\rho \mathbf{u}) + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}
\]

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0
\]

**LAGRANGIAN CONTINUITY EQ.**

In an incompressible flow, fluid elements maintain a constant density, i.e.

\[
\frac{D\rho}{Dt} = 0
\]

We can now see that incompressible flows must be divergence free, $\nabla \cdot \mathbf{u} = 0$. 

\[\]
Pressure

Consider only collisional fluids where there are forces within the fluid due to particle-particle interactions. Thus, there can be momentum flux across surfaces within the fluid even in the absence of bulk flows.

In a fluid with uniform properties, the momentum flux through a surface is balanced by an equal and opposite momentum flux through the other side of the surface. \[ \Rightarrow \text{no net acceleration even for non-zero pressure} \]

Since pressure is defined as the momentum flux on one side of the surface.

If the particle motions within the fluid are isotropic and the fluid motion is isotropic, the momentum flux is locally independent of the orientation of the surface, and the components parallel to the surface cancel out. Then, the force per unit area acting on one side of a surface is

\[
F_i = p \cdot \frac{ds_i}{ds_1}
\]

In the more general case, forces across surfaces are not perpendicular to the surface, and we have

\[
F_i = \sigma_{ij} \frac{ds_j}{ds_1}
\]

Stress tensor - the force in direction \( i \) acting on a surface with normal \( \mathbf{n} \).

Isotropic pressure corresponds to \( \sigma_{ij} = p \delta_{ij} \).
Momentum equation for a fluid

Consider a fluid element that is subject to a gravitational field \( g \) and internal pressure forces. Let the fluid element have volume \( V \) and surface \( S \).

Pressure acting on surface element gives force \(-pds\).
Pressure force on element projected in direction \( \hat{n} \) is \(-p\hat{n}\cdot ds\).
So, net pressure force in direction \( \hat{n} \) is

\[
F\cdot\hat{n} = -\int_S \rho \hat{n} \cdot ds = -\int_V \nabla \cdot (\rho \hat{n}) dV = -\int_V \hat{n} \cdot \nabla p dV
\]

Rate of change of momentum of fluid element in direction \( \hat{n} \) is the total force in that direction

\[
\left(\frac{D}{Dt} \int_V \rho u dV\right) \cdot \hat{n} = -\int_V \hat{n} \cdot \nabla p dV + \int_V \rho g \hat{n} dV
\]

In limit that \( S dV \rightarrow SV \) we have

\[
\frac{D}{Dt} \left( \rho u SV \right) \cdot \hat{n} = -SV \hat{n} \cdot \nabla p + SV \rho g \hat{n}
\]

\[
\Rightarrow \hat{n} \cdot u \frac{D}{Dt} \left( \rho SV \right) + \rho SV \hat{n} \cdot \frac{Du}{Dt} = -SV \hat{n} \cdot \nabla p + SV \rho g \hat{n}
\]

= 0 by mass conservation
Must be true for all \( \hat{\mathbf{i}} \). So,

\[
P \frac{Du}{Dt} = -\nabla p + \rho g
\]

LAGRANGIAN
MOMENTUM EQUATION

or

\[
P \frac{\partial u}{\partial t} + \rho (u \cdot \nabla) u = -\nabla p + \rho g
\]

EULERIAN
MOMENTUM EQUATION

Now consider the Eulerian rate of change of momentum density \( \rho u \) and introduce more compact notation

\[
\frac{\partial}{\partial t} (\rho u_i) = \partial_t (\rho u_i)
\]

\[
= \rho \partial_t u_i + u_i \partial \rho
\]

\[
= -\rho u_j \partial_j u_i - \partial_j \rho \delta_{ij} + \rho g_i - u_i \partial_j (\rho u_j)
\]

where we have used notation

\[
\partial_j \equiv \frac{\partial}{\partial x_j}
\]

and employed summation convention (summation over the repeated index). This gives

\[
\partial_t (\rho u_i) = -\partial_j (\rho u_i u_j + \rho \delta_{ij}) + \rho g_i = -\partial_j \delta_{ij} + \rho g_i
\]

\( \rho \) Stress tensor
due to bulk flow
\( \rho \) Stress tensor
due to random thermal motion

In component free language we write

\[
\partial_t (\rho u) = -\nabla \cdot (\rho u u + \rho I) + \rho g
\]

\( \rho \) flux of momentum density
Example: flow in a pipe in the y-direction

Any surface will experience a momentum flux $p$ due to pressure. Only surfaces with a normal that has a component $\parallel$ flow will experience ram pressure

$$\sigma_{ij} = \begin{pmatrix} P & 0 & 0 \\ 0 & p + pu^2 & 0 \\ 0 & 0 & p \end{pmatrix}$$

The remaining equation of fluid dynamics is based on the conservation of energy. We will defer a discussion of that until later.
1: Basics

Define the gravitational potential \( \Psi \) s.t. the gravitational acceleration \( g \) is

\[
g = - \nabla \Psi.
\]

If \( \mathcal{L} \) is some closed loop, we have

\[
\oint_{\mathcal{L}} g \cdot dl = \int_{\mathcal{S}} \nabla \times g \cdot ds = - \int_{\mathcal{S}} \nabla \times (\nabla \Psi) \cdot ds = 0
\]

so gravity is a conservative force - the work done around a closed loop is zero.

The work needed to take a mass from point \( r \) to \( \infty \) is

\[
- \int_{r}^{\infty} g \cdot dl = \int_{r}^{\infty} \nabla \Psi \cdot dl = \Psi(\infty) - \Psi(r) \quad \text{independent of path}
\]

A particularly important case is the gravity of a point mass which has

\[
\Psi = - \frac{GM}{r} \quad \text{if mass at origin}
\]

\[
\Psi = - \frac{GM}{|r-r'|} \quad \text{if mass at location } r'
\]

For system of point masses we have

\[
\Psi = - \sum_{i} \frac{GM_i}{|r-r_i|}
\]

\[
\Rightarrow \quad g = - \nabla \Psi = - \sum_{i} \frac{GM_i (r-r_i)}{|r-r_i|^3}
\]

Replacing \( M_i \rightarrow \rho_i S V_i \) and going to the continuum limit we have
\[ g(r) = 4G \int \rho(c') \frac{r-r'}{4\pi r'^3} \, dV \]

Take divergence of both sides:

\[
\nabla \cdot g(r) = -G \int \rho(c') \nabla \cdot \left[ \frac{r-r'}{4\pi r'^3} \right] \, dV
\]

\[
= -4\pi G \int \rho(c') \delta(c-c') \, dV
\]

\[
= -4\pi G \rho(c)
\]

So,

\[
\nabla \cdot g = -\nabla^2 \psi = -4\pi G \rho
\]

Poisson's Equation

We can also express Poisson's equation in integral form: for some volume \( V \) bounded by surface \( S \) we have

\[
\int_V \nabla \cdot g \, dV = -4\pi G \rho_V \, dV
\]

\[
\Rightarrow \int_S g \cdot ds = -4\pi G M
\]

This is useful for calculating \( g \) when the mass distribution obeys some symmetry.

Example 1: Spherical distribution of mass

By symmetry \( g \) is radial and \( 1/|r| \) is constant over \( d \) \( r = \) inner shell. So

\[
\int_S g \cdot ds = -4\pi G M(r) \quad \text{mass enclosed}
\]

\[
\Rightarrow -1/|r| 4\pi r^2 = -4\pi G M(r)
\]

\[
\Rightarrow 1/|r| = \frac{GM(r)}{r^2}
\]

\[
\therefore \quad g = -\frac{GM(r)}{r^2}
\]
Example 2: Infinite cylindrically symmetric mass distribution

By symmetry, \( g \) is uniform and radial on the curved sides of the cylindrical surface, and is zero on the flat side:

\[
\iiint_S g \cdot ds = -4\pi G \int_V \rho dV
\]

\[
\Rightarrow -2\pi RLg_l = -4\pi G M(R) l
\]

\[
\Rightarrow g = -\frac{2GM(R)l^2}{R^2} \text{ enclosed mass per unit length}
\]

Example 3: Infinite planar distribution of mass (infinite and homogeneous in \( x \) and \( y \), \( \rho = \rho(z) \)).

By symmetry, \( g \) is in \( -\hat{z} \) direction and is constant on a \( z = \text{constant} \) surface. So, if we also have reflection symmetry about \( z = 0 \):

\[
\iiint_S g \cdot ds = -4\pi G \int_V \rho dV
\]

\[
\Rightarrow -2l g_l A = -4\pi G A \int_{-2}^{2} \rho(z) dz
\]

\[
\Rightarrow g = -\frac{4\pi G A}{2l} \int_{-2}^{2} \rho(z) dz \cdot \hat{z}
\]

(For planar distribution of finite height \( z_{\text{max}} \), \( g \) is constant for \( z \geq z_{\text{max}} \)).

Example 4: Finite axysymmetric dish

\[
\iiint_S g \cdot ds = ?
\]

No surface where \( g \) vanishes by symmetry, and no easily determined surface where \( l g_l \) is a constant.
2: Potential of a spherical mass distribution

We found that, for a spherical distribution,

$$
\frac{g}{r} = -\frac{M}{r^2} \quad \text{and} \quad 1g \lambda = -\frac{G}{r^2} \int_0^r 4\pi \rho(r) r^2 \, dr
$$

so,

$$
\psi = \int_0^r \frac{G}{r^2} \left\{ \int_0^r 4\pi \rho(r) r^2 \, dr \right\} \, dr
$$

taking \( \psi(\infty) = 0 \) by convention. Integrate this by parts:

$$
\psi = -\frac{G}{r} \left\{ \int_0^r 4\pi \rho(r) r^2 \, dr \right\} \bigg|_{r=0}^{r=r_0} + \int_0^r \frac{G}{r} \cdot 4\pi \rho(r) r^2 \, dr
$$

\( \Rightarrow \psi = -\frac{G M(r_0)}{r_0} + \int_0^r 4\pi G \rho(r) r \, dr \)

where we have used assumption that \( M(r) = 0 \) as \( r \to \infty \).

We find that \( \psi \) is affected by matter outside of \( r \). So \( \psi \neq -\frac{G M(r)}{r} \) unless there is no mass outside of \( r \).

3: Gravitational Potential Energy

For a given system of point masses,

$$
\psi = -\sum_i \frac{G m_i}{|x_i - \mathbf{r}|}
$$

and the energy required to take unit mass to \( \infty \) is \( -\psi \). Energy required to take a system of point masses to \( \infty \) is

$$
\Omega = -\frac{1}{2} \sum_{i \neq j} \frac{G m_i m_j}{|x_j - \mathbf{r}_i|} = \frac{1}{2} \sum_j m_j \psi_j
$$

avoid double counting
For a continuum matter distribution,
\[ \Omega = \frac{1}{2} \iiint \rho(r) \psi(r) \, dV \]

Specializing to the spherically symmetric case gives
\[ \Omega = \frac{1}{2} \int_0^\infty 4\pi \rho(r) r^2 \psi(r) \, dr \]

Integrate by parts, choosing parts \( u = \psi \), \( dv = 4\pi \rho r^2 \, dr \) so that \( v = \frac{1}{3} 4\pi \rho r^2 = M(r) \)...

\[ \Omega = \frac{1}{2} \left[ M(r) \psi(r) \right]_0^\infty - \int_0^\infty M(r) \frac{d\psi}{dr} \, dr \]

But
\[ \frac{d\psi}{dr} = \frac{GM(r)}{r^2} \]

So,
\[ \Omega = -\frac{1}{2} \int_0^\infty \frac{GM(r)^2}{r^2} \, dr \]

Integrate again by parts choosing \( u = GM(r)^2 \), \( dv = \frac{1}{r^2} \, dr \)

\[ \Omega = \frac{1}{2} GM(r)^2 \frac{1}{r} \bigg|_0^\infty - \frac{1}{2} \int_0^\infty \frac{1}{r} 2GM \frac{dM}{dr} \, dr \]

\[ \Omega = -GM(r) \frac{dM}{dr} \bigg|_0^\infty \]

\[ \Rightarrow \Omega = -GM(r) \frac{dM}{dr} \]

Assembly of spherical shells of mass, each brought from \( \infty \) with potential energy
\[ \frac{GM(r)}{r} \, dM(r) \]
We now come to a powerful result that greatly helps in the understanding of gravitating systems.

Consider the motion of a cloud of particles (atoms, stars, galaxies...). Particle with mass $m$ at $r$ is acted upon by a force

$$ F = m \frac{d^2r}{dt^2} $$

Consider the 2nd derivative of the moment of inertia, $I_i = m r^2$

$$ \frac{1}{2} \frac{d^2I_i}{dt^2} = m \frac{d}{dt} \left( r \cdot \frac{dr}{dt} \right) $$

$$ = m \cdot d^2r/dt^2 + m \left( \frac{dr}{dt} \right)^2 $$

$$ = \Sigma \cdot F + m \left( \frac{d^2r/dt}{dt} \right)^2 $$

Summing over all particles:

$$ \frac{1}{2} \frac{d^2I}{dt^2} = \Sigma ( \Sigma \cdot F ) + 2T $$

$$ = V, \text{ the virial} $$

(R. Clausius)

Consider any two particles with $m_i$ and $m_j$ at $r_i$ and $r_j$. Newton's 3rd law says

$$ F_{ij} = - F_{ji} $$

and so their contribution to the virial is $F_{ij} \cdot (r_i - r_j)$. In absence of external forces, we then have

$$ V = \Sigma \Sigma F_{ij} \cdot (r_i - r_j) $$
If there are no collisions except for $\mathbf{r}_i = \mathbf{r}_j$, all forces other than gravitational can be neglected and

$$F_{ij} = -\frac{Gm_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \quad r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$$

$$\Rightarrow V = -\sum_i \sum_{j > i} \frac{Gm_i m_j}{r_{ij}}$$

work done to separate each pair of particles to infinity against gravity

And so, $V = -\Omega$, and we can use above to write

$$\frac{1}{2} \frac{d^2 r_i}{dt^2} = 2T + \Omega$$

If the system is in a steady state ("relaxed") then $I$ constant and we can say

$$2T + \Omega = 0$$

**The Virial Theorem**

Here, the kinetic energy $T$ has contributions from local flows and random/thermal motions.

Virial theorem $\Rightarrow$ gravitational potential sets the "temperature" or velocity dispersion of the system.
1: The equation of state

To close the mass continuity and momentum equations, we need to find relations between $\Psi$, $P$ and the other fluid variables such as $\rho$ and $u$.

$\Psi$ and $P$ are related via Poisson equation (and/or we sometimes consider an externally imposed gravitational potential).

$P$ and the other thermodynamic properties of the system are related by the equation of state (EOS). Only valid for collisional fluids.

Most astrophysical fluids are quite dilute and can be well approximated as ideal gases. The corresponding EOS is

$$P = n k_{B} \rho T = \frac{k_{B} \rho}{m_{p}} T$$

$m$ = mean particle mass in units of the proton mass $m_p$.

(Exceptions, where significant deviation from ideal gas behaviour occurs, can be found in high density environments of planets, neutron stars, and white dwarfs).

To close the equations with an ideal gas EOS, one needs to know temperature $T$. In general we need to solve another PDE that describes heating and cooling processes. More on this soon - see "Energy equation".

However, for special cases, we can relate $T$ to $P$ without the need to solve a separate energy equation. EOS's for which $P$ is only a function of $\rho$ are known as barotropic EOS's.
Example I: Isothermal EOS

\[ T \text{ is constant so that } P \propto \rho. \text{ Valid when the fluid is locally in thermal equilibrium with strong heating and cooling processes that are in balance.} \]

Example II: Adiabatic EOS

Ideal gas undergoes reversible thermodynamic changes such that

\[ P = \kappa / \rho^\gamma \quad (\kappa \text{ and } \gamma \text{ constant}) \]

Derivation:

First law of thermodynamics

\[ d\mathcal{Q} = dE + pdV \]

- heat absorbed by unit mass of fluid from surroundings
- change in internal energy of unit mass of fluid
- work done by unit of mass of fluid

Here \( \mathcal{Q} \) is a Pfaffian operator - change in quantity depends on the path taken through the thermodynamic phase space.

For an ideal gas, we can write

\[ P = \frac{R_* \rho T}{\mu}, \quad E = \mathcal{E}(T) \]

\( R_* \) is a modified gas constant

So, first law reads

\[ d\mathcal{Q} = \frac{dE}{dT} dT + pdV \]

\[ = C_v dT + \frac{R_* T}{\mu} dV \]

where we define specific heat capacity at constant volume...
as \( C_V \equiv dE/dT \) and have noted that for unit mass we have \( p = \nu V \).

For a reversible change we have \( dQ = 0 \), so

\[
C_v dT + \frac{R* T}{M} dV = 0
\]

\[
\Rightarrow C_v d(ln T) + \frac{R*}{M} d(ln V) = 0
\]

\[
\Rightarrow V \propto T - \frac{C_v}{R* M}
\]

\[
\Rightarrow p \propto T I + \frac{C_v M}{R*}
\]

\( C_v \) depends on the number of degrees of freedom with which the gas can store kinetic energy, \( f \) s.t.

\[
C_v = f \cdot \frac{R*}{2 M}
\]

Monatomic gas has \( f = 3 \) \( \Rightarrow C_v = \frac{3}{2} \cdot \frac{R*}{M} \)

diatom has \( f \text{=} 5 \) \( \Rightarrow C_v = \frac{5}{2} \cdot \frac{R*}{M} \)

Return to the ideal gas law,

\[
P = \frac{R*}{M} p T
\]

\( \Rightarrow pV = \frac{R* T}{M} \)

\[
p dV + V dp = \frac{R*}{M} dT
\]

but

\[
dQ = \frac{dE}{dT} dT + pdV
\]

\[
= (\frac{dE}{dT} + \frac{R*}{M}) dT - V dp
\]

Specific heat capacity at constant pressure, \( C_P \)

So,

\[
C_P - C_v = \frac{R*}{M}
\]
Define
\[ \gamma = \frac{C_p}{C_v} \]
so that, for the reversible/adiabatic processes discussed above, we have
\[ P \propto T^{1 + \frac{C_v}{C_p}} \Rightarrow P \propto T^{\gamma} \]
\[ V \propto T^{-\frac{C_v}{C_p}} \Rightarrow V \propto T^{-\frac{1}{\gamma}} \]
which we can combine to give
\[ P \propto T^\gamma \]

We say that a fluid element behaves \textit{adiabatically} if \( P = K \rho^\gamma \) with \( K = \text{constant} \). A fluid is \textit{isentropic} if all fluid elements behave adiabatically with the same value of \( K \). \( K \) is related to the entropy per unit mass.

\[ \boxed{2: \text{The Energy Equation}} \]

In general, the equation of state will not be barotropic and we will need to solve a separate differential equation which follows the heating and cooling processes in the gas, the energy equation.

From the first law of thermodynamics we have
\[ \Delta Q = dE + p dV \]  
(in absence of dissipative processes)
so,
\[ \frac{dE}{dt} = \frac{dW}{dt} + \frac{dW}{dt} \]
\[ dW = -p dV \]
with
\[ \frac{dW}{dt} = -P \frac{D}{Dt} (V_p) = -P \frac{D}{Dt} \]
\[ \frac{dQ}{dt} = -Q_{\text{cool}} \quad \text{cooling function per unit mass} \]
\[
\frac{DE}{Dt} = \frac{P}{\rho^2} \frac{DP}{Dt} - \dot{Q}_{\text{con}}.
\]

The total energy per unit volume is

\[
E = \rho \left( \frac{1}{2} u^2 + \Psi + \epsilon \right)
\]

\[\text{kinetic} \quad \text{potential} \quad \text{internal}\]

so,

\[
\frac{DE}{Dt} = \frac{DP}{Dt} \rho + \rho (u \cdot \frac{Du}{Dt} + \frac{D\Psi}{Dt} + \frac{P}{\rho^2} \frac{DP}{Dt} - \dot{Q}_{\text{con}})
\]

where

\[
\frac{DE}{Dt} = \frac{DE}{\delta t} + u \cdot \nabla E
\]
\[
\frac{DP}{Dt} = -\rho \nabla \cdot u
\]
\[
\rho \frac{Du}{Dt} = -\nabla P + \rho g = -\nabla P - \rho \nabla \Psi
\]
\[
\frac{D\Psi}{Dt} = \frac{\partial \Psi}{\delta t} + u \cdot \nabla \Psi
\]

Putting it all together

\[
\frac{DE}{Dt} = -\frac{E}{\rho} \cdot \rho u \cdot u - u \cdot \nabla P - \rho u \cdot \nabla \Psi + \rho \frac{D\Psi}{Dt}
\]
\[
+ \rho u \cdot \nabla \Psi - \frac{\rho}{\rho} \rho \nabla u - \rho \dot{Q}_{\text{con}}
\]
\[
= -(E + P) \nabla \cdot u - u \cdot \nabla P - \rho \dot{Q}_{\text{con}}
\]
\[
= \frac{DE}{\delta t} + u \cdot \nabla E
\]

\[
\Rightarrow \quad \frac{DE}{\delta t} + \nabla \cdot \left[ (E + P) u \right] = \rho \frac{D\Psi}{\delta t} - \rho \dot{Q}_{\text{con}} \quad \text{ENERGY EQUATION}
\]

In many settings \( \frac{D\Psi}{\delta t} = 0 \), i.e. \( \Psi \) depends only on position only.
Common astrophysical heating/cooling/transport mechanisms are:

1. Cosmic rays - heating and energy transport via high-energy (often relativistic) particles that are diffusing/streaming through the fluid.
   - can ionize atoms in fluid, excess energy put into freed e-, ends up as heat in fluid
   
   ionization rate per unit volume \( \propto CR \times \text{flux} \times \rho \)

2. Thermal conduction - transport of thermal energy by diffusion of the hot e- into cooler regions. Relevant in, for example,
   - interiors of white dwarfs
   - supernova shock fronts
   - LMR plasma

   heat flux per unit area \( F_{\text{cond}} = - \kappa \nabla T \) 
   thermal conductivity

   rate of change of E per unit volume \( = - \nabla \cdot F_{\text{cond}} = \kappa \nabla^2 T \)

3. Convection - transport of energy due to fluctuating or circulating fluid flows in presence of temperature gradient. Important in cores of massive stars, or interiors of some planets, or envelopes of low-mass stars.
Radiation - energy carried by photons. Consider two limits:

- Optically thin limit: photons readily escape the fluid since system is transparent to the emitted radiation. So becomes a simple cooling term. Examples:
  
  ① Energy loss by recombination
  
  ② Energy loss by free-free emission
  
  \[ \frac{\text{LL}}{\text{C}} \propto n_e n_p T^{1/2} \]

  ③ Collisionally excited atomic line radiation
  (e^- collides with atom in ground state → produces excited atomic state which returns to ground state by emitting a photon with energy \( \gamma \).
  
  \[ \frac{\text{LC}}{\text{C}} \propto N_e N_{\text{HII}} e^{-\gamma/kT} T^{1/2} \]

  In cold gas cloud with \( T \sim 10^4 K \), H cannot be excited so cooling occurs through trace species (\( O^+, O^{++}, N^+ \)).

④ Collisionally excited molecular cooling for \( T \sim 100 K \) (cooling via \( \text{CO}, \text{O}_2, \text{H}_2\text{O} \)).

- Optically thick limit: photons re-absorbed or scattered locally. Only escape after diffusing through the medium. Photon spectrum is in thermodynamic equilibrium with the fluid (i.e. blackbody).

We can parametrize \( \dot{Q}_{\text{cool}} \) as:

\[ \dot{Q}_{\text{cool}} = A \rho T^\alpha - H \]

optically thin cooling, CR heating
I: Hydrostatic Equilibrium

A fluid system is in hydrostatic eqn if

\[ y = 0, \ \partial y/\partial t = 0 \]

Then,

continuity eqn is trivially satisfied

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

momentum eqn gives

\[ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \rho \nabla \psi = 0 \]

\[ \Rightarrow \ \frac{1}{\rho} \nabla p = -\nabla \psi \]

\[ \text{EQUATION OF HYDROSTATIC EQUILIBRIUM} \]

If system is self-gravitating (rather than having an externally imposed gravitational field) we also have

\[ \nabla^2 \psi = 4\pi G \rho \]

Assuming a barotropic equation of state \( P = P(\rho) \), this system of equations can be solved.

Example: Isothermal slab

Consider static, isothermal slab in x and y which is symmetric about \( z = 0 \) (i.e., two clouds collide and generate a shocked slab of gas between them)
Isothermal \( \Rightarrow p = \frac{R* T}{M} \) \( \Rightarrow p = pT \). Also \( \nabla \psi = \nabla \theta \), due to symmetry, \( p = p(z), \psi = \psi(z) \).

Then the eqn of hydrostatics becomes

\[
A \frac{1}{\rho} \nabla p = - \nabla \psi
\]

\[
\Rightarrow A \frac{d}{dz} (\ln \rho) = - \frac{d\psi}{dz}
\]

\[
\Rightarrow \psi = - A \ln (P/\rho_0) + \psi_0 \quad (\rho_0 = \rho(z=0))
\]

\[
\Rightarrow \rho = \rho_0 e^{- (\psi-\psi_0)/A}
\]

Poisson's eqn is

\[
\frac{d^2 \psi}{dz^2} = 4\pi \sigma \rho_0 e^{- (\psi-\psi_0)/A}
\]

Let's change variables to \( x = -(\psi-\psi_0)/A \), \( z_i = \sqrt{\frac{2\pi \sigma \rho_0}{A}} z \)

so that Poisson's eqn becomes

\[
\frac{d^2 x}{dz^2} = -2e^x \\
\text{at } x=0 \text{ and } x=0
\]

\[
\Rightarrow \frac{dx}{dz} \frac{d^2 x}{dz^2} = -2 \frac{dx}{dz} e^x
\]

\[
\Rightarrow \frac{1}{2} \frac{d}{dz} \left[ \left( \frac{dx}{dz} \right)^2 \right] = -2 \frac{d}{dz} (e^x)
\]

\[
\Rightarrow \left( \frac{dx}{dz} \right)^2 = C_1 - 4e^x
\]

but boundary condition \( dx/dz = 0 \) when \( x = 0 \) \( \Rightarrow C_1 = 4 \)

\[
\Rightarrow dx = 2 \sqrt{1-e^x} \\
\Rightarrow \int \frac{dx}{\sqrt{1-e^x}} = 2 \int dz
\]

Change variables \( e^x = \sin^2 \theta \)

\[
\Rightarrow e^x dx = 2 \sin \theta \cos \theta d\theta \\
\Rightarrow dx = \frac{2 \cos \theta d\theta}{\sin \theta}
\]
So, we can evaluate the integral
\[
\int \frac{dx}{\sqrt{1-e^{-x}}} = \int \frac{2 \cot \theta \, d\theta}{\sin \theta \sqrt{1-\sin^2 \theta}}
\]
\[
= \int \frac{2 \, d\theta}{\sin \theta}
\]
\[
= \left\{ 2 \cdot \frac{1}{2} \frac{1+t^2}{t} \, d\theta \right\}
\]
\[
= 2 \int \frac{dt}{t}
\]
\[
= 2 \ln t + C_2
\]

Setting \( t = \tan \frac{\theta}{2} \),
\[
\Rightarrow dt = \frac{1}{2} (1+t^2) \, d\theta
\]
and \( \sin \theta = 2 \sin \frac{\theta}{2} \cot \frac{\theta}{2} = \frac{2t}{1+t^2} = e^\theta \)

So, Poisson's equation becomes
\[
2 \int \frac{dt}{t} = 2 \theta + C_2
\]

Now, \( \chi = 0 \) at \( z = 0 \) \( \Rightarrow \theta = \pi/2, \ t = 1 \)
\( \Rightarrow C_2 = 0 \)

So, \( t = e^z \)
\( \Rightarrow \sin \theta = e^{z/2} = \frac{2e^z}{1+e^{2z}} = \frac{1}{\cosh^2 z} \)

\( \Rightarrow \psi - \psi_0 = 2A \ln \cosh \left( \sqrt{\frac{2\pi G}{A}} z \right) \)

\( \rho = \frac{\rho_0}{\cosh^2 \left( \sqrt{\frac{2\pi G}{A}} z \right)} \)
Example: Isothermal atmosphere with constant (externally imposed) $g$

Suppose $g = -g \hat{z}$. Then $\nabla^2 \psi$ of hydrostatic eqm with isothermal equation of state reads

$$A \cdot \nabla P = -\nabla \psi = -g \hat{z}$$

$$\Rightarrow \ln P = -\frac{g z}{A} + \text{const}$$

$$\Rightarrow P = P_0 \exp\left(\frac{-M g}{R \cdot T} z\right)$$

is exponential atmosphere

Example of this is the Earth's atmosphere: $T \approx 300K$ and $M \approx 28 \Rightarrow$ e-folding $\approx 9$ km. The highest astronomical observatories are at $z \approx 4$ km, so have $P$ and $P \sim 60\%$ of sea level.

2: Stars as self-gravitating polytropes

Consider a spherical self-gravitating system in hydrostatic eqm; from now on we will refer to this as a "star". We have

$$\nabla P = -P \nabla \psi$$

$$\Rightarrow \frac{dP}{dr} = -P \frac{d\psi}{dr} \quad \text{(spherical polars)}$$

Now, $P > 0$ within star

$$\Rightarrow P$$ is monotonic function of $\psi$

Also,

$$\frac{dP}{dr} = \frac{dP}{d\psi} \frac{d\psi}{dr} = -P \frac{d\psi}{dr} \Rightarrow P = -\frac{dP}{d\psi}$$
so $p$ is monotonic function of $\Psi$

$\therefore p = p(\Psi), \quad \rho = \rho(\Psi) \Rightarrow p = p(\rho)$

ie non-rotating stars are barotropes!

A barotropic EOS can be written as polytropic

$$p = k \rho^{1+\frac{1}{n}}$$

which ends up being a good approximation of real stars even with a single value of $n$.

It is important to note that in general we will have

$$1 + \frac{1}{n} \neq \gamma$$

We do only have $1 + \frac{1}{n} = \gamma$ (ie $p \propto \rho^\gamma$) if the star is isentropic (constant energy throughout).

Now, eq. of hydrostatic eqm gives

$$-\nabla \Psi = \frac{1}{\rho} \nabla (k \rho^{1+\frac{1}{n}}) = (n+1) \nabla (k \rho^{\frac{1}{n}})$$

$$\Rightarrow p = \left( \frac{\Psi - \Psi_s}{[n+1] \Psi} \right)^n$$

$\Psi = \Psi_s$ at $\rho = 0$, the surface.

If the central density is $\rho_c$ and central potential is $\Psi_c$ we have

$$\rho_c = \left( \frac{\Psi - \Psi_s}{[n+1] \Psi} \right)^n$$

so,

$$p = \rho_c \left( \frac{\Psi - \Psi_s}{\Psi - \Psi_c} \right)^n$$

Feeding into Poisson gives

$$\nabla^2 \Psi = 4\pi G \rho_c \left( \frac{\Psi - \Psi_s}{\Psi - \Psi_c} \right)^n$$
Set \( \Theta = \frac{\Psi_t - \Psi}{\Psi_t - \Psi_c} \), we then get
\[
\nabla^2 \Theta = - \frac{4\pi GP_c}{\Psi_t - \Psi_c} \Theta^n
\]

In spherical pores, this becomes
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Theta}{dr} \right) = - \frac{4\pi GP_c}{\Psi_t - \Psi_c} \Theta^n
\]

Set \( s = \sqrt{\frac{4\pi GP_c}{\Psi_t - \Psi_c}} r \), we finally get
\[
\frac{1}{s^2} \frac{d}{ds} \left( s^2 \frac{d\Theta}{ds} \right) = -\Theta^n
\]

\text{LANE-EMDEN EQ OF INDEX } n

The appropriate boundary conditions are
\( \Theta = 1 \) at \( s = 0 \)
\( \frac{d\Theta}{ds} = 0 \) at \( s = 0 \) (zero force at \( s = 0 \), enclosed mass \( \rightarrow 0 \) or \( s \rightarrow 0 \))

The Lane-Emden eq. can be solved analytically for \( n = 0, 1 \) and \( s \); otherwise solve numerically.

\text{Solution for } n = 0:
\[
\frac{1}{s^2} \frac{d}{ds} \left( s^2 \frac{d\Theta}{ds} \right) = -\Theta^n = -1
\]
\[ \Rightarrow \frac{d}{ds} \left( s^2 \frac{d\Theta}{ds} \right) = -s^2 \]
\[ \Rightarrow s^2 \frac{d\Theta}{ds} = -\frac{1}{3}s^3 - C \]
\[ \Rightarrow \Theta = -\frac{s^2}{6} + \frac{C}{s} + D \]
We need $\Theta = 1$ at $\xi = 0 \Rightarrow \psi = 0 + D = 1$

$\therefore \Theta = 1 - \xi^2 / 6$

For solutions for $n = 1$ or $n = 5$ cases, see the book (sections 5.5.2 and 5.5.3).

3: Isothermal spheres; the case $n \to \infty$

The isothermal case $p = k\rho$ corresponds to $n \to \infty$.

Let's combine

$$\frac{dp}{dr} = -p \frac{d\psi}{dr} \quad \text{and} \quad p = k\rho$$

$\Rightarrow \frac{d\psi}{dr} = -\frac{k}{p} \frac{dp}{dr}$

$\Rightarrow \psi - \psi_c = -k \ln \frac{\rho}{\rho_c}$

From Poisson's eqn,

$$\nabla^2 \psi = 4\pi G \rho$$

$\Rightarrow \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) = 4\pi G \rho$

$\Rightarrow \frac{k}{r^2} \frac{d}{dr} \left( r^2 \frac{1}{\rho} \frac{dp}{dr} \right) = -4\pi G \rho$

Let $p = p_c e^{-\psi}$ (imposing $\psi_c = 0$)

$r = a\xi$

$a = \sqrt{\frac{k}{4\pi G \rho_c}}$
Then \( \frac{1}{s^2} \frac{d}{ds} \left( s^2 \frac{d\psi}{ds} \right) = e^{-\psi} \)

with \( \psi = 0 \) at \( s = 0 \)
\( \frac{d\psi}{ds} = 0 \) at \( s = 0 \)

This replaces the Lane-Emden equation in the case where the EOS is isothermal.

At large radii, this has solutions of the form \( P \propto r^{-2} \), so the enclosed mass \( \propto r \). Thus, the mass of an isothermal sphere of self-gravitating gas tends to 0 as the radius tends to \( \infty \).

So, to be physical, isothermal spheres need to be truncated at some finite radius. There needs to be some containing pressure by an external medium. These are called Bonnor-Ebert spheres; density profile depends on \( \frac{\rho}{\rho_c} \).

Eq. dense gas core in a molecular cloud is well described by a Bonnor-Ebert sphere.

4: Scaling Relations

In many circumstances, stars behave as polytropes, e.g., fully convective stars with \( P - P_s \) due to the adiabatic relation. In such a star, assuming monoatomic gas with \( \gamma = 5/3 \), we have \( P = k \rho^{5/3} \), \( \Rightarrow n = 5/2 \).

All stars with a given polytropic index \( n \) belong to a family characterized by the value of \( P_c \) and \( k \), but set the same of all stars in a given family.
Thus one can find how mass and radius vary as a function of $P_c$ and hence relate mass and density to produce scaling relations.

All stars with given $n$ have same $O(5)$ since the Lane-Emden eqn does not depend on $P_c$. Recall

$$P = \left[ \frac{\psi_T - \psi}{(n+1) \psi} \right]^n \Rightarrow \psi_T - \psi_c = K(n+1) P_c^{1/n}$$

$$\zeta = \sqrt{\frac{4\pi G P_c}{\psi_T - \psi_c}} r \Rightarrow \zeta = \sqrt{\frac{4\pi G P_c}{K(1+n)}} r$$

$$P = P_c \left[ \frac{\psi_T - \psi}{\psi_T - \psi_c} \right]^n = P_c \zeta^n$$

The surface of the polytrope is at $\zeta = \zeta_{\text{max}}$ defined as location where we have $O(5) = 0$. Let $r_{\text{max}}$ be the corresponding physical radius. Then the total mass of the polytrope is

$$M = \int_0^{r_{\text{max}}} 4\pi r^2 P \, dr$$

$$= 4\pi P_c \left[ \frac{4\pi G P_c}{K(1+n)} \right]^{-\frac{3}{2}} \int_0^{\zeta_{\text{max}}} \zeta^{n-1} \zeta^2 \, d\zeta$$

$$\Rightarrow M \propto P_c \frac{1}{2} (\frac{3}{n} - 1)$$

From definition of $\zeta$ above, we also know that

$$r_{\text{max}} \propto P_c \frac{1}{2} (\frac{4}{n} - 2)$$

Eliminating $P_c$ gives

$$M \propto \frac{3-n}{1-n} \times R$$

MASS-RADIUS RELATION FOR POLYTROPIC STARS.
For $\sigma = 5/3$, $n = \frac{3}{2}$ this gives $M \propto R^{-3}$ or $R \propto M^{-1/3}$. This suggests more massive stars have smaller radii.

This relation actually works well for white dwarfs (where the polytropic EOS is due to e-degeneracy pressure rather than gas pressure). But for most main-sequence stars we observe $M \propto R$! Reason is that stars do not share the same polytropic constant $k$. Let's write the temperature at the core in terms of the central density and $k$...

\[
\begin{align*}
  p &= k\rho^{1+\frac{1}{n}} \\
  p &= \frac{R_*=\rho_T}{\frac{M}{R_*=\rho_T}} \implies T_c = \frac{mk}{R_*} \rho_c^{\frac{1}{n}}
\end{align*}
\]

Now, nuclear reactions in the core tend to keep $T_c$ similar in the cores of stars of different masses. So we can say that

\[ k \propto \rho_c^{-\frac{1}{n}} \]

Substitute this into above expression for mass when $n = \frac{3}{2}$; gives

\[ M \propto \rho_c^{-\frac{1}{2}}, \quad R \propto \rho_c^{-\frac{1}{2}} \implies M \propto R. \]

When can the $M = \text{const}$ relation be applied? Answer: when new mass is added to a star adiabatically and the nuclear processes have not had time to adjust... for Sun we have

- time to adjust to new hydrostatic equilibrium is

\[ t_h \sim R/c_s \sim 1 \text{ day} \]
- time to lose significant energy

\[ t_{\text{th}} \sim t_{\text{Edd}} \sim \frac{GM^2}{RL} \sim 30 \text{Myr} \]

So, mass loss/gain is followed by rapid re-adjustment of hydrostatic equilibrium but true thermal equilibrium is reached after a much longer time.

**Example:** Spherical rotating star with angular velocity \( \Omega \) gains non-rotating mass. How does \( \Omega \) evolve?

Conservation of angular momentum \( \Rightarrow MR^2\Omega = \text{const.} \)

So, if \( \Omega \to \Omega + \Delta \Omega \) then

\[ MR^2\Delta \Omega + \Omega \Delta (MR^2) = 0 \]

\[ \Rightarrow \frac{\Delta \Omega}{\Omega} = - \frac{\Delta (MR^2)}{MR^2} \]

But, we can use

\[ R \propto M^{1-n/3-n} \]

to say

\[ \frac{\Delta \Omega}{\Omega} \propto - \Delta (M^{n-3n/3-n}) \]

\[ \Rightarrow \frac{\Delta \Omega}{\Omega} \propto - \left( \frac{5}{3-n} \right) \Delta M \]

So,

\[ \Delta M > 0 \Rightarrow \begin{cases} \Delta \Omega < 0 & \text{if } \frac{5}{3-n} > 0 \quad (\text{eq. } n = \frac{3}{2}) \text{ spin down} \\ \Delta \Omega > 0 & \text{if } \frac{5}{3-n} < 0 \quad (\text{eq. } n = 2) \text{ spin up} \end{cases} \]
Example: Star in a binary system loses mass to its companion.

Donor star loses mass, $\Delta M < 0$. So since $R \propto M^{1-n} n$, the radius will increase if $3 > n > 1$. So there is the potential for unstable (runaway) mass transfer (need to look at evolution of the size of the Roche lobe to conclusively decide whether process is unstable).
1: Sound Waves

We now start discussing how disturbances can propagate in a fluid. We begin by talking about sound waves in a uniform medium (no gravity). We proceed by conducting a first-order perturbation analysis of the fluid equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P
\]

The equilibrium around which we will perturb is:

\[\begin{align*}
\rho &= \rho_0 \quad \text{(uniform and constant)} \\
P &= P_0 \quad \text{(uniform and constant)} \\
\mathbf{u} &= \mathbf{0}
\end{align*}\]

We consider small perturbations; write in Lagrangian terms:

\[\begin{align*}
P &= P_0 + \Delta P \\
P &= P_0 + \Delta P \\
\mathbf{u} &= \Delta \mathbf{u}
\end{align*}\]

Lagrangian ≡ change of quantities for a given fluid element.

The relation between Lagrangian and Eulerian perturbations is:
In present example, $\nabla \rho_0 = 0$ and so $\delta \rho = \Delta \rho$. But the distinction between Lagrangian and Eulerian perturbations will be important for other situations that we will address later.

Substitute the perturbations into fluid eq" and ignore terms that are 2nd order (or higher) in the perturbed quantities:

**Continuity eq:**

$$\frac{\partial}{\partial t} (\rho_0 + \Delta \rho) + \nabla \cdot [(\rho_0 + \Delta \rho) \cdot \Delta \mathbf{u}] = 0$$

$$\Rightarrow \frac{\partial \rho_0}{\partial t} + \frac{\partial \Delta \rho}{\partial t} + \nabla \rho_0 \cdot \Delta \mathbf{u} + \nabla (\Delta \rho) \cdot \Delta \mathbf{u}$$

$$+ \rho_0 \nabla \cdot (\Delta \mathbf{u}) + \Delta \rho \cdot \nabla (\Delta \mathbf{u}) = 0$$

$$\Rightarrow \frac{\partial}{\partial t} (\Delta \rho) + \rho_0 \nabla \cdot (\Delta \mathbf{u}) = 0 \quad \ldots \ 1$$

and similarly, the momentum eq":

$$\frac{\partial}{\partial t} (\Delta \mathbf{u}) = -\frac{1}{\rho_0} \nabla (\Delta \rho)$$

$$\Rightarrow \frac{\partial}{\partial t} (\Delta \mathbf{u}) = -\left. \frac{d \rho}{d \rho} \right|_{\rho = \rho_0} \frac{\nabla (\Delta \rho)}{\rho_0} \quad \text{assuming} \ \text{barotropic EDS}.$$...\ 2
Now, take \( \varphi(t, \rho) \):

\[
\frac{\partial^2}{\partial t^2} (\Delta \rho) = -\rho_0 \frac{\partial}{\partial t} \left[ \nabla \cdot (\Delta \mathbf{u}) \right]
\]

\[
= -\rho_0 \nabla \cdot \left[ \frac{\partial}{\partial t} (\Delta \mathbf{u}) \right]
\]

\[
= \left. \frac{\partial \rho}{\partial \rho} \right|_{\rho = \rho_0} \nabla^2 (\Delta \rho)
\]

\[
\therefore \frac{\partial^2 (\Delta \rho)}{\partial t^2} = \left. \frac{\partial \rho}{\partial \rho} \right|_{\rho = \rho_0} \nabla^2 (\Delta \rho)
\]

**WAVE EQUATION**

This admits solutions of the form \( \Delta \rho = \Delta \rho_0 e^{i(kz - \omega t)} \). Substituting into the wave equation we get

\[
(-i\omega)^2 \Delta \rho_0 = \left. \frac{\partial \rho}{\partial \rho} \right|_{\rho = \rho_0} (i k)^2
\]

\[
\Rightarrow \omega^2 = \left. \frac{\partial \rho}{\partial \rho} \right|_{\rho = \rho_0} k^2
\]

The (phase) speed of the wave is \( V_\rho = \omega/k \), so the sound wave travels at speed

\[
C_s = \sqrt{\left. \frac{\partial \rho}{\partial \rho} \right|_{\rho = \rho_0}}
\]

**Sound speed as the derivative of \( \rho(\rho) \)**

Consider a 1D wave and substitute

\( \Delta \rho = \Delta \rho_0 e^{i(kz - \omega t)} \)

\( \Delta \mathbf{u} = \Delta \mathbf{u}_0 e^{i(kz - \omega t)} \)

Into 0. We get

\[
-\omega \Delta \rho + \rho_0 \omega k \Delta \mathbf{u} = 0
\]

\[
\Rightarrow \Delta \mathbf{u} = \frac{\omega}{k} \frac{\Delta \rho}{\rho_0} = C_s \frac{\Delta \rho}{\rho_0}
\]
So we learn that

- fluid velocity and density perturbations are in phase (since $\Delta U/\Delta P \in i\mathbb{R}$)

- disturbance propagates at a much higher speed than that of the individual fluid elements, provided density perturbations are small

$$\Delta U_0 = C_s \frac{\Delta P_0}{\rho_0} \ll C_s.$$

Sound waves propagate because density perturbations give rise to a pressure gradient which then causes accelerations of the fluid elements $\rightarrow$ induce further density perturbations $\rightarrow$ disturbance propagates.

Sound speed depends on how the pressure forces react to density changes. If the EOS is "stiff" (i.e., high $\frac{\partial p}{\partial \rho}$) then restoring force is large and propagation is rapid.

Examples of $\frac{\partial p}{\partial \rho}$:

1. Isothermal case: $C_s^2 = \frac{\partial p}{\partial \rho} \bigg|_T$

   In this case, compressions and rarefactions are effective at passing heat to each other to maintain constant $T$. Then

   $$p = \frac{\gamma R*}{M} \rho T$$

   $\Rightarrow C_{s,i} = \sqrt{\frac{\gamma R* T}{M}}$

2. Adiabatic case: $C_s^2 = \frac{\partial p}{\partial \rho} \bigg|_s$

   No heat exchange between fluid elements: compressions heat up and rarefactions cool down from pdV work.
\[ p = K \rho \sigma \]
\[ \Rightarrow \frac{\partial p}{\partial \rho} \bigg|_{ls} = \gamma K \rho^{\gamma - 1} = \frac{\gamma p}{\rho} \]
\[ \Rightarrow c_s,A = \sqrt{\frac{\gamma R_T T}{\mu}} \]

Notes:
- We see that \( c_s,\Sigma \) and \( c_s,A \) differ by only \( \sqrt{\gamma} \)
- Thermal behaviour of the perturbations does not have to be the same as that of the unperturbed structure!
  - Earth's atmosphere \( \Rightarrow \) background is isothermal
  - Sound waves are adiabatic
- Waves for which \( c_s \) is not a function of \( \xi \) are called non-dispersive. The shape of a wave packet is preserved.

**2: Sound waves in a stratified atmosphere**

We now move to the more subtle problem of sound waves propagating in a fluid with background structure. For concreteness, let's consider an isothermal atmosphere with constant \( g = -g \Sigma \).

Horizontally travelling sound waves are unaffected by the (vertical) structure. So let's just focus on \( z \)-dependent terms.

\[ \frac{\partial p}{\partial t} + \frac{\partial}{\partial z} (\rho u) = 0 \quad \cdots 1 \]
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \cdots 2 \]
The equilibrium is
\[
\begin{align*}
    U_0 &= 0 \\
    P_0(z) &= \bar{\rho} e^{-z/H}, \quad H = \frac{2r+1}{g/M} \\
    \rho_0(z) &= \frac{2r+1}{M} P_0(z) = \bar{\rho} e^{-z/H}
\end{align*}
\]

Consider Lagrangian perturbation:
\[
\begin{align*}
    U &\to \Delta U \\
    P_0 &\to P_0 + \Delta P \\
    \rho_0 &\to \rho_0 + \Delta \rho
\end{align*}
\]

Remember that \( \delta \rho = \Delta \rho - \bar{\rho} \cdot \Delta \rho \). So we have
\[
\begin{align*}
    \delta \rho &= \Delta \rho - \bar{\rho} \cdot \frac{\partial P_0}{\partial z} \\
    \delta \rho &= \Delta \rho - \bar{\rho} \cdot \frac{\partial P_0}{\partial z} \\
\end{align*}
\]

\( \Delta U = \Delta u \)

and
\[
\Delta u = \frac{d\delta}{dt} = \frac{\partial \delta}{\partial t} + \bar{u} \cdot \nabla \delta = \frac{\partial \delta}{\partial t}
\]

Substituting perturbed quantities into Eulerian continuity equation,
\[
\frac{\partial}{\partial t} (\rho_0 + \delta \rho) + \frac{\partial}{\partial z} \left( (\rho_0 + \delta \rho) \delta u_2 \right) = 0
\]

\[
\Rightarrow \frac{\partial}{\partial t} (\rho_0 + \Delta \rho - \bar{\rho}_z \frac{\partial \rho_0}{\partial z}) + \frac{\partial}{\partial z} (\rho_0 \Delta u_2) = 0 \quad \text{(ignoring 2nd order terms)}
\]

\[
\Rightarrow \frac{\partial \rho_0}{\partial t} + \frac{\partial \Delta \rho}{\partial t} - \bar{\rho}_z \frac{\partial \rho_0}{\partial z} - \bar{\rho}_z \frac{\partial \Delta u_2}{\partial z}
\]

\[
+ \bar{\rho}_0 \Delta u_2 + \rho_0 \frac{\partial \Delta u_2}{\partial z} = 0
\]
\[ \frac{\partial \rho}{\partial t} - \Delta u_z \frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial z} \Delta u_z + \rho \frac{\partial u_z}{\partial z} = 0 \]

\[ \frac{\partial \rho}{\partial t} = 0 \]

\[ \Rightarrow \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_z}{\partial z} = 0 \quad \cdots \quad (3) \]

A similar calculation for the momentum eqn gives

\[ \frac{\partial u_z}{\partial t} = - \frac{1}{\rho} \frac{\partial \rho}{\partial z} \]

\[ \Rightarrow \frac{\partial u_z}{\partial t} = - \frac{\alpha u}{\rho} \frac{\partial \rho}{\partial z} \quad , \quad \alpha u = \frac{\partial P}{\partial \rho} \quad \cdots \quad (4) \]

To perform this calculation (which we leave as an exercise!) you need a relation that is obtained from the Lagrangian continuity eqn:

\[ \frac{\partial \rho}{\partial t} + \rho \nabla \cdot u = 0 \]

\[ \Rightarrow \Delta \rho + (\rho \nabla \cdot \frac{\partial \rho}{\partial t}) \Delta t = 0 \quad \text{ (Integrating over a short time \( \Delta t \))} \]

\[ \Rightarrow \Delta \rho + \rho \nabla \cdot \Delta t = 0 \]

Let's now derive the wave equation and dispersion relation. Take \( \frac{\partial}{\partial t} \) of (3):

\[ \frac{\partial^2 \rho}{\partial t^2} + \rho \frac{\partial}{\partial z} \left( \frac{\partial u_z}{\partial t} \right) = 0 \]

\[ \Rightarrow \frac{\partial^2 \rho}{\partial t^2} - \rho \frac{\partial}{\partial z} \left( \frac{\alpha u}{\rho} \frac{\partial \rho}{\partial z} \right) = 0 \]

If medium is isothermal, then \( \alpha u \) is independent of \( z \).
\[ \frac{\partial^2 \Delta p}{\partial t^2} - \frac{\rho}{\rho_0} \frac{c_u^2}{\rho_0} \frac{\partial^2 \Delta p}{\partial z^2} + \frac{\rho}{\rho_0} \frac{c_u^2}{\rho_0} \frac{\partial^2 \Delta p}{\partial z^2} \frac{\partial \Delta p}{\partial z} = 0 \]

\[ \Rightarrow \frac{\partial^2 \Delta p}{\partial t^2} - \frac{c_u^2}{H} \frac{\partial^2 \Delta p}{\partial z^2} + \frac{c_u^2}{H} \frac{\partial \Delta p}{\partial z} = 0 \]

normal sound wave equation

Extra piece associated with stratification

Now,
\[ \frac{\partial \rho}{\partial z} = \frac{2}{\partial z} (\tilde{\rho} e^{-2z/H}) \]
\[ = -\frac{1}{H} \tilde{\rho} e^{-2z/H} \]
\[ = -\rho_0 / H \]

So,
\[ \frac{\partial^2 \Delta p}{\partial t^2} - \frac{c_u^2}{H} \frac{\partial^2 \Delta p}{\partial z^2} + \frac{c_u^2}{H} \frac{\partial \Delta p}{\partial z} = 0 \]

look for solutions of the form \( \Delta p \propto e^{i(kz - wt)} \)

\[ \Rightarrow -w^2 = -c_u^2 k^2 + \frac{c_u^2 i k}{H} \]

\[ \Rightarrow w^2 = c_u^2 \left( k^2 - \frac{i k}{H} \right) \]

\[[\text{Dispersion Relation}]\]

We can also write this as
\[ k^2 - \frac{ik}{H} - \frac{w^2}{c^2 u} = 0 \]

and solve the quadratic for \( k(w) \):
\[ k = \frac{i}{2H} \pm \sqrt{\frac{w^2}{c^2 u} - \frac{1}{4H^2}} \]

We have two cases to examine if we wish to understand the implications of this dispersion relation, assuming that \( w \gg R \)
Case I: \( W > C_u/2h \)

Examine the real and imaginary parts of \( k \):

\[
\text{Im}(k) = \frac{1}{2} W
\]
\[
\text{Re}(k) = \pm \sqrt{\left(\frac{W}{C_u}\right)^2 - \left(\frac{1}{2} W\right)^2}
\]

So the density perturbation is

\[
\Delta \rho \propto e^{-2/2h} e^{i \left(\pm \sqrt{\left(\frac{W}{C_u}\right)^2 - \left(\frac{1}{2} W\right)^2} - Wt\right)}
\]

Exponentially decaying amplitude with increasing height

Wave with phase velocity

\[V_{ph} = \frac{W}{k}, \quad k = \pm \sqrt{\left(\frac{W}{C_u}\right)^2 - \left(\frac{1}{2} W\right)^2}\]

\(V_{ph}\) is function of \( W \), meaning that the wave is dispersive. Wave packet consisting of different \( W \) will change shape as it propagates.

As before, we can relate \( \Delta u \) to \( \Delta \rho \):

\[
\Delta u \propto \frac{\Delta \rho}{\rho_0} \frac{W}{k}
\]

with

\[
\Delta \rho \propto e^{-2/2h}
\]
\[
\rho_0 \propto e^{-2/2h}
\]

Giving

\[
\Delta u \propto e^{+2/2h}, \quad \frac{\Delta \rho}{\rho_0} \propto e^{+2/2h}
\]

Thus the perturbed velocity and the fractional density variation both increase with height. In the absence of dissipation (e.g., viscosity), the kinetic energy flux is conserved and the amplitude of the wave increases until

\[\Delta u \sim C_s, \quad \frac{\Delta \rho}{\rho_0} \sim 1\]

The linear treatment breaks down and the sound wave "steepens" into a shock. So, in the absence of dissipation, an upward propagating sound wave from a hand clapping would generate shocks in the upper atmosphere.
Case II: \( W < \frac{c_n}{2H} \)

In this case, we find that \( k \) is purely imaginary.

So,

\[ \Delta p \propto e^{ikz} e^{iwt} \]

This is a non-propagating, evanescent wave. In essence the wave cannot propagate since the properties of the atmosphere change significantly over one wavelength giving rise to reflection.

3: Transmission of sound waves at interfaces

Consider two non-dispersive media with a boundary at \( x = 0 \). Suppose we have a sound wave travelling from \( x < 0 \) to \( x > 0 \). Let the incident wave have unity amplitude (in, say, the density perturbation), and denote by \( r \) and \( t \) the amplitudes of the reflected and transmitted waves respectively:

\[ \begin{align*}
  & e^{i(k_1x - wt)} \\
  & \rightarrow \\
  & r e^{i(k_3x - wt)} \\
  & \leftarrow \\
  & t e^{i(k_2x - wt)} \\
  & \rightarrow \\
  & x = 0
\end{align*} \]

At the boundary \( x = 0 \), variables must be single valued and the accelerations finite \( \Rightarrow \) oscillations in the second medium must have the same frequency.

\[ \therefore W_1 = W_2 = W_3 = W \]

The reflected wave is in the same medium as the incident

\[ \therefore k_3 = -k_1 \quad \text{(phase speed reversed)} \]
Amplitude of sound wave continuous at \( x=0 \):
\[
1 + r = t
\]
and the derivative of the amplitude is continuous at \( x=0 \):
\[
h_1 (1-r) = h_2 t
\]
We can combine these relations to get:
\[
t = \frac{2h_1}{h_1 + h_2}, \quad r = \frac{h_1 - h_2}{h_1 + h_2}
\]

From these relations we can see that the reflection/transmission of sound waves strongly depends on the relative sound speeds in the two media:

1. If \( c_{s,2} > c_{s,1} \) \( \Rightarrow h_2 < h_1 \)
   \( \Rightarrow r > 0 \)
   \( \Rightarrow \text{reflected wave in phase with incident wave} \)

2. If \( c_{s,2} < c_{s,1} \) \( \Rightarrow h_2 > h_1 \)
   \( \Rightarrow r < 0 \)
   \( \Rightarrow \text{reflected wave is \( 180^\circ \) out of phase with incident wave} \)

3. If \( c_{s,2} \ll c_{s,1} \) \( \Rightarrow h_2 \ll h_1 \)
   \( \Rightarrow t \ll 1 \)
   \( \Rightarrow \text{wave almost completely reflected} \)
Shocks occur when there are disturbances in the fluid caused by compression by a large factor, or acceleration to velocities comparable to or exceeding $C_s$. The linear theory applied to sound waves breaks down.

When thinking about the sound speed, recall that the chemical composition of the fluid matter, $C_s \propto M^{-1/2}$

$C_s \text{ in } H \Rightarrow C_s \text{ in } N \text{ for given } T$

\[
\begin{align*}
M &= 1 & \text{eq } 15m \\
\mu &= 28 & \text{eq Earth atm} \\
\end{align*}
\]

Disturbances in a fluid always propagate at the sound speed relative to the fluid itself. Consider an observer at the centre of a spherical disturbance, watching the fluid flow past it at speed $U$.

The velocity of the disturbance is the vector sum $V$.

The velocity of the disturbance relative to the observer, $U'$, is the vector sum of the fluid velocity and the disturbance velocity relative to the fluid.

- **Subsonic case**: $V'$ sweeps 4π steradians
- **Supersonic case**: disturbance always to the right.

If we continuously produce a disturbance, the envelope of the disturbances will define a cone with opening angle $\alpha$ given by

\[
\sin \alpha = \frac{C_s}{U} \quad \text{MACH CONE}
\]
The ratio of the flow speed to the sound speed is called the Mach number.

\[ M = \frac{v}{c_s} \]

\[ \sin \alpha = \frac{1}{M} \]

Imagine an obstacle in a supersonic flow - disturbances cannot propagate upstream from the obstacle so the flow cannot adjust to presence of obstacle. The flow properties must change discontinuously once the obstacle is reached \( \Rightarrow \) shock!

**5: The Rankine-Hugoniot relations**

We analyze a shock by applying conservation of mass, momentum and energy across the shock front.

In the frame of the shock, let's assume following geometry:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \]

\[ \Rightarrow \frac{\partial}{\partial t} \left( \int_{\alpha x/2}^{\alpha x/2} \rho \, dx \right) + \rho u \frac{d\alpha x}{dx} \bigg|_{x = \frac{\alpha x}{2}} - \rho u \frac{d\alpha x}{dx} \bigg|_{x = -\frac{\alpha x}{2}} = 0 \]

Where we have integrated over a small region \( dx \) around the shock.
Let's take $dx \to 0$ and assume that mass does not continually accumulate at $x=0$. Then

$$\frac{\partial}{\partial t} (S \rho dx) = 0$$

$$\Rightarrow \boxed{p_1 u_1 = p_2 u_2}$$

1ST RANKINE-HUGONIOT RELATION

Apply similar analysis to the momentum equation:

$$\frac{\partial}{\partial t} (\rho u_x) = -\frac{\partial}{\partial x} (\rho u_x u_x + \rho) - \rho \frac{\partial y}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho u_x dx) = - (\rho u_x u_x + \rho) \bigg|_{dx/2} + (\rho u_x u_x + \rho) \bigg|_{-dx/2}$$

(assuming $y$ is continuous across the shock)

$$\Rightarrow \boxed{p_1 u_1^2 + p_1 = p_2 u_2^2 + p_2}$$

2ND R-H RELATION

We note that $u_y$ and $u_z$ do not change across the shock front (can be immediately seen by looking at the $y$- and $z$-cpts of the momentum equation).

Now for the energy equation. Start with the adiabatic case so that the gas cannot cool and hence we have $Q_{\text{adi}} = 0$. Also take gravitational potential to have no time-dependence. Then

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+p)u] = -P \frac{\dot{\rho}}{\rho} \omega_{\text{rot}} + \rho \frac{\partial y}{\partial t}$$

$$\Rightarrow \frac{\partial E}{\partial t} + \nabla \cdot [(E+p)u] = 0$$

$$\Rightarrow \frac{\partial}{\partial t} (SE dx) + (E+p)u_x \bigg|_{dx/2} - (E+p)u_x \bigg|_{-dx/2} = 0$$

$$\Rightarrow (E_1 + p_1) u_1 = (E_2 + p_2) u_2$$
Since \( E = \rho (\frac{1}{2}u^2 + \xi + \psi) \), this becomes

\[
\rho \left( \frac{1}{2}u_1^2 + \rho, \xi, u_1 + \rho_1 \psi u_1 + \rho_1 u_1 \right) = \rho_2 \left( \frac{1}{2}u_2^2 + \rho_2 \xi_2 u_2 + \rho_2 \psi_2 u_2 + \rho_2 u_2 \right)
\]

But \( \psi_1 = \psi_2 \) and \( \rho_1 u_1 = \rho_2 u_2 \), so terms involving \( \psi \) cancel out. We are left with

\[
\frac{1}{2}u_1^2 + \xi_1 + \frac{\rho_1}{\rho_1} = \frac{1}{2}u_2^2 + \xi_2 + \frac{\rho_2}{\rho_2}
\]

3RD R-H RELATION

For an ideal gas, we have

\[
\begin{align*}
\xi &= \text{Cv T} \\
P &= \frac{\text{R} \times \text{T}}{\text{m}}
\end{align*}
\]

\[
\Rightarrow \xi = \frac{\text{Cv M}}{\text{R}} \times \frac{P}{\rho}
\]

\[
\gamma = \frac{\text{Cp/Cv}}{} \quad \left\{ \begin{array}{c}
\text{Cp/Cv} = \frac{\text{R} \times \text{M}}{\text{m}} \\
\end{array} \right\} \Rightarrow \text{Cv} \left( \gamma - 1 \right) = \frac{\text{R} \times \text{M}}{\text{m}}
\]

Which combine to give

\[
\xi = \frac{1}{\gamma - 1} \times \frac{P}{\rho}
\]

(in internal energy per unit mass)

If we assume that \( \gamma \) does not change across the shock (e.g., there are no dissociation of molecules), the 3rd RH relation becomes

\[
\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1} \times \frac{\rho_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1} \times \frac{\rho_2}{\rho_2}
\]

\[
\Rightarrow \frac{1}{2}u_1^2 + \frac{c_{s1}^2}{\gamma - 1} = \frac{1}{2}u_2^2 + \frac{c_{s2}^2}{\gamma - 1}
\]

since, for adiabatic case, the sound speed is

\[
c_{s}^2 = \left. \frac{\partial P}{\partial \rho} \right|_s = \frac{\partial P}{\rho}
\]
Using all three RH relations and after some algebra we get

\[
\frac{P_2}{P_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)P_2 + (\gamma-1)P_1}{(\gamma+1)P_1 + (\gamma-1)P_2}
\]

In the limit of strong shocks, \( P_2 \gg P_1 \), we get

\[
\frac{P_2}{P_1} \Rightarrow \frac{\gamma+1}{\gamma-1}
\]

For \( \gamma = 5/3 \), this gives \( P_2 = 4P_1 \). So there is a maximum possible density contrast across an adiabatic shock — with stronger and stronger shocks, the thermal pressure of the shocked gas increases and prevents further compression.

Note that, since \( P_2 \gg P_1 \) and \( P_2 \leq 4P_1 \), we have

\[
\frac{P_1}{P_1} \neq \frac{P_2}{P_2} \quad \text{i.e.} \quad K_1 \neq K_2
\]

The gas has jumped adiabatically during its passage through the shock. Shocking the gas produces a non-reversible change. Due to various processes operating within shock.

It is interesting that we can derive RH conditions using the inviscid equations.

Not all shocks are adiabatic! To consider the other extreme, let's discuss isothermal shocks. Here we have \( Q_{\text{iso}} \neq 0 \) such that the shocked gas cools to produce \( T_2 = T_1 \). Whether a shock is isothermal or adiabatic depends on whether the "cooling length" is smaller or larger than the system size, respectively.
For isothermal shocks, the first two R-H eqns are unchanged:

\[ p_1 u_1 = p_2 u_2 \]
\[ p_1 u_1^2 + p_1 = p_2 u_2^2 + p_2 \]

but the 3rd R-H eqn is replaced by

\[ T_1 = T_2 \]

Now,

\[ c_{s,1} = \frac{\sqrt{R \cdot T}}{\mu} \quad \Rightarrow \quad c_{s,1} = c_{s,2} \]
\[ = \sqrt{\frac{p}{\rho}} \quad \Rightarrow \quad p = c_{s,1}^2 \rho \]

So, 2nd R-H eqn becomes

\[ p_1 (u_1^2 + c_{s}^2) = p_2 (u_2^2 + c_{s}^2) \]

\[ \Rightarrow \quad u_1 + \frac{c_{s}^2}{u_1} = u_2 + \frac{c_{s}^2}{u_2} \quad \text{(since } p_1 u_1 = p_2 u_2) \]

\[ \Rightarrow \quad c_{s}^2 = \left( \frac{1}{u_1} - \frac{1}{u_2} \right) = u_2 - u_1 \]
\[ c_s^2 \frac{U_2 - U_1}{U_1 U_2} = U_2 - U_1 \]
\[ \Rightarrow c_s^2 = U_1 U_2 \]

Thus we see that

\[ \frac{P_2}{P_1} = \frac{U_1}{U_2} = \left( \frac{U_1}{c_s} \right)^2 = M_1^2 \]

where \( M_1 \) is the Mach number of the upstream flow. So the density compression can be very large.

Note that since \( c_s^2 = U_1 U_2 \) and \( U_1 > c_s \) (condition for a shock), we must have \( U_2 < c_s \). So flow behind the shock is subsonic. In fact this is always true for any shock and is necessary to preserve causality (the post shock gas must know about the shock!).
An important application of shock wave theory is to supernova explosions in the interstellar medium (ISM). A supernova (SN) deposits about $10^{51}$ erg ($= 10^{44}$ J) of energy into the surrounding medium. The shock medium expands, sweeps up more gas, and creates large bubbles in the ISM.

Consider the following system:

- Uniform density medium with density $\rho_0$
- Point explosion with energy $E$
- Ignore temperature of the ambient ISM ($T_0 = 0$)

$\Rightarrow$ no confinement of explosion by an external pressure.

Given that $T_0 = 0$, the shock has $M \to \infty$. Assuming an adiabatic shock, we sweep mass into a shell with density $\rho_1$ given by

$$\rho_1 = \rho_0 \frac{5+1}{5-1}$$

![Diagram](attachment:image.png)
If all mass is swept up into shell then
\[ \frac{4\pi}{3} \rho_0 R^3 = 4\pi \rho_1 R^2 D \quad \text{(assuming } D \ll R) \]

\[ \Rightarrow D = \frac{1}{3} \left( \frac{\gamma-1}{\gamma+1} \right) R \]

For \( \gamma = 5/3 \), we have \( D \approx 0.08 R \) which justifies the assumption \( D \ll R \).

Assume that all gas in the shell moves with a common velocity. In the frame of a local patch of the shock we have

\[ \begin{align*}
\rho_0 & \quad \Rightarrow \quad \rho' \\
\frac{u_0}{u_1} & \quad \Rightarrow \quad \frac{u_0}{u_1} \\
\end{align*} \]

and so
\[ \rho_0 u_0 = \rho_1 u_1 \]

\[ \Rightarrow u_1 = \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma-1}{\gamma+1} u_0 \] \( \ldots \) \( 0 \)

Thus, relative to the unshocked gas, the velocity of the shocked gas \( U \) is

\[ U = u_0 - u_1 = \frac{2u_0}{\gamma+1} \]

Then, the rate of change of momentum of the shocked shell is

\[ \frac{d}{dt} \left[ \frac{4\pi}{3} \rho_0 R^3 \frac{2u_0}{\gamma+1} \right] \]

This momentum gain is provided by pressure acting on the inside surface of the shell—call this \( P_{in} \). Let's make the ansatz that this is related to the pressure within the shell by

\[ P_{in} = \alpha P_1 \]
Let's now relate \( p_1 \) and \( u_0 \) using the R-H jump conditions: we have

\[
P_0 + p_0 u_0^2 = p_1 + p_1 u_1^2
\]

\[
\Rightarrow p_1 = p_0 u_0^2 \left[ 1 - \frac{p_1 u_1}{p_0 u_0} \right]
\]

(since \( p_0 = 0 \) by assumption)

\[
= p_0 u_0^2 \left[ 1 - \frac{1}{\frac{\gamma-1}{\gamma+1}} \right]
\]

(assuming a strong shock)

\[
= \frac{2}{\gamma+1} p_0 u_0^2 \quad \text{(2)}
\]

So, equating rate of change of momentum of the shocked shell to the pressure acting on the inside surface of the shell, we have

\[
\frac{d}{dt} \left[ \frac{4\pi}{3} \rho_0 R^3 \frac{2u_0}{\gamma+1} \right] = 4\pi R^2 p_1
\]

\[
= 4\pi R^2 \alpha p_1
\]

\[
= 4\pi R^2 \alpha \cdot \frac{2}{\gamma+1} p_0 u_0^2
\]

\[
\Rightarrow \quad \frac{d}{dt} [R^3 u_0] = 3\alpha R^2 u_0^2
\]

\[
\Rightarrow \quad \frac{d}{dt} [R^3 R] = 3\alpha R^2 R^2
\]

since \( u_0 = R \)

Assume solution of the form \( \text{R} \propto t^b \)

\[
\Rightarrow \quad \frac{d}{dt} (t^b \text{bt}^{b-1}) = 3\alpha t^{b} \text{(bt}^{b-1})^2
\]

\[
\Rightarrow \quad b (4b-1) t^{4b-2} = 3\alpha b^2 t^{4b-2}
\]

\[
\Rightarrow \quad b = 0 \quad \text{(not physical)}
\]

or \( b = \frac{1}{4-3\alpha} \)

\[
\Rightarrow \quad \text{R} \propto t^{\frac{1}{4-3\alpha}} \quad , \quad u_0 \propto t^{\frac{3\alpha-3}{4-3\alpha}} \propto R^{3\alpha-3}
\]
To determine $\alpha$, we need to consider energy conservation. For an adiabatic shock, the explosive energy is conserved and transformed into kinetic and internal energy...

- Kinetic energy of the shell is $\frac{1}{2} \frac{4\pi}{3} \rho_0 R^3 U^2$

- Internal energy per unit mass is $\epsilon = \frac{1}{\gamma-1} \frac{P}{\rho}$

and so the internal energy per unit volume is $\rho \epsilon = \frac{1}{\gamma-1} P$.

Since the shell is very thin, it has small volume and so most of the internal energy is in the central cavity which contains little mass

$$\text{Int. Energy of cavity} \approx \frac{4\pi}{3} R^3 \rho_0 u_0 = \frac{4\pi}{3} R^3 \alpha \frac{P_1}{\gamma-1}$$

So, energy conservation says that

$$E = \frac{1}{2} \frac{4\pi}{3} \rho_0 R^3 U^2 + \frac{4\pi}{3} R^3 \alpha \frac{P_1}{\gamma-1}$$

$$= \frac{1}{2} \frac{4\pi}{3} \rho_0 R^3 \left( \frac{2 u_0}{\gamma+1} \right)^2 + \frac{4\pi}{3} R^3 \alpha \frac{2}{\gamma-1} \rho_0 u_0^2 \frac{1}{\gamma-1}$$

$$= \frac{4\pi}{3} R^3 u_0^2 \left[ \frac{1}{2} \rho_0 \frac{4}{(\gamma+1)^2} + \alpha \rho_0 \frac{2}{\gamma+1} \frac{1}{\gamma-1} \right]$$

So, $E \propto R^3 u_0^2 \propto t^{6\alpha-3/4-3\alpha}$

but $E$ must be conserved. So we need $\alpha = \frac{1}{2}$.

$$\Rightarrow R \propto t^{7/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}$$
Similarity solutions: The above problem only has 2 parameters, $E$ and $\rho_0$. Look at their dimension:

$$[E] = \frac{M \cdot L^2}{T^2}, \quad [\rho_0] = \frac{M}{L^3}$$

These cannot be combined to give quantities with the dimension of length or time. So, there is no natural length scale or time scale in the problem!

Given some time $t$, the only way to combine $E, \rho_0$ and $t$ to give a length scale is

$$\lambda = (Et^2/\rho_0)^{1/5}$$

We can define a dimensionless distance parameter,

$$\xi \equiv \frac{r}{\lambda} = r \left(\frac{\rho_0}{Et^2}\right)^{1/5}$$

Then, for any variable in the problem $X(r,t)$, we will have

$$X = x_1(t) \tilde{X}(\xi)$$

i.e., $X$ as a function of scaled distance $\xi$ always has same shape scaled up/down by the time dependent factor $x_1(t)$

and so,

$$\frac{\partial X}{\partial r} = x_1 \frac{\partial \tilde{X}}{\partial \xi} \frac{\partial \xi}{\partial r}$$

$$\frac{\partial X}{\partial t} = \tilde{X}(\xi) \frac{dx_1}{dt} + x_1 \frac{d\tilde{X}}{d\xi} \frac{\partial \xi}{\partial t}$$

$\xi$ is neither a Lagrangian or an Eulerian coordinate. It labels a particular feature in the flow (e.g., shock wave) that can move through the fluid. So, we can write

$$R_{\text{shock}} \propto (E/\rho_0)^{1/5} t^{2/5}$$
Let's put some numbers in for the case of supernova explosions,

\[ R(t) = \xi_0 \left( \frac{E}{\rho_0} \right)^{\frac{1}{15}} t^{\frac{2}{15}} \quad \text{(we will assume } \xi_0 \approx 1) \]

\[ u_0(t) = \frac{dR}{dt} = \frac{2}{5} \xi_0 \left( \frac{E}{\rho_0 t^3} \right)^{\frac{1}{5}} = \frac{2}{5} \frac{R}{t} \]

In a supernova we have

\[ E \approx 10^{44} J = 10^{51} \text{ erg} \]

\[ \rho_0 = \rho_{\text{ISM}} \approx 10^{-21} \text{ kg m}^{-3} \]

So similarity solution gives

\[ R \approx 0.3 \ t^{\frac{2}{15}} \text{ pc} \]
\[ u_0 \approx 10^5 t^{-\frac{1}{15}} \text{ km s}^{-1} \]

where \( t \) is measured in yr.

The original explosion injects the stellar debris at about \( 10^4 \text{ km s}^{-1} \). So the above solution is valid for-

\[ t > 100 \text{ yr} \quad \text{(when } u_0 < U_{\text{inj}}) \]
\[ t < 10^5 \text{ yr} \quad \text{(after which energy losses become important)} \]

Structure of the blast wave: we can, in principle, write each variable \( p, \rho, u, r \) in terms of separated functions of \( t \) and \( \xi \). We can then substitute into the Eulerian equations of fluid dynamics (in spherical coordinates with \( \partial / \partial \rho = \partial / \partial \theta = 0 \), i.e., spherical symmetry).

The result is a set of ODEs where \( \xi \) is the only dependent variable — the time dependence cancels out! (Sedov 1946)
Solution for $\gamma = 7/5$

These solutions tell us that:
- Most of mass is swept up in a shell just behind the shock (from form of $\widetilde{p}$).
- Post-shock pressure is indeed a multiple of $p_i$ (from form of $\widetilde{p}$; just like $p_i = \alpha p$, assumption).
- Shell material not really moving at a single velocity, but arguments above are resolved by taking some weighted average (form form of $\widetilde{u}$).

Breakdown of the similarity solution: the self-similar flow breaks down when the surrounding medium pressure $p_0$ becomes significant, $p_1 \approx p_0$.

From the strong shock solution, we derived:
\[ p_1 = \frac{2}{\gamma+1} p_0 u_0^2, \quad \sqrt{\gamma} = \frac{\gamma p_0}{\rho_0} \]

So if $p_1 \approx p_0$ then:
\[ \frac{2}{\gamma+1} p_0 u_0^2 \approx p_0 \frac{1}{\gamma} \frac{p_0}{\rho_0} \]
\[ \Rightarrow u_0 \approx \frac{1}{\gamma} \frac{p_0}{\rho_0} \]

ie shell not moving supersonically anymore.
The blast wave weakens to a sound wave

\[ P \rightarrow P_0 \]

\[ r \]

\[ \text{cavity} \]

\[ \text{shell} \]

\[ \text{max radius of cavity} \]

\[ \text{sound wave} \]

At a sound wave, disturbance passes into the undisturbed gas as a mild compression followed by a rarefaction. After the sound wave passes, gas returns to the original state.

For SN, the maximum bubble/cavity size is set by the radius when blast wave becomes sonic and \( p \sim p_0 \). We've just shown that this implies

\[ U_0^2 \sim \frac{\gamma + 1}{2 \gamma} C_s^2 \]

We showed above that energy conservation gives

\[ E = \frac{4\pi}{3} R^3 \left[ \frac{1}{2} \rho_0 \left( \frac{2U_0}{\gamma + 1} \right)^2 + \frac{\alpha}{\gamma - 1} \frac{2\rho_0 U_0^2}{\gamma + 1} \right] \quad \text{with} \quad \alpha = \frac{1}{2} \]

\[ = \frac{4\pi}{3} R^3 \rho_0 U_0^2 \left[ \frac{2}{(\gamma + 1)^2} + \frac{1}{\gamma - 1} \cdot \frac{1}{\gamma + 1} \right] \]

\[ = \frac{4\pi}{3} R^3 \rho_0 U_0^2 \left[ \frac{2(\gamma - 1) + \gamma + 1}{(\gamma + 1)^2 (\gamma - 1)} \right] \]

\[ = \frac{4\pi}{3} R^3 \rho_0 U_0^2 \frac{3\gamma - 1}{(\gamma + 1)^2 (\gamma - 1)} \]

\[ \Rightarrow U_0^2 = \frac{(\gamma + 1)(\gamma^2 - 1)}{3\gamma - 1} \cdot \frac{3E}{4\pi \rho_0 R^3} \sim \frac{\gamma + 1}{2 \gamma} C_s^2 \]

when blast wave becomes sonic and \( p \sim p_0 \)

\[ \Rightarrow E \sim \frac{4\pi}{3} \rho_0 R_{\text{max}}^2 \frac{C_s^2}{2\gamma} \frac{3\gamma - 1}{\gamma^2 - 1} \]
Internal energy initially contained within $R_{\text{max}}$ is

$$E_{\text{init}} = \frac{4\pi}{3} R_{\text{max}}^3 \frac{P_0}{\gamma - 1} = \frac{4\pi}{3} R_{\text{max}}^3 P_0 \frac{c_s^2}{\gamma (\gamma - 1)}$$

So, when $P_0 \sim p$, we have $E \sim E_{\text{init}}$.

$\Rightarrow$ Blast wave propagates until the explosion energy is comparable to the internal energy in the sphere!

Some numbers:
- timescale on which the bubble reaches $R_{\text{max}}$ is roughly the sound crossing time
  $$t_s \sim R_{\text{max}} / c_s$$

For ISM: $T \sim 10^4 K$, $p \sim 10^{-21} \text{kg m}^{-3}$
  $$\Rightarrow R_{\text{max}} \sim \text{few} \times 100 \text{pc}$$
  $$t_{\text{max}} \sim 10 \text{Myr}$$

- SN rate is about $10^{-7} \text{Myr}^{-1} \text{pc}^{-3}$. So, over a duration $t_{\text{max}}$, can find 1 SN in $\sim 10^6 \text{pc}^{-3}$. But
  $$\frac{4\pi}{3} R_{\text{max}}^3 > 10^6 \text{pc}^3$$

So filling factor of SN driven bubbles is $>1$. This would seem to suggest that the entire ISM would be heated to SN to $>10^6 K$. NOT OBSERVED!

We need to account for cooling + the finite height of the Galactic disk (i.e. bubble “blowout”). After $10^5 \text{yr}$, when $R \sim 20 \text{pc}$, cooling losses become important and so the bubble grows more slowly than $R \propto t^{3/5}$. Simulations show that $R \propto t^{0.3}$ and $R_{\text{max}} \sim 50 \text{pc}$
  $\Rightarrow$ filling factor < 1. Thus, due to cooling, only a small fraction of $E$ is deposited into ISM.