NATURAL SCIENCES TRIPOS Part II

Monday 6 June 2011 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

Before you begin read these instructions carefully.

Candidates may attempt not more than 6 questions.

Each question is divided into Part (i) *and Part* (ii), *which may or may not be related. Candidates may attempt either or both Parts.*

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A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS
Script Paper
Blue Cover Sheets
Yellow Master Cover Sheets
1 Rough Work Pad
Tags

SPECIAL REQUIREMENTS
Formulae Booklet
Approved Calculators Allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1Z – Relativity

(i) A rocket propels itself rectilinearly by converting its rest mass into radiation which it emits in the direction opposite to its motion. If the initial rest mass is M_i and the final mass is M_f, show that the final speed u of the rocket relative to its initial rest frame satisfies

$$\frac{M_\mathrm{i}}{M_\mathrm{f}} = \left(\frac{1+u}{1-u}\right)^{1/2}$$

in units with $c = 1$.

(ii) A particle moves rectilinearly with constant proper acceleration α in Minkowski spacetime. Show that the speed $u(\tau)$ of the particle relative to its initial rest frame, S, after proper time τ has elapsed is

$$u(\tau) = \tanh(\alpha\tau)\,,$$

in units with $c = 1$.

Show further that, with a suitable choice of origin, the wordline of the particle in terms of coordinates in the S frame is

$$\begin{aligned} t(\tau) &= \frac{1}{\alpha}\sinh(\alpha\tau) \\ x(\tau) &= \frac{1}{\alpha}\left[\cosh(\alpha\tau) - 1\right]\,, \end{aligned}$$

and sketch the motion in the x–t plane.

Write down the components of the acceleration four-vector in the instantaneous rest frame of the particle.

By transforming to the frame S, or otherwise, show that

$$\frac{\mathrm{d}A^\mu}{\mathrm{d}\tau} = \alpha^2 U^\mu\,,$$

where A^μ and U^μ are the acceleration and velocity four-vectors respectively.

Question 2X – Astrophysical Fluid Dynamics

(i) A self-gravitating gas, which is a polytrope of index $n = 1$, is distributed in a slab that is infinitely extended in the x, y plane and is symmetrical about $z = 0$. If all properties of the gas are functions of z only, determine the density, $\rho(z)$, of the slab in the case that $\rho(0) = \rho_0$.

Show that the thickness of the slab is $2H_0$, where

$$H_0^2 = \frac{\pi K}{8G},$$

with $K = P/\rho^2$ and P the pressure.

Is the slab convectively stable?

(ii) The slab described in **(i)** above is used as a model for a galactic disc which additionally contains a thin uniform layer of stars in the plane $z = 0$. These stars modify the structure of the gas disc by providing a uniform gravitational acceleration of magnitude g_0 directed towards the mid-plane. Calculate the vertical structure of the modified gas disc, $\rho(z)$.

If the modified disc has the same column density of gas as in the case when $g_0 = 0$, show that the disc has thickness $2H_1$, where:

$$\cos\left(\sqrt{\frac{2\pi G}{K}} H_1\right) = \left[1 + \sqrt{\frac{8\pi G \rho_0^2 K}{g_0^2}}\right]^{-1}.$$

Hence, show that in the limit that $g_0^2 \ll GK\rho_0^2$

$$H_0 - H_1 = \frac{g_o}{4\pi G \rho_0},$$

where H_0 is the disc semi-thickness in the case $g_0 = 0$, as given in **(i)** above.

TURN OVER...

Question 3Z – Cosmology

(i) Consider a homogeneous and isotropic universe with scale factor $a(t)$, curvature k, mass density $\rho(t)$ and pressure $P(t)$, satisfying the Friedmann and energy-conservation equations

$$\begin{aligned} H^2 + \frac{kc^2}{a^2} &= \frac{8\pi G\rho}{3}\,, \\ \dot\rho + 3H\left(\rho + P/c^2\right) &= 0\,, \end{aligned}$$

with Hubble parameter $H \equiv \dot a/a$, where overdots denote derivatives with respect to t. Use these equations to derive the acceleration equation for the universe,

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right).$$

If matter satisfies the strong energy condition $\rho c^2 + 3P \geq 0$, show that the quantity aH is necessarily non-increasing in an expanding universe (i.e. $\mathrm{d}(aH)/\mathrm{d}t \leq 0$).

Given the density parameter $\Omega(t) \equiv 8\pi G\rho/(3H^2)$, show that it satisfies

$$\Omega - 1 = \frac{kc^2}{a^2H^2}\,.$$

Hence, or otherwise, briefly explain the flatness problem of the standard cosmology.

(ii) Consider a flat universe ($k = 0$) filled with two major fluid components, each with an equation of state $P_i = w_i\rho_i c^2$ (with w_i constant): a radiation fluid ($w_R = 1/3$) with density parameter today Ω_{R0}; and dark energy with $w_\Lambda = -1$ and density parameter $\Omega_{\Lambda 0}$. Assuming that each component independently obeys the energy conservation equation, show that the total mass density satisfies

$$\rho(t) = \frac{3H_0^2}{8\pi G}\frac{\Omega_{R0}}{a^4}\left(1 + \frac{1-\Omega_{R0}}{\Omega_{R0}}a^4\right),$$

where H_0 is the current value of the Hubble parameter and we have taken $a = 1$ at the present.

Hence solve the Friedmann equation and show that the scale factor can be expressed in the form

$$a(t) = \alpha\left(\sinh\beta t\right)^{1/2},$$

where α and β should be specified in terms of Ω_{R0} and H_0.

Demonstrate the expected asymptotic solutions at both early and late times.

[*Hint*: You may assume that $\int \mathrm{d}x/\sqrt{1+x^2} = \sinh^{-1} x\,.$]

Question 4Y – Structure and Evolution of Stars

(i) What are the four pressure contributions in a star which are most relevant to stellar structure and evolution?

Make a sketch of the $\log T$ versus $\log \rho$ plane, with the $\log T$ axis extending from the typical surface temperature of a brown dwarf to the typical central temperature of a massive star just before it becomes a type II supernova.

Use the equations of state to draw lines in this plane and divide it into four distinct regions, with each region corresponding to a particular dominant equation of state, and give brief physical explanations for the dividing lines.

Sketch and label a line which corresponds to the interior of the sun.

Briefly state what a *helium flash* is. Sketch and label a line which shows the central T and ρ of a star which experiences a helium flash. What is the typical mass of such a star?

(ii) The total luminosity of a $1M_\odot$ white dwarf is $0.03L_\odot$. If the star radiates as a black body of surface temperature $T = 27,000$ K, determine its radius and mean density.

Suppose a white dwarf has a thin non-degenerate atmosphere with negligible mass compared to M. The opacity in this atmosphere is given by Kramer's law

$$\kappa \propto \rho T^{-7/2}.$$

Show that in this atmosphere

$$\rho \propto \left(\frac{M}{L}\right)^{1/2} T^{13/4},$$

where L is the luminosity of the white dwarf (which is constant throughout the atmosphere).

By considering the interface between the atmosphere and the degenerate interior (or otherwise), show that

$$L \propto T_c^{7/2},$$

where T_c is the central temperature.

Show that the central temperature and the luminosity vary with time t according to

$$T_c(t) = T_0(1 + \alpha t)^{-2/5} \text{ and } L(t) = L_0(1 + \alpha t)^{-7/5},$$

where L_0 and T_0 are the values of L and T_c at $t = 0$ and α is a constant.

If the star has a central temperature of 3×10^7 K, how long will it take to cool by a factor of 10?

TURN OVER...

Question 5X – Statistical Physics

(i) Describe the physical relevance of the microcanonical, canonical and grand canonical ensembles.

Explain briefly the circumstances under which all ensembles are equivalent.

The Gibbs entropy for a probability distribution $p(n)$ over states is (with Boltzmann's constant $k_{\mathrm{B}} = 1$)

$$S = -\sum_n p(n) \log p(n).$$

By imposing suitable constraints on $p(n)$, show how maximising the entropy gives rise to the probability distribution for the microcanonical and canonical ensembles.

(ii) A system consists of N non-interacting particles fixed at points in a lattice. Each particle has three states with energies $\epsilon, 0, -\epsilon$. If the system is at a fixed temperature T, determine the average energy E and the heat capacity C.

Evaluate E and C in the limits $T \to 0$ and $T \to \infty$.

Compute the entropy S of the system in terms of E and N.

Describe a configuration of the system that has negative temperature.

Does the system obey the third law of thermodynamics? Explain.

TURN OVER...

Question 6X – Stellar Dynamics and the Structure of Galaxies

(i) Show directly from the Poisson Equation that for a spherical density distribution, $\rho(r)$, the gravitational acceleration is in the radial direction and is of the form

$$a_r = -\frac{GM(< r)}{r^2} ,$$

where

$$M(< r) = \int_0^r 4\pi r'^2 \rho(r') dr' .$$

The Sun is on a circular orbit a distance $R_0 = 8$ kpc from the centre of the Galaxy and with speed $V_0 = 220$ kms^{-1}. Assuming the Galactic density distribution is spherically symmetric, calculate the mass $M\,(< R_0)$ within the Solar circle in units of solar mass.

Old stars in the Galactic disc have a scaleheight of $z_0 = 300$ pc. Assuming that the Galactic density distribution is spherically symmetric, and noting that $z_0 \ll R_0$, give an estimate of their root mean square speed σ_z perpendicular to the plane.

The actual value of σ_z is found to be $\sigma_z \approx 19$ kms^{-1}. Comment on this result.

(ii) Give an expression for the escape velocity $V_{\rm esc}\,(\mathbf{r}_0)$ from a point $\mathbf{r}_0$ in a fixed gravitational potential $\Phi\,(\mathbf{r})$, making a suitable assumption about the behaviour of $\Phi\,(\mathbf{r})$ as $|\mathbf{r}| \to \infty$.

A star of mass m is in a circular orbit at radius a about a binary pair of black holes with total mass $M \gg m$, and separation $d \ll a$. Find the star's orbital velocity V and its escape velocity $V_{\rm esc}$ to zeroth order in d/a.

First, let us suppose that the binary black holes merge and lose mass instantaneously to gravitational radiation. After the merger, the mass of the black hole is fM, where $0 < f < 1$. Find the condition on f that the star remains bound to the hole.

Second, let us imagine an alternative scenario in which the fraction of mass lost is negligible ($f \ll 1$), but the black hole is given an instantaneous kick with velocity V_0. If the kick is perpendicular to the star's orbital plane, find the condition on V_0 that the star remains bound to the hole.

TURN OVER...

If, instead, the kick is in a random direction in the orbital plane, show that if

$$\frac{V_0}{V} < \sqrt{2} - 1,$$

the star will always remain bound.

If $V = V_0$, find the probability that the star remains bound.

Question 7Y – Topics in Astrophysics

(i) The eclipsing binary X-ray source SMC X-1 lies in the Small Magellanic Cloud, roughly 60 kpc distant. A telescope of aperture 0.4 m^2 in Earth orbit detects approximately 50 X-ray photons per second. The X-rays have a typical energy of 5 keV and the telescope is 80% efficient. Estimate the X-ray luminosity of SMC X-1.

Compare this with the luminosity of the Sun.

(ii) A stationary circular ring of material surrounds the supernova SN1987A. Ultraviolet radiation from the supernova ionises material in the ring producing radiation visible from Earth. The ring is elliptical as viewed from the Earth, with major axis diameter $a = 1.66$ arcsec and minor axis diameter $b = 1.21$ arcsec. Radiation from material in the ring is first detected at time t_0 after the supernova explosion and the ring reaches maximum luminosity at later time $t_{\rm max}$. If $t_0 = 95$ days, and $t_{\rm max} = 415$ days, what is the radius of the ring?

Use your answer to estimate the distance to SN1987A.

If the illuminated ring is not stationary, but rather expanding uniformly, will this affect your calculations?

How might spectroscopic observations lead to measuring the expansion velocity of the ring?

Sketch the spectra you would observe at three times: t_0, $t_{\rm max}$, and $t_0 + 2(t_{\rm max} - t_0)$.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Wednesday 8 June 2011 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

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Question 1Z – Relativity

(i) What is meant by an affine parameter λ for a geodesic $x^\mu(\lambda)$?

By reparameterising $\Lambda = \Lambda(\lambda)$ in the equation for an affinely-parameterised geodesic, show that the geodesic equation for a general parameter Λ is of the form

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\Lambda^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\Lambda} \frac{\mathrm{d}x^\rho}{\mathrm{d}\Lambda} = f(\Lambda) \frac{\mathrm{d}x^\mu}{\mathrm{d}\Lambda} \,,$$

for some function $f(\Lambda)$.

A conformal transformation of a spacetime metric is given by

$$g_{\mu\nu}(x^\rho) \to \tilde{g}_{\mu\nu}(x^\rho) = \Omega^2(x^\rho) g_{\mu\nu}(x^\rho) \,.$$

Show that the connection coefficients, $\tilde{\Gamma}^\mu_{\nu\rho}$, for the new metric are related to those for the original metric by

$$\tilde{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + \delta^\mu_\nu \frac{\partial \ln \Omega}{\partial x^\rho} + \delta^\mu_\rho \frac{\partial \ln \Omega}{\partial x^\nu} - g_{\nu\rho} g^{\mu\tau} \frac{\partial \ln \Omega}{\partial x^\tau} \,.$$

(ii) Using the results in part **(i)**, show that if $x^\mu(\lambda)$ is an affinely-parameter -ised *null* geodesic in the metric $g_{\mu\nu}$, then it is also a null geodesic of the conformal metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ but λ is, generally, no longer affine.

Show further that an affine parameter $\tilde{\lambda}$ in the metric $\tilde{g}_{\mu\nu}$ can be obtained by solving

$$\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\lambda} = \Omega^2$$

along the geodesic.

Determine the null geodesics of the two-dimensional space with line element

$$\mathrm{d}s^2 = \frac{1}{t^2}\left(\mathrm{d}t^2 - \mathrm{d}x^2\right) \,.$$

Question 2X – Astrophysical Fluid Dynamics

(i) A stationary, infinitesimally thin, contact discontinuity separates regions I and II of a hydrodynamical flow. The temperature is maintained at fixed values in each region so that the sound speeds are respectively $c_{\rm I}$ and $c_{\rm II}$. Gas flows from region I to region II, with densities $\rho_{\rm I}$ and $\rho_{\rm II}$ on each side of the contact discontinuity. If the gas velocities at the contact discontinuity are respectively $u_{\rm I}$ and $u_{\rm II}$ and are normal to the contact discontinuity in both regions, write down two expressions relating quantities on either side of the boundary.

Show that

$$\mathcal{M}_{\rm I} + \frac{1}{\mathcal{M}_{\rm I}} = \frac{c_{\rm II}}{c_{\rm I}}\left(\mathcal{M}_{\rm II} + \frac{1}{\mathcal{M}_{\rm II}}\right),$$

where $\mathcal{M}_{\rm I}$ and $\mathcal{M}_{\rm II}$ are the isothermal Mach numbers of the flow in each region.

(ii) A steady spherically symmetric isothermal wind with sound speed c_s is generated by a star of mass M. Assuming without proof that the Bernoulli function is constant in the flow, show that the Mach number in the wind $\mathcal{M}$ satisfies

$$\frac{1}{2}\mathcal{M}^2 + 2\ln\left(\frac{r_s}{r}\right) - \ln\mathcal{M} - 2\left(\frac{r_s}{r}\right) = -\frac{3}{2},$$

where $r_s = GM/(2c_s^2)$.

Explain why at $r \ll r_s$ it is possible to neglect one of the terms in the above equation and hence show that at small radii

$$\mathcal{M} \sim e^{3/2-2(r_s/r)}\left(\frac{r_s}{r}\right)^2.$$

Hence, determine the Mach number of the flow at a radius of 1.2×10^{12}m for a one solar mass star and a wind temperature of 2000 K.

TURN OVER...

Question 3Y – Structure and Evolution of Stars

(i) A small element of material within a star is initially in equilibrium with its surroundings. It is then perturbed vertically upwards through an infinitesimal distance. Assuming that the element changes abiabatically show that the condition for convection to occur is

$$\frac{\gamma - 1}{\gamma} < \frac{p}{T}\frac{dT}{dp},$$

where p and T are the pressure and temperature of the element's initial surroundings and γ is the ratio of the specific heats.

(ii) In a stellar atmosphere, the temperature T satisfies

$$T^4 = \frac{1}{4}(3\tau + 2)T_{\rm e}^4,$$

where $T_{\rm e}$ is the effective temperature of the star and the optical depth τ at radius r is defined by

$$\tau = \int_r^\infty \kappa(r')\rho(r')dr',$$

where $\rho(r')$ is the density and $\kappa(r')$ is the opacity. The atmosphere of a cool main sequence star has opacity

$$\kappa = \kappa_0 p^{1/2} T^8,$$

where κ_0 is a constant and p is the pressure. Show that for a star of mass M and radius R

$$p^{3/2} = \frac{8GM}{R^2 \kappa_0 T_{\rm e}^8}\left(\frac{1}{2} - \frac{1}{3\tau + 2}\right).$$

Hence, or otherwise, show that the outermost regions of the star (i.e. where $\tau \sim 0$) are stable against convection.

TURN OVER...

Question 4X – Statistical Physics

(i) An isothermal gas at temperature T is composed of molecules of mass m, which lie in a cylinder of radius R_0 and length L_0. The cylinder is rotating about its axis with an angular velocity Ω. Write down the distribution of energies of the gas molecules.

What is the distribution of speeds?

Show that the number density distribution within the cylinder as a function of cylindrical radius R is

$$n(R) = \frac{Nm\Omega^2}{2\pi T L_0} \frac{\exp\left(\frac{m\Omega^2 R^2}{2T}\right)}{\exp\left(\frac{m\Omega^2 R_0^2}{2T}\right) - 1} ,$$

where N is the total number of molecules in the cylinder and Boltzmann's constant is set equal to unity.

(ii) Write down the partition function for a single classical particle of mass m moving in three dimensions in a spherical potential $U(r)$ in equilibrium with a constant temperature heat bath at temperature T.

A system of N non-interacting classical particles in equilibrium at temperature T move in the potential

$$U(r) = \frac{r^{2n}}{V^{2n/3}} ,$$

where n is a positive integer and V is a constant. Using the partition function, show that the free energy is

$$F = -NT\left(\log V + \frac{3(n+1)}{2n}\log T + \log I_n - \frac{1}{N}\log N!\right) ,$$

where

$$I_n = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \int_0^\infty 4\pi r'^2 \exp(-r'^{2n}) dr' ,$$

and Boltzmann's constant is unity. Viewing V as an external parameter, akin to volume, compute the conjugate pressure p and show that the equation of state coincides with that of an ideal gas.

Compute the energy E and the heat capacity at constant volume C_{V} of the gas.

What is the particle number density law $n(r)$?

Question 5X – Stellar Dynamics and the Structure of Galaxies

(i) A star is orbiting in the potential

$$\Phi(r) = -GM\left(\frac{1}{r} + \frac{a}{r^2}\right) , \qquad (*)$$

where $a > 0$ is a constant. Show that the orbit lies in a plane.

Let (r, ϕ) be polar coordinates in the orbital plane. Show further that the equation of the orbit is

$$\frac{1}{r} = C\cos\left(\frac{\phi - \phi_0}{K}\right) + \frac{GMK^2}{h^2},$$

with

$$\frac{1}{K^2} = 1 - \frac{2GMa}{h^2},$$

where h is the specific angular momentum, and ϕ_0 and C are constants.

(ii) A star is in a circular orbit, with angular velocity Ω, in a spherically symmetric gravitational potential $\Phi(r)$. Write down an expression for $\Omega(r)$ in terms of $\Phi(r)$.

The star is now perturbed slightly in its orbital plane. Show that the star undergoes radial oscillations with period $T_r = 2\pi/\kappa$, where

$$\kappa^2 = r\frac{d\Omega^2}{dr} + 4\Omega^2 .$$

You may assume without proof that $\kappa^2 > 0$.

Suppose the star is moving in the potential given by $(*)$ of part **(i)**. Find $\Omega^2(r)$ and $\kappa^2(r)$.

Show that in a frame rotating with angular velocity

$$\Omega_\text{p} = \Omega - \frac{n\kappa}{m} ,$$

where $m(> 0)$ and n are integers, nearly-circular orbits are closed.

Sketch the orbits for $n = 1$, $m = 2$ in the rotating frame.

TURN OVER...

Question 6Y – Topics in Astrophysics

(i) The haloes of galaxies near the sight lines to background quasars are thought to give rise to quasar absorption systems. Hubble Space Telescope observations of absorption systems with redshifts $0 < z < 0.1$ indicate there are, on average, three systems which show neutral hydrogen absorption per quasar. If all these systems are associated with galaxies, and the number density of galaxies is 1.3×10^{-2} Mpc^{-3}, show that the size of galaxies responsible for the absorption are ~ 500 kpc.

(ii) One possible model for the emission line region in quasars involves a steady, spherically symmetric outflow of low density ionized material. Show that in this model the gas density, ρ, and velocity, v, at a distance r from the central source satisfy

$$\rho v r^2 = \text{constant}.$$

The fraction of gas that is ionized depends on the ionisation parameter, U, which is proportional to the number of ionising photons per atom. Suppose the broad-line region consists of emission line clouds, pressure-confined and carried along by the low-density outflowing medium. Assuming the clouds and the outflowing medium are maintained at constant, but different, temperatures throughout the broad-line region, show that U is proportional to v.

Hence, in an accelerating flow, show that the emission line clouds become more highly ionised as they get further from the central continuum source.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Thursday 9 June 2011 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 4

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Question 1Z – Relativity

(i) Derive the equations of motion for a massive point-particle moving in the equatorial plane ($\theta = \pi/2$) of the Schwarzschild metric (with $c = 1$) as

$$\begin{aligned} \dot{t}\,(1 - 2\mu/r) &= k\,, \\ r^2\dot{\phi} &= h\,, \\ \dot{r}^2 + \frac{h^2}{r^2} - \frac{2\mu h^2}{r^3} - \frac{2\mu}{r} &= k^2 - 1\,, \end{aligned}$$

where $\mu \equiv GM$ and overdots denote derivatives with respect to proper time τ.

Interpret the constants k and h physically.

Briefly, contrast the properties of circular orbits about a central mass in general relativity with those in Newtonian theory.

(ii) A static observer in the Schwarzschild spacetime at radius r_0 ($> 2\mu$) launches a point-particle in the outwards radial direction initially with speed $v_{\rm esc}$ so that the particle just reaches infinity. Determine the escape speed $v_{\rm esc}$ and comment on your result with reference to the Newtonian analogue.

If the particle is instead launched tangentially with (proper) angular velocity $\Omega = \dot{\phi}$, show that the critical value for the particle just to reach infinity satisfies

$$\Omega^2 = \frac{2\mu}{r_0^3(1 - 2\mu/r_0)}\,.$$

Calculate the initial speed, relative to the static observer at radius r_0, of a particle with this critical value of Ω and compare your result to Newtonian theory.

Question 2X – Astrophysical Fluid Dynamics

(i) Optically thin hydrogen gas of density $\rho = 10^{-23}$ kg m^{-3} is heated by cosmic rays at a constant rate of 3.8×10^{-34} W per particle and cooled by optically thin, thermal bremsstrahlung at a rate per particle Γ where

$$\Gamma = 1.4 \times 10^{-13} \left(\frac{\rho}{\mathrm{kgm^{-3}}}\right) \left(\frac{T}{\mathrm{K}}\right)^{1/2} \mathrm{W}\,.$$

Determine the temperature T of the gas in thermal equilibrium and show that this equilibrium is unstable.

Describe how such a system is expected to evolve and estimate the timescale for its evolution.

(ii) Explain what is meant by a *dispersion relation* and briefly outline the steps involved in deriving the dispersion relation for a perturbed hydrodynamical system.

Describe the information that can be obtained from a dispersion relation, illustrating your answer with reference to the following example

$$\omega^2 = c_\mathrm{s}^2 k^2 - 2\pi G\Sigma|k| + \Omega^2, \qquad (*)$$

which applies to the axisymmetric perturbation of a Keplerian disc with angular velocity Ω, sound speed c_s and surface density Σ. Your answer should explain the meaning of ω and k, as well as distinguishing between the nature of the solution in the cases where $\omega^2 < 0$ and $\omega^2 > 0$.

Show that $\omega^2 > 0$ for all k if and only if $Q_\mathrm{T} > 1$ where

$$Q_\mathrm{T} = \frac{c_\mathrm{s}\Omega}{\pi G\Sigma}.$$

Find the wavenumber of the fastest growing mode in the case that $Q_\mathrm{T} < 1$ and show that in this case the mode grows at a rate $\Omega(Q_\mathrm{T}^{-2} - 1)^{1/2}$.

Provide a physical interpretation of each term in the dispersion relation $(*)$ and state which of these effects provide a restoring force and which reinforce the growth of perturbations.

TURN OVER...

Question 3Z – Cosmology

(i) In the Zel'dovich approximation, the perturbed motion of cold dark matter particles (pressure-free matter) in an expanding universe is parameterised by the trajectory $\mathbf{r}(\mathbf{q},t) = a(t)[\mathbf{q} + \boldsymbol{\psi}(\mathbf{q},t)]$, where $a(t)$ is the scale factor, $\mathbf{q}$ is the unperturbed comoving position and $\boldsymbol{\psi}(\mathbf{q},t)$ is the comoving displacement. The particle equation of motion is $\ddot{\mathbf{r}} = -\boldsymbol{\nabla}\Phi$, where Φ is the Newtonian potential satisfying the Poisson equation $\boldsymbol{\nabla}^2\Phi = 4\pi G\rho$ with mass density ρ. Here, $\boldsymbol{\nabla}$ is the gradient with respect to $\mathbf{r}$ and overdots denote derivatives with respect to t. Argue that the conservation of cold dark matter particles in a volume $\mathrm{d}^3\mathbf{r}$ implies that $\rho(\mathbf{r},t)\mathrm{d}^3\mathbf{r} = a^3(t)\bar{\rho}(t)\mathrm{d}^3\mathbf{q}$, where $\bar{\rho}(t)$ is the homogeneous background density.

Hence show that the fractional density perturbation δ satisfies

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} = -\boldsymbol{\nabla}_{\mathbf{q}} \cdot \boldsymbol{\psi}\,. \qquad (*)$$

Here, $\boldsymbol{\nabla}_{\mathbf{q}}$ is the gradient with respect to $\mathbf{q}$ and you may assume that the Jacobian $|\partial r^i/\partial q^j|^{-1} = |a\delta^i{}_j + a\partial\psi^i/\partial q^j|^{-1} \approx a^{-3}(1 - \boldsymbol{\nabla}_{\mathbf{q}} \cdot \boldsymbol{\psi})$ if the magnitude of all components of $\partial\psi^i/\partial q^j$ are $\ll 1$.

Use the result $(*)$ to integrate the Poisson equation to find $\boldsymbol{\nabla}\Phi$ and, using the particle equation of motion, show that

$$\ddot{\boldsymbol{\psi}} + 2\frac{\dot{a}}{a}\dot{\boldsymbol{\psi}} - 4\pi G\bar{\rho}\boldsymbol{\psi} = 0\,.$$

Show further that the evolution equation for density perturbations in a cold dark matter model is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\bar{\rho}\delta = 0\,. \qquad (\dagger)$$

[*Hint*: you may assume that the integral of $\boldsymbol{\nabla}^2\Phi = 4\pi G\bar{\rho}$ is $\boldsymbol{\nabla}\Phi = 4\pi G\bar{\rho}\mathbf{r}/3$ and the pressure-free acceleration equation $\ddot{a}/a = -4\pi G\bar{\rho}/3$.]

(ii) Consider a flat, matter-dominated universe with scale factor $a(t) = (t/t_0)^{2/3}$ filled with a pressure-free homogeneous component (H) of mass density $\rho_{\mathrm{H}}(t)$ which cannot clump, as well as cold dark matter (C) with density $\rho_{\mathrm{C}}(\mathbf{r},t)$. Argue that the perturbation equation $(\dagger)$ from part **(i)** still holds, but that $\bar{\rho}$ should not include the homogeneous component (i.e. it is replaced by $\bar{\rho}_{\mathrm{C}}$).

Seek power-law solutions of the form $\delta \propto t^\alpha$ to find modes with

$$\alpha = \frac{1}{6}\left(-1 \pm \sqrt{25 - 24\Omega_{\rm H}}\right) ,$$

where $\Omega_{\rm H}$ is the (constant) relative density $\Omega_{\rm H} \equiv \rho_{\rm H}/\bar{\rho}$.

With matter domination ($t = t_{\rm eq}$) beginning at a redshift $1 + z_{\rm eq} \approx 10^5$, and an initial perturbation $\delta(t_{\rm eq}) \approx 10^{-5}$, explain briefly why $\Omega_{\rm H} = 2/3$ could not be compatible with observed large-scale structure today.

If the H component is composed of massive particles with number density today comparable to that of cosmic microwave background (CMB) photons, estimate what particle mass would correspond to $\Omega_{\rm H} = 2/3$. (Take $\bar{\rho} = 9.2 \times 10^{-27}\,{\rm kg\,m^{-3}}$ and the temperature of the CMB to be $T = 2.725\,{\rm K}$ today.)

[*Hint*: use the Planck radiation law and note that $\int_0^\infty x^2(e^x - 1)^{-1}\,dx = 2\zeta(3) \approx 2.404$.]

TURN OVER...

Question 4Y – Structure and Evolution of Stars

(i) A star in hydrostatic equilibrium has a total radius R and a central density ρ_c. The density varies as

$$\rho(r) = \rho_c \left(1 - \frac{r^2}{R^2}\right) .$$

Derive expressions for the total mass M and the average stellar density.

Write down an explicit expression for the total gravitational potential energy of the star.

Find the total kinetic energy of the star and verify that the virial theorem is exactly satisfied.

(ii) On the assumption that a star is in hydrostatic equilibrium, show that the minimum central pressure p_{c} is given by

$$p_{\mathrm{c}} > \frac{GM^2}{8\pi R^4},$$

for a star of mass M and radius R.

Find the central pressure if the radial density distribution is (a) uniform, and (b) has the form given in Part **(i)**.

Assuming the equation of state for an ideal gas, what is the central temperature T_{c} in these two cases (a) and (b)?

Find the ratio of the radiation pressure to the gas pressure at the center of these two stars, cases (a) and (b), as a function of the total stellar mass (expressed in units of $M_{\odot}$).

At what mass does the radiation pressure become comparable to the ideal gas pressure?

Question 5X – Statistical Physics

(i) The pressure P and volume V of a gas are related by the equation of state

$$PV^n = a,$$

where a and n are constants. Find the work done on the gas on compression from volume V_1 to V_2.

Show further that the quantity of heat Q gained is (with Boltzmann's constant $k_{\mathrm{B}} = 1$)

$$Q = a\left(C_V + \frac{1}{1-n}\right)\left(V_2^{1-n} - V_1^{1-n}\right),$$

where C_V is the heat capacity per particle at constant volume.

(ii) A gas of non-interacting particles has an energy-momentum relationship

$$E = Ap^{\alpha},$$

where A and α are constants. Determine the density of states $g(E)$ in a three dimensional volume V.

If the particles are bosons at fixed temperature T and chemical potential μ, write down an expression for the number of particles that are not in the ground state.

For which values of α does there exist a Bose-Einstein condensate at low temperatures?

Can a gas of photons undergo Bose-Einstein condensation? Explain your answer.

TURN OVER...

Question 6X – Stellar Dynamics and the Structure of Galaxies

(i) Show that the gravitational potential $\Phi(z)$ of an infinitesimally thin layer with constant surface density Σ_0 lying in the $z = 0$ plane is

$$\Phi(z) = 2\pi G \Sigma_0 |z| \, .$$

Consider a slab of uniform density ρ_0 and thickness a, so that

$$\rho(z) = \begin{cases} 0, & |z| > a \, , \\ \rho_0, & |z| < a \, . \end{cases}$$

Hence, show that in the region $|z| < a$, the gravitational potential of the slab is given by

$$\Phi(z) = 2\pi G \rho_0 \left(z^2 + a^2 \right) \, .$$

(ii) A gravitating layer in the region $-a \leq z \leq a$ has uniform density ρ_0 and is made up of stars with distribution function $f(z, v)$, where v is the velocity in the z-direction.

Expain why we may look for a distribution function of the form

$$f(z, v) = f(\xi) \, ,$$

where

$$\xi = \frac{1}{2}\omega^2 \left(a^2 - z^2 \right) - \frac{1}{2} v^2 \, ,$$

and $\omega^2 = 4\pi G \rho_0$.

Deduce that $f(\xi)$ satisfies

$$\sqrt{2} \int_0^{\frac{1}{2}\omega^2 \left(a^2 - z^2 \right)} \frac{f(\xi) d\xi}{\sqrt{\frac{1}{2}\omega^2 \left(a^2 - z^2 \right) - \xi}} = \rho_0 \, .$$

Hence, find $f(z, v)$ and sketch the velocity distribution $f(z = 0, v)$ on the plane $z = 0$.

Hint: You may assume that if

$$F(\eta) = \int_0^\eta \frac{E(\xi) d\xi}{\sqrt{\eta - \xi}}, \text{ then } E(\xi) = \frac{1}{\pi} \frac{d}{d\xi} \int_0^\xi \frac{F(\eta) d\eta}{\sqrt{\xi - \eta}} \, .$$

Question 7Y – Topics in Astrophysics

(i) The Earth is hit by an asteroid falling from rest at infinity. If the orbital velocity of the Earth relative to the Sun is 30 kms^{-1}, what is the velocity of the asteroid on impact?

If the asteroid has diameter ~ 1 km, estimate the kinetic energy of the asteroid relative to that of the Earth.

The energy liberated by the Hiroshima atomic bomb was $\sim 8 \times 10^{13}$ J. How does this compare to the energy released by the asteroid impact?

(ii) There are about 1000 asteroids with diameters > 1 km and with orbits that cross the orbit of the Earth. Such *Near-Earth Asteroids* have inclinations in the range $-25°$ to $25°$ to the ecliptic. The perihelia are greater than 0.8 astronomical units and their aphelia are less than 1.2 astronomical units. Estimate how often the Earth is hit by such asteroids.

Assuming that the reservoir of Near-Earth asteroids is not replenished, estimate how many asteroids will have hit the Earth since it was formed 4.65×10^9 years ago.

If all the asteroids were 1 km diameter and had the same density as the Earth, what fraction of the Earth's mass is from such asteroidal material?

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Friday 10 June 2011 09:00am – 12:00pm

ASTROPHYSICS - PAPER 5

Before you begin read these instructions carefully.

Candidates may attempt not more than 6 questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1Z and 3Z** *should be in one bundle and* **2X, 5X and 6X** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1Z – Relativity

(i) The interior of a static, spherically-symmetric relativistic star has line element (in units with $c = 1$)

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) .$$

Compute the connection coefficients Γ^i_{00} where $i = r, \theta, \phi$.

The star is composed of a perfect fluid with energy density $\rho(r)$ and pressure $p(r)$ in hydrostatic equilibrium (i.e. at rest with respect to the r, θ and ϕ coordinates). The stress-energy tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} ,$$

where u^μ is the four-velocity of the fluid and $g^{\mu\nu}$ the metric tensor. Using the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$, show that

$$\frac{A'}{A} = -2\frac{p'}{\rho + p} , \qquad (*)$$

where primes denote derivatives with respect to r.

(ii) The non-trivial Einstein equations for the setup in part **(i)** are

$$\begin{aligned} R_{00} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = -4\pi GA(\rho + 3p) , \\ R_{rr} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = -4\pi GB(\rho - p) , \\ R_{\theta\theta} &= -1 + \frac{1}{B} + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = -4\pi Gr^2(\rho - p) . \end{aligned}$$

The $R_{\phi\phi}$ equation is equivalent to that for $R_{\theta\theta}$. By forming $R_{00}/A + R_{rr}/B + 2R_{\theta\theta}/r^2$, or otherwise, find a first-order differential equation satisfied by B and show that it is solved by

$$B(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1} \qquad \text{where} \qquad M(r) = 4\pi\int_0^r r'^2\rho(r')\, dr' .$$

Interpret $M(r)$ physically.

Using $(*)$ from part **(i)** and the solution for B in $R_{\theta\theta}$, show that the relativistic equation of hydrostatic support is

$$\frac{dp}{dr} = -\frac{GM(r)\rho}{r^2}\left(1 + \frac{p}{\rho}\right)\left(1 - \frac{2GM(r)}{r}\right)^{-1}\left(1 + \frac{4\pi r^3 p}{M(r)}\right) .$$

Show that this reduces to the correct Newtonian equation in the non-relativistic limit.

Question 2X – Astrophysical Fluid Dynamics

(i) A linear shear flow with velocity field $\mathbf{v} = (0, v_y(x), 0)$ consists of gas particles of mass m, charge q, radius a and thermal velocity dispersion σ. Estimate the kinematic viscosity, ν, of the gas for the case that $m = 10^{-27}$ kg, $q = 1.6 \times 10^{-19}$C, $a = 10^{-10}$m, $\sigma = 10^4$ m s $^{-1}$ and density $\rho = 10^{-4}$ kg m^{-3}.

A uniform magnetic field of magnitude $B = 10^{-4}$ Tesla is applied in the y direction. Calculate the radius of curvature of helical motion (the gyroradius) and discuss the viscous properties of the flow.

(ii) The rate of change of kinetic energy of an incompressible viscous flow in the absence of gravitational or magnetic fields is given by:

$$\frac{\partial}{\partial t}\int \frac{1}{2}\rho u^2 dV = -\int \left[\rho u_i \left(\frac{1}{2}u^2 + \frac{P}{\rho}\right) - u_j \sigma'_{ij}\right] dS_i - \int \sigma'_{ij}\frac{\partial u_i}{\partial x_j} dV, \qquad (*)$$

where σ'_{ij} is the viscous stress tensor. Give a brief physical explanation of each of the terms on the right hand side of this equation.

Which term on the right-hand side of $(*)$ does not necessarily vanish if the domain of integration is extended so as to enclose the entire fluid within a free boundary surface across which there is no flow?

The viscous diffusion equation for the evolution of the surface density Σ of an accretion disc around a star with mass M and radius R_* is given by

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2})\right].$$

Show that the following solution corresponds to a steady state disc with accretion rate of $\dot{M}$:

$$\nu\Sigma = \left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]\frac{\dot{M}}{3\pi}.$$

What boundary condition has been applied at $R = R_*$ and how is this justified?

Calculate the rate of change of mechanical (that is, kinetic plus potential) energy of gas flowing across an annulus in the disc between radii $R + dR$ and R.

TURN OVER...

Suppose that the rate of viscous dissipation per unit area in such a disc is given by

$$D(R) = \frac{9\nu\Sigma GM}{4R^3}.$$

Consider the case that the annulus is expanded to contain the whole disc from R_* to ∞. Compare the rate of change of mechanical energy with the total rate of energy dissipated.

Comment on these results.

Question 3Z – Cosmology

(i) The equilibrium number density of fermions of mass m at temperature T is given by (using units with Boltzmann's constant $k_{\rm B} = 1$)

$$n = \frac{4\pi g_{\rm s}}{h^3} \int_0^\infty \frac{p^2\,{\rm d}p}{\exp[(E(p)-\mu)/T]+1}\,,$$

where $g_{\rm s}$ is the degeneracy, μ is the chemical potential and $E(p) = c\sqrt{p^2 + m^2c^2}$ is the total energy. For a non-relativistic gas with typical $pc \ll mc^2$ and $T \ll mc^2 - \mu$, show that the number density becomes

$$n = g_{\rm s}\left(\frac{2\pi mT}{h^2}\right)^{3/2} \exp[(\mu - mc^2)/T]\,. \qquad (\dagger)$$

Before recombination, chemical equilibrium is maintained between neutral hydrogen, free electrons, protons and photons through the interaction

$$p + e^- \leftrightarrow H + \gamma\,.$$

Using the non-relativistic number density ($\dagger$), deduce Saha's equation for the ratio between the electron and hydrogen number densities,

$$\frac{n_{\rm e}^2}{n_{\rm H}} = \left(\frac{2\pi m_{\rm e}T}{h^2}\right)^{3/2} \exp(-I/T)\,,$$

where $I = (m_{\rm p} + m_{\rm e} - m_{\rm H})c^2 \approx 13.6\,{\rm eV}$ is the ionization energy of hydrogen, stating clearly any assumptions you make.

[*Hint*: You may assume that $\int_0^\infty x^2 e^{-x^2/\alpha}\,{\rm d}x = \sqrt{\pi\alpha^3}/4$.]

(ii) From Saha's equation in part **(i)**, derive an expression for the ionization fraction $X_{\rm e} \equiv n_{\rm e}/(n_{\rm p} + n_{\rm H})$ at temperature T in terms of the total number of hydrogen nuclei $(n_{\rm p} + n_{\rm H})$.

If the current values of the baryon density parameter and Hubble parameter are $\Omega_{\rm b} = 0.05$ and $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$ respectively, compute the current number density of hydrogen nuclei ignoring the effects of helium and other light elements.

Estimate $X_{\rm e}$ at: (a) $T = I$; and (b) at redshift $z = 1000$ (taking the current temperature to be $2.725\,{\rm K}$) assuming that equilibrium holds in both cases. Comment on your results.

TURN OVER...

Question 4Y – Structure and Evolution of Stars

(i) Main-sequence stars with masses in the range $0.3M_\odot < M < 3M_\odot$ are modelled as a homologous series with luminosities $L \propto M^4$ and radii $R \propto M$. Determine their effective surface temperatures and their main-sequence lifetimes relative to that of the Sun.

Why are there few, if any, red giant stars with masses of $0.3M_\odot$ observed in the Galaxy?

(ii) A homologous series of stars of the same chemical composition has energy generation rate $\epsilon = \epsilon_0 \rho T^{17}$ and opacity $\kappa = \kappa_0 \rho T^{-3.5}$ where ρ is the density and T the temperature. The energy transport within the stars is purely radiative. The radius r, density, luminosity L, temperature and pressure p through a star of mass M obey relations of the form

$$
\begin{aligned}
r &= M^a r^*(m),\\
\rho &= M^b \rho^*(m),\\
L &= M^c L^*(m),\\
T &= M^d T^*(m),\\
p &= M^e p^*(m),
\end{aligned}
$$

where m is the ratio of the enclosed mass to the total mass. Prove the following relations

$$
\begin{aligned}
4a + e &= 2,\\
3a + b &= 1,\\
c - b - 17d &= 1,\\
8a - 2b - 2c + 15d &= 2,\\
b + d - e &= 0.
\end{aligned}
$$

Hence, find a, b, c, d and e.

Show that the total luminosity $\propto T_{\rm e}^{268/49}$, where $T_{\rm e}$ is the effective temperature.

Question 5X – Statistical Physics

(i) Explain why the heat capacity at constant volume C_V of a gas is given by

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V ,$$

where S is the entropy, T the temperature and V the volume.

By transforming to new variables P and T, show that

$$C_V = C_P - T\frac{\left(\frac{\partial S}{\partial P}\right)_T\left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial V}{\partial P}\right)_T} ,$$

where C_P is the heat capacity at constant pressure.

Show further that

$$C_P - C_V = -T\frac{\left[\left(\frac{\partial V}{\partial T}\right)_P\right]^2}{\left(\frac{\partial V}{\partial P}\right)_T}$$

and hence argue that C_P is always greater than C_V.

(ii) The van der Waals' equation of state is (with Boltzmann's constant $k_{\mathrm{B}} = 1$)

$$\left(P + \frac{aN^2}{V^2}\right)(V - Nb) = NT .$$

Explain the significance of the terms involving the constants a and b.

Sketch the isothermals of the van der Waals' equation.

Show that the pressure, temperature and volume at the critical point are

$$T_{\mathrm{c}} = \frac{8a}{27b}, \qquad V_{\mathrm{c}} = 3bN, \qquad P_{\mathrm{c}} = \frac{a}{27b^2} .$$

Describe the Maxwell construction to determine the condition for phase equilibrium.

Show, using the results of part **(i)** or otherwise, that the difference in heat capacities for a van der Waals' gas is

$$C_P - C_V = \frac{N}{1 - 2Na(V - Nb)^2/(TV^3)} .$$

Question 6X – Stellar Dynamics and the Structure of Galaxies

(i) Describe the morphology of galaxies in terms of the Hubble "tuning fork" diagram.

What is the Hubble type of the Milky Way Galaxy ?

Draw a schematic side view of the Milky Way Galaxy, including the thin disc, the neutral hydrogen gas disc, the bulge, the halo and the distribution of globular clusters.

Mark on your diagram the approximate lengthscales of the Galactic bulge, disc and halo.

(ii) A galactic disc has a surface density $\Sigma(R,\phi)$ in polar coordinates (R,ϕ). Show that the potential $\Phi(R,\phi)$ at any point in the plane of the disc is

$$\Phi(R,\phi) = -G\int_0^\infty dR'R'\int_0^{2\pi} d\phi' \frac{\Sigma(R',\phi')}{\sqrt{R'^2+R^2-2RR'\cos(\phi-\phi')}}\,.$$

Define a new radial coordinate $u = \ln R$, and introduce a reduced potential $V(u,\phi)$ and reduced surface density $S(u,\phi)$ by

$$V(u,\phi) = R^{\frac{1}{2}}\Phi, \qquad\qquad S(u,\phi) = R^{\frac{3}{2}}\Sigma\,.$$

Show that

$$V(u,\phi) = -G\int_{-\infty}^{\infty} du'\int_0^{2\pi} d\phi' K(u-u',\phi-\phi')\,S(u',\phi')\,,$$

where

$$K(u-u',\phi-\phi') = \frac{1}{\sqrt{2}\sqrt{\cosh(u-u')-\cos(\phi-\phi')}}\,.$$

What surface density configuration is represented by the real part of

$$S(u,\phi) = e^{i(\alpha u+m\phi)}\,,$$

where α is real and m is an integer?

Deduce that the corresponding potential is given by

$$V = -GN(\alpha,m)e^{i(\alpha u+m\phi)}\,,$$

where $N(\alpha,m)$ is an integral constant depending only on α and m.

Show that $N(-\alpha,-m) = N(\alpha,m)$ and deduce that N is real.

What does this imply about the azimuthal offset between density maxima and potential minima at a given radius R?

Question 7Y – Topics

(i) A classical nova is a binary star system consisting of a white dwarf (mass $M_1 = 1M_\odot$, radius $R_1 = 5 \times 10^6$m, temperature $T_1 = 50,000$K) and a main-sequence star (mass $M_2 = 0.2M_\odot$, radius $R_2 = 1.4 \times 10^8$m, temperature $T_2 = 3000$K). The binary period is 110 min and the orbit is circular. Find the distance between the two stars, the orbital velocity of each star, and the luminosity of each star.

The widths of spectral lines in the white dwarf are $\sim$ 10,000 km s^{-1} and in a main sequence star they are $\sim$ 10 km s^{-1}. Describe what problems might arise in trying to detect the orbital velocities of the stars.

(ii) A white dwarf of mass M and radius R is accreting material from an accretion disc which deposits material at the white dwarf surface with angular velocity

$$\Omega_{\rm K} = \left(\frac{GM}{R^3}\right)^{1/2} .$$

Explain why the angular velocity Ω of the white dwarf cannot exceed $\Omega_{\rm K}$.

If the mass of the white dwarf was M_0 when accretion started and the white dwarf was not rotating initially, show that

$$\frac{\Omega}{\Omega_{\rm K}} = \frac{3}{5k^2}\left[1 - \left(\frac{M_0}{M}\right)^{5/3}\right] .$$

You may assume that the moment of inertia of the white dwarf is $I = k^2MR^2$, where k is the radius of gyration, and the mass-radius relation for the white dwarf is $R \propto M^{1/3}$.

Estimate the maximum value of $\Omega/\Omega_{\rm K}$ for a white dwarf.

END OF PAPER