NATURAL SCIENCES TRIPOS Part II

Monday 31 May 2010 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

Before you begin read these instructions carefully.

Candidates may attempt all 6 questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **5Z and 6Z** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X - Relativity

(i) A photon travels at an angle θ to the x-axis in an inertial frame S. Another inertial frame S' is moving relative to S in the x-direction with velocity v, and the x' axis of S' is parallel to the x-axis of S. When observed in S', the photon has energy E' and is travelling at angle θ' relative to the x' axis. Setting $c = 1$, show that

$$E' = \frac{E(1 - v\cos\theta)}{\sqrt{1 - v^2}},$$

and that

$$\cos\theta' = \frac{\cos\theta - v}{1 - v\cos\theta}.$$

(ii) Particles A, B and C have masses M_A, M_B and M_C respectively. Suppose A decays according to

$$A \rightarrow B + C.$$

In the laboratory frame in which A is at rest, show that particle B has energy

$$E_B = \frac{M_A^2 + M_B^2 - M_C^2}{2M_A},$$

using units in which $c = 1$.

If A decays while moving in the laboratory frame, find a relationship between the angle at which B is emitted, and the energies of A and B.

If an atom of mass M decays to a state of rest energy $M - \Delta$ by emitting a photon of energy $h\nu$, show that $h\nu < \Delta$.

Why is Δ not exactly equal to $h\nu$?

Question 2X - Astrophysical Fluid Dynamics

(i) Explain what is meant by the term 'pressure'.

Give two examples of microphysical effects that give rise to an effective pressure.

Suppose all internal stresses within a fluid can be modelled in terms of an effective pressure, $p_{\rm eff}$. Show that longitudinal disturbances within the fluid propagate at a speed $c_{\rm s}$ given by

$$c_{\rm s}^2 = \frac{dp_{\rm eff}}{d\rho},$$

where ρ is the fluid density.

Why are transverse waves not possible in this case?

(ii) In ideal magnetohydrodynamics, the force per unit volume produced by a magnetic field $\mathbf{B}$ is given by

$$\mathbf{f}_{\rm mag} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B},$$

and the magnetic field evolves according to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}),$$

where $\mathbf{u}$ is the velocity field. Explain carefully what conditions have to be satisfied for each expression to be valid.

A uniform fluid with thermal sound speed $c_{\rm th}$ is threaded by a uniform magnetic field, $\mathbf{B_0}$ with Cartesian components $(B_0, 0, 0)$. A linear longitudinal disturbance is set up in the y-direction. Show that the effect of the magnetic field can be modelled in terms of an effective pressure, $p_{\rm eff}$.

Determine the propagation speed of longitudinal disturbances perpendicular $v_{\rm perp}$ and parallel $v_{\rm par}$ to the field.

Consider the case of a uniform toroidal field, $\mathbf{B}_\phi$. Use Stokes theorem to show that at radius R

$$f_{\rm mag} = \frac{|\mathbf{B}_\phi|^2}{\mu_0 R}.$$

TURN OVER...

Suppose a fluid of uniform density ρ is threaded by an initially uniform magnetic field in the x-direction. The field is then distorted so that the field lines are parallel to the sinusoidal function $y = y_0 \sin(kx)$, where $y_0 \ll 2\pi/k$ and k is a constant. By considering the effective radius of curvature of the distorted field, determine f_{mag} and the frequency of the fluid's oscillation in the y-direction.

Hence, deduce the speed of transverse disturbances propagating in the x-direction, v_{trans}, and show that

$$v_{\text{perp}}^2 = v_{\text{trans}}^2 + v_{\text{par}}^2.$$

Question 3X - Structure and Evolution of Stars

(i) A star has density ρ at radius r given by

$$\rho(r) = \rho_0\Big(1 - \frac{r}{R}\Big),$$

where R is the radius of the star. Find an expression for the central density ρ_0 in terms of R and the mass M of the star.

Write down the equation of hydrostatic equilibrium and use it to find the pressure as a function of radius as

$$P(r) = \frac{5GM^2}{4\pi R^4}\Big[1 - \frac{4}{5}\left(6x^2 - 7x^3 + \frac{9}{4}x^4\right)\Big],$$

where $x = r/R$.

Give the dependence of the central pressure P_c in terms of M and R in solar units, and, find the central temperature T_c, assuming the equation of state for an ideal gas with a mean molecular weight $\mu = 0.6$.

(ii) For the star in **(i)**, show that the total gravitational potential energy Ω is

$$\Omega = -\frac{26}{35}\frac{GM^2}{R}.$$

For matter with a general equation of state (i.e., not just an ideal monatomic non-relativistic gas), explicitly evaluate the total kinetic energy K and verify that the virial theorem is satisfied.

Using results from **(i)**, estimate the time taken for a large solar-mass cloud to contract to a radius $R_\odot$, assuming that nuclear reactions have not yet ignited and that the cloud cooled via emission of blackbody radiation from its surface at a temperature $T_{\rm eff} = 2500$ K.

Now sketch on an Hertzsprung-Russell diagram the following two curves:

(a) the zero-age main sequence,

(b) the track of a contracting cloud of mass $M_\odot$.

TURN OVER...

Question 4Y - Cosmology

(i) Linear, pressureless, density perturbations of overdensity $\delta = (\rho - \overline{\rho})/\overline{\rho}$, superimposed on a homogeneous and isotropic background universe with scale factor $a(t)$ and mean density $\overline{\rho}(t)$ satistfy the equations:

$$\begin{aligned}
\left(\frac{\partial \delta}{\partial t}\right)_x &= -\frac{1}{a(t)}\boldsymbol{\nabla}_x.\mathbf{v}_p, \\
\left(\frac{\partial \mathbf{v}_p}{\partial t}\right)_x + \frac{\dot{a}}{a}\mathbf{v}_p &= -\frac{\boldsymbol{\nabla}_x\hat{\phi}}{a}, \\
\boldsymbol{\nabla}_x^2\hat{\phi} &= 4\pi G\overline{\rho}a^2\delta,
\end{aligned}$$

where $\mathbf{v}_p$ is the peculiar velocity ($\dot{\mathbf{r}} = \dot{a}\mathbf{x} + \mathbf{v}_p$), $\boldsymbol{\nabla}_x$ is the gradient operator with respect to the comoving coordinates $\mathbf{x}$, and dots denote differentiation with respect to time. Assume that the dominant growing solution to these equations is

$$\delta(\mathbf{x}, t) = A(\mathbf{x})D(t),$$

and that the peculiar velocity is irrotational, *i.e.* it is related to a scalar potential ψ_v via,

$$\mathbf{v}_p = -\boldsymbol{\nabla}_x\psi_v.$$

Show that the peculiar velocity can be expressed as

$$\mathbf{v}_p = \frac{1}{4\pi G\overline{\rho}}\frac{\dot{D}}{D}\mathbf{g},$$

where $\mathbf{g}$ is the peculiar gravitational acceleration, $\mathbf{g} = -a^{-1}\boldsymbol{\nabla}_x\hat{\phi}$.

Show that for a spherical density perturbation of radius r and mean overdensity Δ, the peculiar velocity at radius r is

$$\mathbf{v}_p = -\frac{1}{3}\frac{\dot{D}}{D}\Delta\mathbf{r}. \qquad (*)$$

Give a physical interpretation of this result.

(ii) A spherical pressureless perturbation of mass M and radius r superimposed on an Einstein de-Sitter universe (*i.e.* spatially flat universe with zero cosmological constant) satisfies the equation of motion:

$$\ddot{r} = -\frac{GM}{r},$$

where dots denote differentiation with respect to time. The parametric solution of this equation is

$$r = A(1 - \cos\theta), \quad t = B(\theta - \sin\theta), \quad A^3 = GMB^2. \qquad (**)$$

Show that for $\theta \ll 1$,

$$\theta^3 = s^3\left(1 + \frac{1}{20}s^2 + \mathcal{O}(s^4)\right), \qquad s = \left(\frac{6t}{B}\right)^{1/3}.$$

Hence show that the overdensity of the perturbation is

$$\Delta = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}, \quad \theta \ll 1.$$

Show that the peculiar velocity at radius r is consistent with the result (*) of part (i) if $\theta \ll 1$.

An observer measures redshifts of galaxies in a nearby rich cluster of galaxies. If the observer interprets redshift as distance via $d = cz/H_0$, sketch the three-dimensional appearance of the cluster inferred by the observer if it is described by the solution (**): (a) for $\theta = \pi/4$; (b) $\theta = \pi/2$.

TURN OVER...

Question 5Z - Stellar Dynamics and the Structure of Galaxies

(i) Suppose that the frame of reference (x, y, z) in which a gravitational potential ϕ is steady rotates uniformly at angular velocity $\mathbf{\Omega}$ about the z-axis. The equations of motion are

$$\ddot{\mathbf{r}} = -\nabla\phi - 2(\mathbf{\Omega} \times \dot{\mathbf{r}}) - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}).$$

Explain the origin of each term in the equations of motion.

Show that Jacobi's constant

$$E_{\mathrm{J}} = \frac{1}{2}|\dot{\mathbf{r}}|^2 + \phi - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2,$$

is an integral of the motion.

Show further that Jacobi's constant is related to the energy E and angular momentum $\mathbf{L}$ in an inertial coordinate system by

$$E_{\mathrm{J}} = E - \mathbf{\Omega} \cdot \mathbf{L}.$$

(ii) A star is moving in an axisymmetric potential $\Phi(R, z)$ where (R, z) are cylindrical polar coordinates. If the angular momentum component parallel to the symmetry axis is L_z, then show that the star's motion takes place in a meridional plane (R, z) with effective potential

$$\Phi_{\mathrm{eff}}(R, z) = \Phi(R, z) + \frac{L_z^2}{2R^2}.$$

Is the meridional plane uniformly rotating?

Suppose the star is moving on a nearly circular orbit in the equatorial plane $(z = 0)$. Show that the star's motion is well-approximated by the superposition of retrograde motion at angular frequency κ around an ellipse, and prograde motion of the ellipse's centre at angular frequency Ω around a circle.

Show further that the frequencies are related by

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2,$$

and find the axis ratio of the ellipse.

In the solar neighbourhood of our Galaxy, Oort's constants A and B are defined as

$$A = -\frac{R}{2}\frac{d\Omega}{dR} \approx 14.5\,\mathrm{km\,s^{-1}kpc^{-1}},$$

$$B = -\frac{R}{2}\frac{d\Omega}{dR} - \Omega \approx -12\,\mathrm{km\,s^{-1}kpc^{-1}}.$$

Estimate how many radial oscillations the Sun does in the time it takes to complete one orbit around the Galactic Centre.

TURN OVER...

Question 6Z - Topics

(i) There is considerable interest in performing a census of the relatively small (asteroid size) bodies believed to exist in the Kuiper Belt, at distances of 45 AU from the Sun. Assuming that Kuiper Belt objects are fully illuminated and viewed face-on and that they are visible only through reflected light from the Sun, write down the expression relating the apparent magnitude, absolute magnitude and distance.

Assume that all Kuiper Belt objects lie at 45 AU. The number of Kuiper Belt objects, $N(> R)$, with radii $> R$, is given by

$$N(> R) \propto R^{1-q},$$

with $q > 1$. What is the dependence of the cumulative number-magnitude counts on the index q?

If observations show that 5 Kuiper Belt object per square degree are seen with apparent magnitude $m < 25.0$ and the index $q = 2.5$, how many object per square degree would be seen in a survey to $m = 28.0$?

(ii) If a Kuiper Belt object of radius $R \sim 100$ km has an apparent magnitude of $m = 23$ at a distance of 45 AU and the cumulative number-magnitude counts satisfy

$$\frac{d \log n(< m)}{dm} = 0.35$$

estimate the size of the Kuiper Belt objects that contain most of the mass of the belt.

Assume the objects are uniformly distributed in a ring of width 10 AU at a distance of 45 AU with a thickness 0.1 AU. Estimate the characteristic collision timescale between Kuiper Belt objects of size 10 km, assuming a velocity dispersion of $1\,\mathrm{kms}^{-1}$.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Wednesday 2 June 2010 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

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Question 1X - Relativity

(i) Let R_{abcd} be the Riemann tensor in a four-dimensional spacetime. By considering the symmetries

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{badc},$$

show that there are 6 ways of choosing the pair (a, b) and 6 ways of choosing the pair (c, d) that give independent results.

The Riemann tensor also possesses the pair-interchange symmetry

$$R_{abcd} = R_{cdab}.$$

Show that this reduces the number of independent components of the Riemann tensor to 21.

When the final symmetry of the Riemann tensor

$$R_{abcd} + R_{adbc} + R_{acdb} = 0,$$

is considered, how many independent components remain?

(ii) Suppose a vector field ξ_a satisfies Killing's equation

$$S_{ab} = \xi_{b;a} + \xi_{a;b} = 0.$$

By considering the quantity $S_{ab;c} + S_{ca;b} + S_{bc;a}$, show that

$$\xi_{c;ba} = R^d_{abc}\xi_d.$$

If U^a is the tangent vector to a geodesic, show that $U^a\xi_a$ is constant along a geodesic.

Show that Killing's equation becomes

$$\xi_{b,a} + \xi_{a,b} = 0,$$

in Minkowski spacetime.

Hence, find 10 linearly independent vector fields that satisfy Killing's equation in Minkowski spacetime and interpret the results physically.

Question 2X - Astrophysical Fluid Dynamics

(i) The magnitude of the viscous torque G exerted at radius R in an axisymmetric disc with angular velocity Ω, kinematic viscosity ν and surface density Σ is given by:

$$G(R) = 2\pi\nu\Sigma R^3 \frac{d\Omega}{dR}.$$

Explain, without detailed derivation, the physical basis for this expression.

Suppose that the kinematic viscosity ν depends on the disc scale height H, density ρ and sound speed $c_{\rm s}$ according to

$$\nu \propto H^a \rho^b c_{\rm s}^c,$$

where a, b and c are constants. Use dimensional analysis to determine the values of a, b and c.

In the case of a gas pressure supported disc in hydrostatic equilibrium, show that $\nu \propto T/\Omega$, where T is the disc temperature.

(ii) Using any of the results given in **(i)** or otherwise, show that the evolution of a thin, axisymmetric viscous accretion disc is governed by

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{(R^2\Omega)'}\frac{\partial}{\partial R}(\nu\Sigma R^3\Omega')\right),$$

where $'$ denotes differentiation with respect to R.

A disc surrounds a white dwarf of radius $R_{\rm wd}$ and is subject to a constant rate of mass input $\dot{M}$ at radius $R_{\dot{M}}$. The angular velocity of the disc makes a rapid transition from Keplerian rotation to the (lower) angular velocity of the white dwarf in a narrow region around $R_{\rm wd}$. State the boundary condition that applies at the inner edge of the disc.

Write down the radial dependence of $\nu\Sigma$ in a steady state.

If the disc is tidally truncated by a binary companion immediately beyond $R_{\dot{M}}$, find an expression for the viscous torque at $R_{\dot{M}}$ and explain the sign of this torque.

Determine the rate at which angular momentum is transferred to or from the disc by the tidal effect, and explain how angular momentum is globally conserved.

TURN OVER...

The rate of viscous dissipation per unit area is given by $F_{\rm diss} = \nu\Sigma R^2\Omega'^2$. If the disc radiates as a black body, locate the radius in the disc at which the effective temperature, T_e, is a maximum.

What is the radial dependence of T_e at radii $\gg R_{\rm wd}$?

Question 3Y - Statistical Physics

(i) Show that the partition function for a harmonic oscillator of frequency ν is

$$Z = \frac{\exp\left(-\frac{h\nu}{2kT}\right)}{\left[1 - \exp(-\frac{h\nu}{kT})\right]}. \qquad (*)$$

A system consists of N independent, distinguishable, harmonic oscillators each of frequency ν. Derive expressions for the internal energy and entropy in terms of the partition function Z.

(ii) Use the partition function (*) of part (i) to calculate the contribution of a single mode of molecular vibration to the molar specific heat capacity, C_{vib}, of a gas of molecules.

Show that

$$\begin{aligned} C_{\mathrm{vib}} &\rightarrow \mathcal{R}, \quad \text{for } kT \gg h\nu, \\ C_{\mathrm{vib}} &\rightarrow \mathcal{R}\frac{h^2\nu^2}{k^2T^2}\exp\left(-\frac{h\nu}{kT}\right), \quad \text{for } kT \ll h\nu, \end{aligned}$$

where $\mathcal{R}$ is the gas constant.

CO_2 is a linear symmetric molecule with a bending vibration at $\nu = 2 \times 10^{13}$ Hz and stretching vibrations at 4×10^{13} Hz and 7×10^{13} Hz. The rotational levels are spaced at about 2.5×10^{10} Hz. Sketch the temperature dependence of the molar heat capacity of gaseous CO_2 over the temperature range 200 K to 5000 K.

TURN OVER...

Question 4X - Structure and Evolution of Stars

(i) Give a relationship between the electron pressure P_e and the gas pressure P_g in the following cases:

(a) at the centre of a low mass star just arrived on the main sequence;

(b) in the (non-degenerate) centre of a 2 $M_\odot$ red giant star;

(c) in the stellar atmosphere of a hot star ($T_{eff} \sim 15000$ K);

(d) in the stellar atmosphere of a cool star ($T_{eff} \sim 4000$ K).

In which cases are there non-negligible contributions to P_e from heavier elements, and why?

(ii) Consider a homologous family of stars with radius R, mass M and luminosity L, in which nuclear energy is generated by the CNO cycle so that

$$\epsilon = \frac{dL}{dM} \propto \rho T^{17},$$

where ρ is the density and T the temperature. Suppose also that the opacity κ is dominated by Thomson scattering by electrons, and that the composition and ionisation state is uniform throughout the star so that

$$\kappa = \frac{n_e}{\rho}\sigma_T = \text{constant},$$

where σ_T is the Thomson cross-section and n_e is the electron number density. Using homology relations, and assuming an ideal gas pressure law, show that

$$R \propto M^{4/5}, \qquad L \propto M^3.$$

Suppose instead that radiation pressure dominates. Now show that the homology relations become

$$R \propto M^{19/40}, \qquad L \propto M.$$

Where do these stars reside on the Hertzsprung-Russell diagram?

Question 5Y - Cosmology

(i) Define the 'particle horizon' surrounding each observer in a homogeneous isotropic universe with scale factor $a(t)$.

If the scale factor varies as $a(t) \propto t^{\alpha}$, where α is a constant, show that the proper distance to the particle horizon at time t is

$$d_{\rm ph}(t) = \frac{ct}{(1-\alpha)}.$$

Show that in a spatially flat universe with zero cosmological constant dominated by matter with an equation of state $P = w\rho c^2$ (where w is a constant), the particle horizon is

$$d_{\rm ph}(t) = \frac{(3+w)ct}{(1+w)}.$$

Give a physical explanation of this result as $w \to -1$.

(ii) Show that in a spatially flat homogeneous and isotropic universe filled with matter with an equation of state $P = -\rho c^2$, the scale factor evolves as

$$a(t) \propto \exp(Ht), \qquad H = \text{constant}. \qquad (*)$$

Derive an expression for the particle horizon.

Show that every observer is surrounded by an event horizon of proper radius

$$d_{\rm eh} = \frac{c}{H}.$$

Assume that an inflationary phase described by (*) ends at $t = 10^{-32}$s and that the scale factor evolves as $a(t) \propto t^{1/2}$ thereafter. A perturbation of physical scale $\lambda = 10^{-2}$m at the end of inflation becomes equal in scale to the Hubble radius ct at time t_λ and oscillates as an acoustic wave thereafter. Estimate t_λ.

Estimate the mass enclosed within the Hubble radius at t_λ. (You may assume that the present observable Universe had a physical radius of 1m at the end of inflation.)

TURN OVER...

Question 6Z - Stellar Dynamics and the Structure of Galaxies

(i) Let f be the distribution function of a collisionless, stellar system with density ρ moving in a gravitational potential ϕ. Let us use angled brackets to define averages over the distribution function in velocity space, for example

$$\langle v_i \rangle = \frac{1}{\rho} \int v_i f d^3 v.$$

Derive the Jeans equations in the form

$$\rho \frac{\partial \langle v_j \rangle}{\partial t} + \rho \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\rho \frac{\partial \phi}{\partial x_j} - \frac{\partial (\rho \sigma_{ij}^2)}{\partial x_i},$$

where the velocity dispersion tensor $\sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle$.

What is the analogue of the velocity dispersion in a fluid system?

(ii) The Hohmann interplanetary travel orbit is the ellipse which is in contact with the circular orbit of the Earth at pericentre and the circular orbit of the planet at apocentre. Show that the time of flight of a satellite on a Hohmann orbit from the Earth to a planet with semimajor axis a is

$$T = \frac{1}{4\sqrt{2}} \left[1 + \frac{a}{a_\oplus} \right]^{3/2} \text{yr},$$

where $a_\oplus$ is the Earth's semimajor axis.

If $V_\oplus$ is the circular velocity of the Earth, then the additional velocity required to place a satellite on a Hohmann orbit is

$$V_{\text{add}} = V_{\text{Hoh}} - V_\oplus,$$

where V_{Hoh} is the velocity of the Hohmann orbit. Using energy conservation, find V_{Hoh} and hence show that

$$V_{\text{add}} = V_\oplus \left[\sqrt{2} \left(\frac{a}{a + a_\oplus} \right)^{1/2} - 1 \right].$$

Mars has a semimajor axis $a = 1.524$ AU. Calculate the eccentricity of the Hohmann orbit, the time of flight and the additional velocity V_{add}.

[*Hint: The period of an elliptic orbit of semimajor axis a around a point mass M is* $P = 2\pi a^{3/2}/(GM)^{1/2}$.]

Question 7Z - Topics

(i) Describe and contrast two observational techniques used to detect extrasolar planets and explain why most planets found have masses comparable to that of Jupiter or greater.

A star's observed radial velocity, $V_{\rm star}$, shows a periodic variation of $\pm 1\ {\rm ms}^{-1}$, suggesting the presence of an orbiting planet. The velocity of the planet around the star is,

$$V_{\rm planet} = \sqrt{\frac{GM_{\rm star}}{r}},$$

where r is the planet's distance from the star. Estimate the mass of the planet for a solar mass star, and an orbital period of 1 year.

Under what assumption would this be the true mass and not a lower limit?

What other practical measurement would allow the true mass of the planet to be estimated?

(ii) The first extrasolar planet to be discovered orbits the star 51 Peg. The star has a surface temperature of $T = 5700$K and a radius $R = 1.4\ {\rm R}_\odot$. The planet is in a circular orbit with radius $r = 0.05$ AU and has a mass obeying the constraint

$$M \sin(i) = 0.46 M_{\rm Jup}$$

where i is the inclination of the planet's orbit to the line of sight and $M_{\rm Jup}$ is the mass of Jupiter. Estimate the temperature of the planet assuming that it absorbs all the photons incident on its atmosphere.

Compute the escape velocity, $v_{\rm esc}$, and the thermal velocity, $v_{\rm th}$, of atoms in the the planet's atmosphere.

Can gas typically overcome the gravitational field of the planet and escape?

[The mass of Jupiter is $M_{\rm Jup} = 2 \times 10^{-27}$ kg and the average density is $\rho = 1300\,{\rm kg\ m}^{-3}$.]

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Thursday 3 June 2010 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 4

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Question 1X - Relativity

(i) A solution of Einstein's equations is given by the metric ($c = 1$)

$$ds^2 = dt^2 + \frac{4J}{r}\sin^2\theta dt d\phi - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2,$$

where J is a constant. The Lagrangian of a photon is

$$L = \dot{t}^2 + \frac{4J}{r}\sin^2\theta\dot{t}\dot{\phi} - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2.$$

where a dot denotes differentiation with respect to the affine parameter λ. Derive the equations of motion as

$$\frac{d}{d\lambda}\left[\dot{t} + \frac{2J}{r}\sin^2\theta\dot{\phi}\right] = 0,$$

$$\frac{d}{d\lambda}\left[\dot{\phi}r^2\sin^2\theta - \frac{2J}{r}\dot{t}\sin^2\theta\right] = 0,$$

$$\frac{d}{d\lambda}\left[r^2\dot{\theta}\right] = \left(r^2\dot{\phi}^2 - \frac{4J\dot{\phi}\dot{t}}{r}\right)\sin\theta\cos\theta,$$

$$\ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 + \frac{4J}{r^2}\sin^2\theta\dot{\phi}\dot{t}.$$

(ii) Using the equations of motion in part **(i)**, show that there are three integrals of motion, P_0, P_ϕ and L, given by

$$P_0 = \dot{t} + \frac{2J}{r}\sin^2\theta\dot{\phi},$$

$$P_\phi = \dot{\phi}r^2\sin^2\theta - \frac{2J}{r}P_0\sin^2\theta,$$

$$L^2 = P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta},$$

where $P_\theta = r^2\dot{\theta}$ and J is so small that quantities of order J^2 may be neglected.

Consider a photon orbit with $\theta = \pi/2$ and $\dot{\theta} = 0$. Demonstrate that

$$\frac{d}{d\lambda} = \left(\frac{P_\phi}{r^2} + \frac{2JP_0}{r^3}\right)\frac{d}{d\phi}.$$

By introducing $u = 1/r$ and $\ell = P_0/P_\phi$, show that the orbit equation becomes

$$u'' + u = -6u^2J\ell - 2J\ell u'^2 - 4J\ell uu'',$$

where a dash represents differentiation with respect to ϕ.

Verify that

$$u = \frac{1}{b}\cos\phi - \frac{2J\ell}{b^2},$$

is a solution of this equation, where b corresponds to the impact parameter.

Hence, show that the bending angle of the light ray is

$$\Delta\phi = -\frac{4J\ell}{b}.$$

TURN OVER...

Question 2X - Astrophysical Fluid Dynamics

(i) Let p be the pressure, ρ the density and u the velocity of a fluid. Using subscripts 1 and 2 to denote quantities respectively upstream and downstream of a strong adiabatic shock ($p_1 \approx 0$), show that

$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1},$$

and that

$$p_2 = \frac{2}{\gamma + 1}\rho_1 u_1^2,$$

where γ is the ratio of specific heats.

(ii) Use the results of **(i)** to show that for a strong adiabatic blast wave expanding spherically into a uniform medium:

a) the shocked gas occupies a thin shell;

b) the ratio of the kinetic energy of this shell to the internal energy of hot gas in the bubble is a constant provided that the ratio α of the pressure in the bubble interior to that in the shocked gas is a constant;

c) the total energy is a factor 3 times the kinetic energy in the case of monatomic gas with $\gamma = 5/3$ and $\alpha = 1/2$;

d) the total energy is conserved if the bubble radius R varies with time t according to $R \propto t^{2/5}$.

A supernova explodes in the mid-plane of a galactic disc modelled as a uniform slab with density ρ and thickness $2H$. Calculate the minimum explosion energy, $E_{\rm min}$ such that the bubble is still expanding supersonically when its radius $R = H$.

For a supernova with $E > E_{\rm min}$, the bubble 'breaks out' when $R = H$ and the hot gas in the interior is vented into the intergalactic medium. Explain why the bubble does not conserve energy thereafter and suggest another quantity that is conserved.

What is the total mass of gas that is ultimately swept up by the supernova if $E = 1000E_{\rm min}$?

Question 3Y - Statistical Physics

(i) The equation of state of a non-ideal gas is sometimes approximated by Van der Waals formula,

$$\left(P + \frac{N^2\alpha}{V^2}\right)(V - N\beta) = NkT,$$

where N is the number of gas particles contained within the volume V. Give a heuristic justification of this equation explaining the physical significance of the constants α and β.

If C_V is the specific heat capacity at constant volume, show that

$$\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 P}{\partial T^2}\right)_V .$$

Hence show that for a Van der Waals gas, C_V is a function of temperature only.

(ii) A classical ideal gas is repelled by the walls of its container. The repulsive force can be represented by a step function in the potential energy:

$$\begin{aligned} \Delta\phi &= W, \qquad x < h, \\ \Delta\phi &= 0, \qquad x \geq h, \end{aligned}$$

where x is the distance of a gas particle from the wall. Derive an expression for the additional specific heat, ΔC_V, induced by this repulsive force for gas within a container of volume V and wall area A. (Assume that the distance h is small compared with the dimensions of the container).

Sketch the dependence of ΔC_V as a function of temperature.

Explain physically why $\Delta C_V \to 0$ as $T \to 0$ and as $T \to \infty$.

TURN OVER...

Question 4X - Structure and Evolution of Stars

(i) If 0.7% of the mass of a star on the main sequence is converted to energy via hydrogen burning, show that the total energy liberated is given by

$$E_{\rm MS} = 6.3 \times 10^{14}\, FXM \text{ Joules},$$

where F is the fraction of the mass of hydrogen burnt, X is the initial hydrogen fraction by mass, and M is the mass of the star in kg.

Justify briefly the use of $F = 0.1$ and $X = 0.7$ as appropriate values.

Assuming a mass-luminosity relation of the form

$$L = \left(\frac{M}{\rm M_\odot}\right)^{7/2} \rm L_\odot,$$

and that the star's luminosity on the main sequence is independent of time, show that the main sequence lifetime is given by

$$\tau_{\rm MS} \sim (L_\odot/L)^{5/7} \times 10^{10} \text{ years},$$

where $\tau_{\rm MS,\odot} = 10^{10}$ years is the main sequence lifetime of the Sun.

Estimate the main sequence lifetime of a $20M_\odot$ star.

(ii) Consider a carbon-oxygen white dwarf of mass M with an approximately isothermal core of temperature $T_{\rm c}$. Explain why the internal energy E is given by

$$E = \frac{3}{24}\frac{M}{m_{\rm H}}kT_{\rm c},$$

where k is Boltzmann's constant and $m_{\rm H}$ is the mass of a hydrogen atom.

Suppose the relationship between the emitted luminosity L and $T_{\rm c}$ is

$$T_{\rm c} = 7 \times 10^7 \left(\frac{L/\rm L_\odot}{M/\rm M_\odot}\right)^{2/7} \rm K.$$

Derive the cooling rate $dT_{\rm c}/dt$ for a 0.6 $\rm M_\odot$ white dwarf with initial core temperature $T_{\rm c} = 10^7$ K, and hence show that

$$T_{\rm c} = \left(5.1 \times 10^{-27} t + 3.16 \times 10^{-18}\right)^{-2/5} \rm K,$$

where t is measured in years.

Suppose the mass-radius relation for white dwarfs is

$$\frac{M}{\mathrm{M}_\odot} \simeq 10^{-6}\left(\frac{R}{\mathrm{R}_\odot}\right)^{-3}.$$

Ignoring the thickness of the radiative envelope, derive the effective temperature T_{eff} of the white dwarf surface as a function of time.

What is the T_{eff} for this white dwarf after 10^{10} years?

TURN OVER...

Question 5Y - Cosmology

(i) The number density of fermions in thermal equilibrium at temperature T is given by

$$n = g_s \frac{4\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\left[\exp\left(\frac{E(p)-\mu}{kT}\right) - 1\right]},$$

where g_s is the number of degrees of freedom for the particles with energy $E(p)$ and μ is the chemical potential. Give a brief justification of this equation.

Explain the physical significance of the chemical potential μ.

Derive an expression for the number density, n, for non-relativistic fermions.

(ii) Thermal equilibrium between two species of non-relativistic particle, a and b with rest-masses m_a and m_b, is maintained by the reaction

$$a + \alpha \leftrightarrow b + \beta, \qquad (*)$$

where α and β are massless particles with zero chemical potential. Evaluate the ratio of the number densities, n_a/n_b.

Explain how a reaction such as (*) is relevant to the neutron-proton ratio, $r = n_n/n_p$, in the early Universe.

Why does the neutron-proton ratio not fall rapidly to zero as the Universe cools?

Explain briefly the process of primordial nucleosynthesis by which neutrons are converted into stable helium nuclei.

Let Y_{He} denote the fraction of the mass density of the Universe in the form of helium after nucleosynthesis is completed. How is Y_{He} related to the neutron-proton ratio r at the time of nucleosynthesis?

Question 6Z - Stellar Dynamics and the Structure of Galaxies

(i) Show that the energy E and angular momentum $\mathbf{J}$ are integrals of the motion in a spherical potential $\phi(r)$ and describe the orbits.

What does Jeans theorem say in general about the form of the stellar distribution function f for a collisionless stellar system in equilibrium?

Show that if the distribution function is a function of the energy alone, $f(E)$, the velocity dispersion tensor of a spherical stellar system must be isotropic.

(ii) An equilibrium, non-spherical, axisymmetric distribution of 'test' objects (*i.e.* particles of negligible mass) is embedded within a spherical potential $\phi(r)$. The distribution function of the test objects has the form

$$f(E, J^2, J_z^2),$$

where E and $\mathbf{J}$ are the energy and angular momentum per unit mass respectively and the Cartesian axis z defines the symmetry axis. Does this distribution function provide a satisfactory model of a steady state axisymmetric system if the masses of the test objects cannot be neglected?

Suppose the distribution function has the form

$$f(E, J^2, J_z^2) \propto J^{2\beta} J_z^{2\gamma} \exp\left(-(\alpha + 2\beta + 2\gamma)\frac{E}{\sigma_0^2}\right),$$

where α, β, γ and σ_0 are constants. Using cylindrical polar coordinates in velocity space, show that the velocity dispersions in spherical polar coordinates (with the z-axis defining the pole) are given by,

$$\begin{aligned}
\sigma_r^2 &= \frac{\sigma_0^2}{(\alpha + 2\beta + 2\gamma)}, \\
\sigma_\theta^2 &= \frac{(1 + \beta + \gamma)\sigma_r^2}{(1 + \gamma)}, \\
\sigma_\phi^2 &= (1 + 2\gamma)\sigma_\theta^2.
\end{aligned}$$

Derive an expression for the density distribution in terms of the potential $\phi(r)$.

[You may assume that
$\int_0^\infty x^{2\nu-1}\exp(-\mu x^2)dx = \frac{\Gamma(\nu)}{2\mu^\nu}$, $\quad \int_0^\pi \sin^{2\nu} x dx = \frac{\sqrt{\pi}}{\nu}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu)} \quad \nu > -1/2,\ \mu > 0,$
and Γ denotes the Gamma function, $\Gamma(\nu + 1) = \nu\Gamma(\nu), \Gamma(1/2) = \sqrt{\pi}$.]

TURN OVER...

Question 7Z - Topics

(i) A compact object, C, of mass M, is at a distance d from the Earth. A background star at distance $2d$ is situated a small angle β away from it. A light ray passing at an impact parameter h ($\gg GM/c^2$) from an object is gravitationally deflected by an angle $\alpha = 4(GM/c^2)h^{-1}$.

Show that an observer on Earth usually sees two images of the star which are at apparent angular distances Θ_1 and Θ_2 from C, where Θ_1 and Θ_2 are the roots of the equation

$$\Theta^2 - \beta\Theta - \alpha_0^2 = 0,$$

where $\alpha_0^2 = 2(GM/c^2)d^{-1}$.

What does the observer see when $\beta = 0$?

(ii) Explain why gravitational lensing preserves surface brightness.

For the situation outlined in part **(i)**, by computing the Jacobian of the transformation between the image and source plane, show that lensing of a small compact object produces images with magnifications of

$$\mu_{\pm} = \frac{1}{4}\left(\frac{d}{\sqrt{d^2+4}} + \frac{\sqrt{d^2+4}}{d} \pm 2\right).$$

What does "small" mean in this context?

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Friday 4 June 2010 09:00am – 12:00pm

ASTROPHYSICS - PAPER 5

Before you begin read these instructions carefully.

Candidates may attempt not more than 6 questions.

Each question is divided into Part (i) *and Part* (ii), *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i). *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **3Y and 5Y** *should be in one bundle and* **6Z and 7Z** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X - Relativity

(i) Consider the metric ($c = 1$)

$$ds^2 = dt^2 - \exp(2t/b)\Big[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big].$$

Light signals from a galaxy with coordinate distance r are emitted at an epoch t_1 and received by an observer at epoch t_2. Demonstrate that

$$r = b\Big[\exp(-t_1/b) - \exp(-t_2/b)\Big].$$

For given r, show that there is a maximum epoch t_1 and interpret this result physically.

What is the redshift of the galaxy in terms of t_1 and t_2?

If an observer watches the galaxy for infinitely long, show that the luminosity distance becomes infinite, but the angular diameter distance tends to b.

(ii) A point mass M is a function of the time-like coordinate t. The metric is ($c = G = 1$)

$$ds^2 = \Big[1 - \frac{2M(t)}{r}\Big]dt^2 + 2dtdr - r^2\Big[d\theta^2 + \sin^2\theta d\phi^2\Big].$$

Compute the Christoffel connection Γ^r_{tt}.

The only other non-vanishing connections are

$$\Gamma^t_{tt} = -\frac{M}{r^2} = -\Gamma^r_{tr}.$$

Using this information, calculate the component of the Ricci tensor R_{tt}.

All other components of the Ricci tensor vanish. Hence, compute the Ricci scalar R.

Now using the Einstein field equations, evaluate the only non-vanishing component of the energy-momentum tensor $T_{\mu\nu}$ and interpret your result physically.

Question 2X - Astrophysical Fluid Dynamics

(i) A jet of cold fluid is emitted supersonically at speed v from a nozzle that is located at the origin of a Cartesian coordinate system in which a uniform gravitational acceleration g acts in the negative z-direction. The nozzle rotates in the (x, z) plane so that its angle to the positive x-direction is given by $\theta = \theta_0 \sin(\omega t)$, where θ_0 and ω are constants. Explain why the fluid flow is approximately ballistic.

Show that in the limit $\theta_0 \ll 1$, the equation of the particle path for a fluid element emitted at angle θ is given by:

$$z = -z_0(x) + \theta(x - z_0(x)\theta),$$

where $2z_0(x) = gx^2/v^2$.

With the help of a diagram, explain how particles that are emitted consecutively may subsequently collide, and determine the phase of the oscillation during which this behaviour is possible.

What hydrodynamic phenomenon is associated with colliding particles?

[You may assume that the nozzle length, a, is extremely small, i.e. that $v \gg a\omega$]

(ii) In the jet described in **(i)** above, consider a set of particles that arrive simultaneously at a given horizontal distance x from the origin. Determine, for these particles, the relationship between the difference in the time of their emission (dt) and the difference in the angle of their emission ($d\theta$).

If $\omega x \theta_0^2 \gg v$, explain why particles with a range of emission angles, θ, can arrive simultaneously at a given x.

Define the half-thickness of the jet as the difference between the maximum height of fluid $z_{\rm max}(x)$ and $z_0(x)$. In the case where particles with a range of θ arrive simultaneously at a horizontal distance x from the source, determine how the half-thickness of the jet varies as a function of θ_0 (assumed small).

Denote the minimum value of θ_0 such that particles can collide at distance x by $\theta_{\rm coll}$. Determine $\theta_{\rm coll}$ and if $\theta_0 < \theta_{\rm coll}$, estimate the sound speed in the gas for which pressure effects within the stream at distance x can be ignored.

TURN OVER...

Question 3Y - Statistical Physics

(i) A bosonic gas consisting of N indistinguishable non-interacting particles is in thermal equilibrium at temperature T. Show that the mean number of particles of energy E_i is

$$n_i = \frac{g_i}{\left[\exp\left(\frac{E_i-\mu}{kT}\right) - 1\right]},$$

where g_i is the number of states of energy E_i and μ is a constant.

(ii) Assume that the bosonic gas of part (i) consists of non-relativistic particles of mass m confined within a box of volume V. Derive an expression for the number of states, $G(E)$, with energy less than E, in the limit that E is much larger than the energy difference between states.

The pressure of the bosonic gas is

$$P = \frac{2}{3V}\int_0^\infty \frac{dG}{dE}\frac{EdE}{\left[\exp\left(\frac{E-\mu}{kT}\right) - 1\right]}.$$

Show that if $\mu/kT \ll 1$, the equation of state of the bosonic gas is

$$PV = NkT\left[1 - \frac{Nh^3}{16V(m\pi kT)^{3/2}}\right] = NkT\left[1 - \left(\frac{T_B}{T}\right)^{3/2}\right].$$

What is the physical significance of the characteristic temperature T_B?

[You may assume thatfor $\nu > -1$
$\int_0^\infty z^\nu[\exp(z-u) - 1]^{-1}dz = \Gamma(\nu+1)\sum_{n=0}^\infty \frac{\exp[(1+n)u]}{(1+n)^{1+\nu}}$,
$\Gamma(5/2) = 3/2\Gamma(3/2), \quad \Gamma(3/2) = \sqrt{\pi}/2.$]

Question 4X - Structure and Evolution of Stars

(i) What is the main-sequence lifetime of a $0.4\mathrm{M}_\odot$ star if time on the main-sequence scales as $M^{-2.5}$?

Main-sequence stars with mass below $0.4\mathrm{M}_\odot$ undergo convective energy transport throughout their mass. Since the star is already fully convective, describe its post-main-sequence evolutionary track in the Hertzsprung-Russell diagram.

At the surface of a fully convective low-mass main-sequence star of mass M and radius R, the boundary condition is

$$\kappa P = \frac{GM}{R^2},$$

where P is the pressure and κ is the opacity. Assume that the equation of state is that of an ideal gas and the opacity depends on the density and the temperature as $\kappa = \kappa_0 \rho T^8$, where κ_0 is a constant. Show that the adiabatic constant $K = PT^{-5/2}$ scales as

$$K \propto M^{1/2} R^{-1} T_{\mathrm{e}}^{-6},$$

where T_{e} is the stellar effective temperature.

(ii) Describe the physical basis of the Schonberg-Chandrasekhar limit, and the consequences of exceeding it.

Assuming a perfect gas law, the pressure at the core boundary P_{s} can be written as

$$P_{\mathrm{s}}(R_{\mathrm{c}}) = \frac{3}{4\pi}\frac{kT_{\mathrm{c}}}{\mu_{\mathrm{c}} m_H}\frac{M_{\mathrm{c}}}{R_{\mathrm{c}}^3} - \frac{\alpha G}{4\pi}\frac{M_{\mathrm{c}}^2}{R_{\mathrm{c}}^4},$$

where R_{c} is the radius, M_{c} the mass, μ_{c} the mean molecular weight and T_{c} the temperature of the core, whilst α is a constant. Find the core radius corresponding to maximum and minimum pressure at fixed core mass.

For $R_{\mathrm{c}} \ll R$, the radius of the star, hydrostatic equilibrium gives the pressure in the envelope as

$$P_{\mathrm{env}} > \frac{GM^2}{8\pi R^4},$$

TURN OVER...

whilst homology can be used to derive

$$T_{\mathrm{c}} \propto \frac{\mu_{\mathrm{env}} m_H}{k} \frac{M}{R},$$

where μ_{env} is the mean molecular weight in the envelope. Find an expression for the ratio of the core mass, M_{c}, to the total stellar mass, M, in terms of the chemical composition of the star.

Assume a red giant star of mass M and radius R has a core mass M_{c} and radius R_{c}. Let the density distribution be given by:

$$\rho = \rho_0 - (\rho_0 - \rho_c) \left(\frac{r}{R_c} \right)^2 \qquad \text{for} \qquad 0 \leq r \leq R_c,$$

and

$$\rho = \rho_c \frac{\left(\frac{R_c}{r} \right)^3 - \left(\frac{R_c}{R} \right)^3}{1 - \left(\frac{R_c}{R} \right)^3} \qquad \text{for} \qquad R_c \leq r \leq R,$$

where ρ_0 is the central density and $\rho_c = \rho(R_c)$. Show that

$$\frac{R}{R_c} \simeq \exp \frac{1}{15} \left[(y_1 - 1)(2x_1 + 3) + 5 \right],$$

where $x_1 = \rho_0/\rho_c$ and $y_1 = M/M_c$.

By choosing appropriate values for x_1 and y_1, estimate the size of the envelope of a red giant star.

Question 5Y - Cosmology

The cosmic microwave background (CMB) is described by the blackbody distribution:

$$\rho_\gamma(\nu, T_\gamma)c^2 d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{\left[\exp\left(\frac{h\nu}{kT_\gamma}\right) - 1\right]},$$

where ν is the photon frequency and T_γ is the photon temperature. Show that the energy density, $\rho_\gamma c^2$, of the CMB is

$$\rho_\gamma c^2 = aT_\gamma^4, \qquad a = \frac{8\pi^5 k^4}{15h^3c^3}.$$

Compute the contribution of the CMB to the critical density of a spatially flat universe, $\Omega_\gamma = \rho_\gamma/\rho_c$, at the present day if $T_\gamma = 2.7$ K.

Assume that the Universe also contains three massless neutrino flavours with (non-degenerate) thermal distributions characterised by the temperature $T_\nu = 1.9$ K at the present day. Estimate the redshift at which matter with a present day density parameter Ω_m has the same density as the relativistic components.

[You may assume that $\int_0^\infty x^3[\mathrm{e}^x - 1]^{-1}dx = \frac{\pi^4}{15}$.]

(ii) Primordial black holes (PBHs) of mass M evaporate on a timescale

$$t_{\rm ev} \approx \frac{10240\pi^2}{hc^4}G^2M^3.$$

Estimate the minimum mass of a PBH formed in the early Universe, in units of $M_\odot$, if it is to survive by the present day, $t_0 \approx 13.7 \times 10^9$y.

Assume that the energy density of the Universe is $\rho c^2 = (g^*/2)aT^4$, where g^* is the effective statistical weight at temperature T. Calculate the mass, in units of $M_\odot$, contained within the Hubble radius, ct, at temperature $kT = 200$ GeV when $g^* \sim 100$.

If PBHs formed at $kT = 200$ GeV, calculate the maximum fraction of the energy density of the Universe that could have collapsed into PBHs to avoid exceeding the critical density at the present day.

Discuss your result.

TURN OVER...

Question 6Z - Stellar Dynamics and the Structure of Galaxies

(i) Consider a spherically-symmetric stellar-dynamical system with distribution function

$$f(E) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp[E/\sigma^2],$$

where $E = \Phi - \frac{1}{2}v^2$ is the binding energy per unit mass, Φ is the potential, and σ and ρ_1 are constants. Find the density ρ of the system, and derive a differential equation for the potential $\Phi(r)$, where r is the radial coordinate.

(ii) The equation of hydrostatic support for an isothermal ideal gas of density $\rho(r)$ at temperature T is

$$\frac{kT}{m_0}\frac{d\rho}{dr} = -\rho\frac{GM(r)}{r^2},$$

where m_0 is the mass of a gas particle, and $M(r)$ the total mass interior to radius r. Show that the stellar-dynamical system of part **(i)** and the gaseous system have the same density structure when

$$\sigma^2 = \frac{kT}{m_0}.$$

Show that the mean particle speed is

$$\bar{v} = \sqrt{\frac{8}{\pi}}\,\sigma.$$

Show further that the mean square speed is

$$\bar{v}^2 = 3\sigma^2$$

Show also that the mean square relative velocity of any two particles is

$$\bar{v}^2_{vel} = 6\sigma^2$$

[You may assume that: $\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx = \sqrt{\frac{\pi}{\alpha}}$]

Question 7Z - Topics

(i) A self-gravitating gas cloud has a mass M, radius R, and temperature T. Write down an estimate of α, the ratio of internal thermal energy, U, to the gravitational self-energy, W.

Show that if the cloud is isothermal, $\alpha \propto R$.

What does this imply for the stability of isothermal self-gravitating clouds?

(ii) A giant molecular cloud has a mass of $M_{\rm cl} \sim 3 \times 10^5\ \mathrm{M}_\odot$ and a radius of $R_{\rm cl} \sim 20\,\mathrm{pc}$. The gas temperature is $T_{\rm cl} \sim 20\,\mathrm{K}$. If the cloud is supported against collapse by turbulent motions with mean velocity $v_{\rm t}$, show that such motions are highly supersonic.

If the cloud is supported against collapse by a tangled magnetic field, give an estimate of its root mean square magnitude, $\bar{B}$.

The cloud is very inhomogeneous and the densest gas has a density $\rho_{\rm max}$. Its thermal pressure $\rho_{\rm max} {c_{\rm s}}^2$, where $c_{\rm s}$ is the thermal sound speed, is comparable to the turbulent pressure $\bar{\rho} {v_{\rm t}}^2$, where $\bar{\rho}$ is the mean cloud density. Give an estimate of $\rho_{\rm max}$.

A star of mass M_* forms in the cloud when a sufficient volume of gas of density $\rho_{\rm max}$ has accumulated that its gravitational self energy is comparable to its thermal energy. Give an estimate of M_*.

END OF PAPER