NATURAL SCIENCES TRIPOS Part II

Monday 1 June 2009 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

Before you begin read these instructions carefully.

Candidates may attempt all 6 questions.

Each question is divided into Part (i) *and Part* (ii), *which may or may not be related. Candidates may attempt either or both Parts.*

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STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1Y Theory of Relativity

(i) A particle of rest mass m_0 is moving at a uniform velocity $\mathbf{u}$ in an inertial frame defined by the coordinates $x^\mu = (ct, x, y, z)$. Explain why the quantity

$$\mathbf{P} = m_0 \mathbf{U},$$

is a four-vector, where

$$U^\mu = \frac{dx^\mu}{d\tau},$$

and τ is the proper time.

Show that the components of the four-momentum $\mathbf{P}$ are

$$\mathbf{P} = m_0(\gamma c, \gamma \mathbf{u}),$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

(ii) Explain the basis of Einstein's famous equation relating energy and mass,

$$E = mc^2.$$

A stationary particle of rest mass M_1 decays into a particle of rest mass M_2 and a photon. Calculate the energies of each of the decay products.

A photon collides with a stationary proton of rest mass M. Show that the threshold energy of the photon to produce a neutron of rest mass M and a pion of mass m is

$$E_\gamma = \frac{(m^2 + 2Mm)c^2}{2M}.$$

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Question 2Z Astrophysical Fluid Dynamics

(i) Show that the velocity u of a steady, spherically symmetric, isothermal wind moving in the potential of a point mass M has velocity u, where

$$(u^2 - c_s^2)\frac{d\log u}{dr} = \frac{2c_s^2}{r}\left[1 - \frac{GM}{2c_s^2 r}\right],$$

c_s is the sound speed and r is the distance from the point mass.

Hence determine the radius r_s at which the wind undergoes a sonic transition.

Using the Bernoulli equation, show that $u(r)$ satisfies

$$\frac{u^2}{2c_s^2} - \log\left(\frac{u}{c_s}\right) = 2\log\left(\frac{r}{r_s}\right) + \frac{2r_s}{r} - \frac{3}{2}.$$

Sketch the function $u(r)$.

(ii) The wind described in part (i) is now exposed to an ultraviolet radiation field at radii $\geq r_{IF}$ (where $r_{IF} > r_s$), which heats the gas to a high temperature. The ionisation front is modelled as a stationary contact discontinuity between gas with density, sound speed and flow velocity ρ_1, c_1 and u_1 and gas where these quantities take the values ρ_2, c_2 and u_2 respectively. Derive the expressions

$$\rho_1 u_1 = \rho_2 u_2,$$

and

$$\rho_1(u_1^2 + c_1^2) = \rho_2(u_2^2 + c_2^2).$$

Show that if $u_1 < c_1$ and $u_2 \approx c_2$, then

$$u_1 \approx \frac{c_1^2}{2c_2}.$$

Hence explain why the gas must undergo a shock inward of the ionisation front and sketch $u(r)$ in this case.

Question 3X Structure and Evolution of Stars

(i) The radii and the peak wavelength of the spectral energy distributions of several stars are given in the table below.

Star	Radius ($R_\odot$)	$\lambda_{\rm peak}$(nm)
Sp	5	110
Ve	1	300
Su	1	500
UU	100	1150

What are the effective temperatures of these stars?

Use the Stefan-Boltzman law to infer their luminosities and state whether they are likely to be main-sequence or evolved stars.

(ii) A star has a zero age main sequence mass of $1M_\odot$. Its luminosity on the main sequence is constant until it runs out of hydrogen in its core. At this time its mass is $0.99994M_\odot$. Calculate the star's main sequence lifetime.

Later the star evolves to the red giant branch (RGB) where it initially has a helium core with a mass of $M_c = 0.2M_\odot$ and a luminosity of $10L_\odot$ due to a thin hydrogen burning shell which surrounds the core. While on the RGB the star's luminosity obeys

$$L_{RGB} \propto M_c^8.$$

At the helium flash the star's core mass is $M_c = 0.5M_\odot$. By considering the growth of the helium core, or otherwise, calculate the star's RGB lifetime.

For the next phase of the star's evolution after the helium flash, state

(a) where the star appears in a colour-magnitude diagram

(b) what the star's energy source is and

(c) how long this phase lasts.

[For hydrogen burning the energy available from stellar material with the star's initial composition is 5×10^{14}Jkg^{-1}]

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Question 4Y Cosmology

(i) In a homogeneous and isotropic Universe, the scale factor $R(t)$ obeys the Friedmann equation

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho, \qquad (*)$$

where $\rho(t)$ is the matter density, k is a constant, and dots denote differentiation with respect to time t. Energy conservation requires that the density, $\rho(t)$, and pressure, $P(t)$, must satisfy

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right). \qquad (**)$$

Use (*) and (**) to derive the Raychaudhuri equation

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right). \qquad (***)$$

A spatially flat universe ($k = 0$) is filled with matter with an equation of state $P = w\rho c^2$, $w =$ constant. Show that if $w < -1/3$ an observer is surrounded by an event horizon.

(ii) Conformal time, τ, is defined by $d\tau = dt/R(t)$. Denote

$$\mathcal{H} = \frac{1}{R}\frac{dR}{d\tau} = \frac{R'}{R},$$

where primes denote differentiation with respect to τ. Show that the Friedmann and energy conservation equations (*) and (**) of part (i) for matter obeying the equation of state $P = w\rho c^2$, $w =$ constant, require

$$\mathcal{H}^2 + kc^2 = \frac{8\pi G}{3}\rho_0 R^{-(1+3w)},$$

where $\rho_0 = \rho(t_0)$, and $R(t_0)$ is normalized to unity at the present time t_0.

By using the Raychaudhuri equation (***) of part (i), or otherwise, show that

$$\mathcal{H}' + \frac{1}{2}(1+3w)[\mathcal{H}^2 + kc^2] = 0.$$

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By solving first for $\mathcal{H}$, or otherwise, show that for a closed universe with $kc^2 = 1$, the scale factor satisfies

$$R = \alpha \left(\sin \frac{(1+3w)}{2} \tau \right)^{2/(1+3w)} ,$$

where α is a constant.

For a closed universe composed of pressure-free matter ($P = 0$), find the complete parametric solution

$$R = \frac{1}{2}\alpha(1 - \cos\tau), \qquad t = \frac{\alpha}{2}(\tau - \sin\tau).$$

[You may assume that $\int dx/(1+x^2) = -\cot^{-1} x$.]

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Question 5Z Stellar Dynamics and Structure of Galaxies

(i) Show that the vertical density distribution of an isolated self-gravitating isothermal sheet is

$$\rho(z) = \rho_0 \operatorname{sech}^2\left(z\left[\frac{2\pi G\rho_0}{\sigma^2}\right]^{1/2}\right),$$

where $\rho_0 = \rho(z=0)$ is the midplane density and σ is the vertical velocity dispersion.

(ii) A star is moving in an axisymmetric potential $\Phi(R,z)$ where (R,z) are cylindrical polar coordinates. If the angular momentum component parallel to the symmetry axis is L_z, then the effective potential is

$$\Phi_{\text{eff}}(R,z) = \Phi(R,z) + \frac{L_z^2}{2R^2}.$$

Sketch $\Phi_{\text{eff}}(R,0)$ and mark the location of the circular orbit.

Suppose that the star is moving on a nearly circular orbit in the equatorial plane ($z=0$). Show that the star's motion is well-approximated by the superposition of retrograde motion at angular frequency κ around an ellipse, and prograde motion of the ellipse's centre at angular frequency Ω around a circle.

Show further that the frequencies are related by

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2,$$

and find the axis ratio of the ellipse.

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Question 6X Topics in Astrophysics

(i) A globular cluster lies at a distance of 5.2 kiloparsecs from the Sun. The components of its proper motion, parallel and perpendicular to the Galactic plane, are $\mu_{\rm par} = 0.26$ arcsec per century and $\mu_{\rm perp} = -0.74$ arcsec per century. Show that the transverse component of velocity, v_t, of the globular cluster relative to the Sun is comparable to the observed heliocentric radial velocity $v_r = 233$ km s^{-1}.

Assuming the globular cluster is approximately on a circular orbit of radius 3 kiloparsecs, estimate the mass of the Galaxy within this orbit.

(ii) A non-rotating spherical star cluster of radius R and at a distance D is moving transverse to the line-of-sight with velocity v_t. Show that measurements of radial velocities of stars in the cluster give the impression that it is rotating with angular velocity $\Omega \sim v_t/D$.

Draw a diagram illustrating the relationship between the line-of-sight, the transverse velocity vector, the apparent rotation axis and the apparent rotation direction.

The inner parts of the globular cluster of part (i) display a radial velocity difference of 20 km s^{-1} across a diameter of 20 arcminutes. Use the information in part (i) to determine whether this is caused by a real rotation of the cluster.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Wednesday 3 June 2009 09:00am – 12.00pm

ASTROPHYSICS - PAPER 3

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Question 1Y Theory of Relativity

(i) A particle moves purely under the action of gravity. Let us construct a local freely falling coordinate system ξ^μ,

$$ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu, \qquad \left(\begin{array}{rl} \eta_{\mu\nu} = (+1,-1,-1,-1), & \mu = \nu \\ \eta_{\mu\nu} = 0, & \mu \neq \nu \end{array} \right).$$

Explain carefully why it is justifiable to describe the equation of motion of the particle by

$$\frac{d^2 \xi^\mu}{d\tau^2} = 0.$$

By transforming to a general coordinate system x^μ, show that the equation of motion of the particle is

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \qquad (*)$$

and derive an expression for the affine connection $\Gamma^\lambda_{\mu\nu}$ in terms of the coordinates x^μ and ξ^μ.

(ii) Consider the motion of a slowly moving particle ($v \ll c$) in a static gravitational field. The gravitational field is weak, so the space-time metric is nearly Minkowski with metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

Show that the equation of motion of the particle (∗) of part (i) is well-approximated by the Newtonian equation

$$\frac{d^2 \mathbf{x}}{dt^2} = -\nabla \phi,$$

where $\phi = c^2 h_{00}/2$ is the Newtonian potential.

A satellite containing an atomic clock is in geostationary orbit at a radius of 42,000 km. Estimate the number of orbits that the satellite must traverse for its clock to gain one second relative to an identical clock on Earth.

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Question 2Z Astrophysical Fluid Dynamics

(i) An accretion disc around a star is vertically isothermal with a temperature $T(R)$ and surface density $\Sigma(R)$ as a function of radius R. Show that if the disc is supported by gas pressure, then the mid-plane density ρ and the disc scale-height H behave like

$$\begin{aligned} H &\propto \frac{T^{1/2}}{\Omega}, \\ \rho &\propto \frac{\Sigma\Omega}{T^{1/2}}, \end{aligned}$$

where Ω is the Keplerian angular velocity.

Derive the condition on $T(R)$ such that

$$\frac{d}{dR}\left(\frac{H}{R}\right) > 0.$$

(ii) The accretion disc of part (i) is heated by viscous dissipation at a rate per unit area

$$Q^+ \propto \nu\Sigma\Omega^2,$$

where $\nu \propto c_s H$ is the kinematic viscosity and c_s is the sound speed. In thermal equilibrium, this is balanced by bremstrahlung cooling at a rate per unit area

$$Q^- \propto \kappa\Sigma T^4,$$

where $\kappa \propto \rho T^{-3.5}$ is the opacity.

Show that

$$Q^+ \propto \Sigma T\Omega, \qquad\qquad Q^- \propto \Sigma^2\Omega,$$

and that in thermal equilibrium

$$\nu\Sigma \propto \frac{\Sigma^2}{\Omega}.$$

The thermal stability of an accretion disc is determined by the sign of

$$\frac{\partial \log Q^+}{\partial \log T}\bigg|_{\Sigma,R} - \frac{\partial \log Q^-}{\partial \log T}\bigg|_{\Sigma,R}.$$

Determine the thermal stability of the disc and provide a physical interpretation of your result.

[TURN OVER

Question 3X Statistical Physics

(i) Show that

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

where V is the volume, T is the temperature, S is entropy and p is the pressure.

Show that

$$\left(\frac{\partial T}{\partial p}\right)_H = \frac{1}{C_p}\left[T\left(\frac{\partial V}{\partial T}\right)_p - V\right]$$

where H is the enthalpy and the specific heat capacity at constant pressure is $C_p = T(\partial S/\partial T)_p$.

(ii) The Van der Waals equation of state for a non-ideal gas is

$$\left(p + \frac{aN^2}{V^2}\right)(V - bN) = NkT$$

where a and b are constants and N is the number density of the gas particles. Briefly explain the physical motivation for differences between the Van der Waals and ideal gas equations of state.

Find the volume dependence (at constant temperature) of the internal energy, and the heat capacity $C_{\rm V}$ at constant volume of a Van der Waals gas.

[CONTINUED...

Question 4X Structure and Evolution of Stars

(i) The pressure P of a gas is given by

$$P = \frac{1}{3}\int_0^\infty vpf(p)dp,$$

where p is the momentum of a gas particle, v is the speed of a gas particle and $f(p)dp$ is the number density of particles with momentum between p and $p + dp$. Show that for a free electron gas the pressure is given by

$$P = \frac{8\pi c}{3h^3}\int_0^{p_{\mathrm{F}}} \frac{p/m_e c}{\sqrt{1 + p^2/m_e^2 c^2}} p^3 dp, \qquad (*)$$

where m_e is the rest mass of an electron and p_F is the Fermi momentum.

(ii) From the general expression for the pressure of a free electron gas $(*)$ of part (i) show that in the non-relativistic limit

$$P = \frac{8\pi}{15 m_e h^3} p_F^5.$$

Show further that in the ultra-relativistic limit

$$P = \frac{2\pi c}{3h^3} p_F^4.$$

Derive the critical density for the transition between the above two extremes.

Explain why the radius of a degenerate, non-relativistic white dwarf becomes smaller as its mass increases.

Why is there a maximum stable mass in the ultra-relativistic case?

[TURN OVER

Question 5Y Cosmology

(i) The cosmic microwave background (CMB) radiation is well described as a blackbody, for which the photon number density distribution is

$$N_\gamma(\mathbf{p})\, d^3\mathbf{p} = \frac{2}{h^3} \frac{d^3\mathbf{p}}{\left[\exp\left(\frac{pc}{kT}\right) - 1\right]}, \qquad (*)$$

where $\mathbf{p}$ is the photon momentum, T is the temperature and k is the Boltzmann constant. Show that the energy density contributed by blackbody radiation is

$$\rho_\gamma c^2 = \frac{8\pi^5}{15} \frac{(kT)^4}{(hc)^3}.$$

Energy conservation in an expanding Friedmann-Robertson-Walker universe for material with mean mass density $\rho(t)$ and pressure $P(t)$ can be written as

$$\frac{d(\rho R^3)}{dR} = -3\frac{P}{c^2}\, R^2, \qquad (**)$$

where $R(t)$ is the scale factor. Show that for radiation, (**) requires

$$T(t) \propto \frac{1}{R(t)}.$$

Explain carefully why the blackbody nature of the CMB provides strong evidence in support of the Hot Big Bang model.

(ii) The photon distribution $N_\gamma(\mathbf{p})$ of part (i) transforms as a scalar under Lorentz transformations. Show that an observer moving with a velocity $\mathbf{v}$ through the CMB sees blackbody radiation with a temperature

$$T' = \frac{T}{\gamma(1+\beta\cos\theta)}, \qquad \beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}},$$

where θ is the angle between a CMB photon and the observer's velocity $\mathbf{v}$.

Show that in the limit $\beta \ll 1$,

$$\frac{\Delta T}{T} = \frac{T' - T}{T} = -\frac{1}{6}\beta^2 - \beta P_1(\cos\theta) + \frac{2}{3}\beta^2 P_2(\cos\theta) + \ldots, \qquad (***)$$

[CONTINUED...

and so the measurable effect of the observer's motion is a kinematic dipole anisotropy (term proportional to $P_1(\cos\theta)$) together with a much smaller kinematic quadrupole anisotropy (term proportional to $P_2(\cos\theta)$).

Observations of the CMB have revealed a kinematic dipole with $\beta = 0.0013$. Discuss possible contributions to the Earth's motion through the CMB.

The observed quadrupole anisotropy of the CMB has an amplitude of $\sim 20\mu$K. Why does this differ from the predictions of (***)?

[You may assume that
$\int_0^\infty x^3[e^x - 1]^{-1}dx = \pi^4/15, \qquad P_1(\mu) = \mu, \qquad P_2(\mu) = (3\mu^2 - 1)/2.$]

[TURN OVER

Question 6Z Stellar Dynamics and Structure of Galaxies

(i) Explain what is meant by the term *integral of the motion.*

A particle with position vector $\mathbf{r}$ is moving in a spherically symmetric potential. Show that the angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$$

is an integral of the motion, where the dot denotes differentiation with respect to time.

Show further that the orbit is confined to a plane.

Now suppose the particle is moving in the Keplerian potential given by

$$\phi = -\frac{GM}{r},$$

where $r = |\mathbf{r}|$. Show that the Laplace vector

$$\mathbf{A} = \dot{\mathbf{r}} \times \mathbf{L} - \frac{GM}{r}\mathbf{r},$$

is an integral of the motion.

Why does this imply that every bound orbit is closed and periodic?

(ii) Suppose that a star with energy E and total angular momentum L is moving in a spherically symmetric potential $\phi(r)$. Using polar coordinates (r, θ) in the orbital plane, show that the orbit may be written as

$$\theta = \int \frac{L dr}{r^2\sqrt{2E - 2\phi - L^2/r^2}}.$$

Suppose that a star falls from a finite distance towards the centre of a strongly cusped potential

$$\phi = -\frac{\alpha}{r^n}, \qquad n \geq 2,$$

where α is a constant. Show that close to the centre, the orbit takes the form

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$$\theta \approx \begin{cases} \dfrac{L \log(r/r_0)}{\sqrt{2\alpha - L^2}} + \theta_0, & \text{if } n = 2, \\ \\ \dfrac{2L}{\sqrt{2\alpha}(2-n)} r^{-1+\frac{n}{2}} + \theta_0, & \text{if } n \neq 2, \end{cases}$$

where r_0 and θ_0 are constants.

In each case, sketch the orbit close to the centre.

As $L \to 0$, does the star make a finite number of revolutions around the centre?

[TURN OVER

Question 7X Topics in Astrophysics

(i) A self-gravitating gas cloud has mass M, radius R and average temperature T. Estimate the ratio $\alpha = U/\Omega$ in terms of the cloud parameters M, R and T, where U is the internal thermal energy and Ω is the gravitational self-energy.

Show that if the cloud is isothermal, $\alpha \propto R$.

What does this imply for the stability of isothermal self-gravitating clouds?

(ii) A giant molecular cloud has mass $M_{\rm cl} = 3 \times 10^5 M_\odot$ and a radius of $R_{\rm cl} = 20$ pc. The gas temperature is $T_{\rm cl} = 20$ K. Assume that the cloud is supported against collapse by a tangled magnetic field of mean magnitude B_t. Give an estimate of B_t.

The cloud is very inhomogeneous and the densest clumps in the cloud have a density $\rho_{\rm max}$ such that their thermal pressure, $\rho_{\rm max} c_s^2$ is comparable to their turbulent pressure $\overline{\rho} u_t^2$ (c_s is the sound speed, $\overline{\rho}$ is the mean cloud density and u_t is a characteristic turbulent velocity within the cloud). Show that

$$\rho_{\rm max} \sim \overline{\rho}\frac{u_t^2}{c_s^2},$$

and estimate its value, assuming in this case that the pressure support for the cloud comes from the turbulent motion of the clumps.

A star of mass M_* forms in the cloud when a sufficient clump of gas of density $\rho_{\rm max}$ has accumulated so that its gravitational self-energy is comparable to its thermal energy. Show that

$$M_* = \sqrt{\frac{c_s^6}{\rho_{\rm max} G^3}},$$

and estimate its value.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Thursday 4 June 2009 1:30pm – 4:30pm

ASTROPHYSICS - PAPER 4

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Question 1Y Theory of Relativity

(i) The Schwarzchild metric describing space-time around a point mass M at $r = 0$ can be written as

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \qquad (*)$$

where (r, θ, ϕ) are spherical polar coordinates. Draw a diagram showing the light cone structure of the metric $(*)$.

What is the physical significance of the radius $r_s = 2GM/c^2$?

Show that, as measured by an observer at infinity, an ingoing radial light ray takes an infinite time t to reach the radius r_s.

(ii) For the metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (*) of part (i), show that the proper distance ρ to the horizon r_s,

$$\rho = \int_{r_s}^{r} \sqrt{g_{rr}(r')}\, dr',$$

is given approximately by

$$\rho \approx 2 r_s^{1/2} \sqrt{(r - r_s)},$$

when close to the horizon $(r - r_s \ll r_s)$.

Now replace the angular coordinates (θ, ϕ) of the metric $(*)$ with Cartesian coordinates

$$X = r_s \theta \cos \phi,$$
$$Y = r_s \theta \sin \phi,$$

and introduce a dimensionless time variable $\omega = ct/(2r_s)$. Show that close to the horizon and for small angular separations from the pole $\theta = 0$, the Schwarzchild metric $(*)$ can be written as

$$ds^2 = \rho^2 d\omega^2 - d\rho^2 - dX^2 - dY^2. \qquad (**)$$

Show that the Minkowski metric

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2,$$

is equivalent to $(**)$, where

[CONTINUED...

$$T = \rho \sinh \omega,$$
$$Z = \rho \cosh \omega.$$

What does this result imply concerning the singularity of the Schwarzchild metric $(*)$ at $r = r_s$?

[TURN OVER

Question 2Z Astrophysical Fluid Dynamics

(i) Gas with a constant adiabatic index γ enters a stationary, adiabatic shock front. Explaining clearly any assumptions involved, derive the Rankine-Hugoniot conditions.

Show that the density ρ_1 and pressure p_1 upstream of the shock are related to the density ρ_2 and pressure p_2 downstream by

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2}.$$

(ii) An adiabatic shock as described in part (i) is extremely weak so that

$$\frac{\rho_2}{\rho_1} = 1 + \frac{\Delta\rho}{\rho_1}, \qquad \frac{\Delta\rho}{\rho_1} \ll 1,$$

and that

$$\frac{p_2}{p_1} = 1 + \frac{\Delta p}{p_1}, \qquad \frac{\Delta p}{p_1} \ll 1.$$

Show that

$$\frac{\Delta\rho}{\rho_1} = \frac{1}{\gamma}\frac{\Delta p}{p_1} - \frac{\gamma-1}{2\gamma^2}\left(\frac{\Delta p}{p_1}\right)^2.$$

Hence, show that the change in specific entropy across the shock is

$$\frac{\Delta p}{p_1} - \gamma\frac{\Delta\rho}{\rho_1} \propto \left(\frac{\Delta\rho}{\rho_1}\right)^2.$$

Gas on circular orbits in a disc galaxy enters weak, adiabatic, spiral shocks. Suppose that the energy dissipated in each shock is balanced by radiative cooling before the gas enters the next shock. If the energy radiated between the shocks is doubled and the energy balance is maintained, how does the shock amplitude change?

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Question 3X Statistical Physics

(i) In a system of N weakly interacting atoms at temperature T, the non-degenerate energy levels available to each atom are ϵ_0, ϵ_1, ϵ_2, $\ldots \epsilon_n$. Assuming that the numbers of atoms in each state are distributed according to the Boltzmann distribution $n_i = a\exp(-\epsilon_i/kT)$, where a is a constant, show that the total energy of the system E is given by

$$E = NkT^2 \frac{\partial}{\partial T}(\ln Z),$$

where Z is the partition function,

$$Z = \sum_i \exp(-\epsilon_i/kT).$$

(ii) Consider an ideal Bose gas in an external potential

$$V(x, y, z) = A(x^2 + y^2 + 4z^2)^{1/2},$$

where A is a positive constant. The resulting density of single particle states is given by

$$g(\epsilon) = B\epsilon^{7/2},$$

where B is a positive constant. Derive the critical temperature for Bose-Einstein condensation of a gas of N of these atoms.

What is the internal energy E of the gas in the condensed state as a function of N and T?

Now consider the high temperature limit. How does the internal energy E depend on N and T?

[You may assume that

$$\int_0^\infty \frac{x^{n-1}dx}{e^x - 1} = \Gamma(n)\zeta(n),$$

where $\Gamma(n)$ is the gamma function and $\zeta(n)$ is the Riemann zeta function.]

[TURN OVER

Question 4X Structure and Evolution of Stars

(i) The inter-pulse luminosity in solar units of an assymptotic giant branch star is

$$L_{\rm AGB} \approx 5.925 \times 10^4 (M_{\rm core} - 0.51),$$

where $M_{\rm core}$ is the mass of the star's core in solar units. The Eddington luminosity $L_{\rm Edd}$ (in solar units) for a star of mass M (in solar units) and opacity κ dominated by electron scattering is

$$L_{\rm Edd} = \frac{4\pi c G M}{\kappa} \left(\frac{M_\odot}{L_\odot} \right) = 37000M$$

Stating your assumptions clearly, find the critical core mass to ensure mass loss.

(ii) Provide a short paragraph giving a qualitative physical explaination for each of the following:

(a) How does the radius of a $3M_\odot$ main sequence star change if its opacity is increased by a small amount?

(b) How would this star's luminosity change if the helium abundance were increased by a small amount?

(c) How does the star's main sequence lifetime change if mass loss from the stellar surface is enhanced?

(d) How does a main sequence star of mass $\sim 25M_\odot$ evolve and what are its possible endpoints?

[CONTINUED...

Question 5Y Cosmology

(i) The main reaction responsible for maintaining hydrogen and radiation in equilibrium is

$$p + e^- \rightleftharpoons H + \gamma.$$

Assume that the particle densities are given by the non-relativistic Maxwell-Boltzmann formula,

$$n_i = g_i \left(\frac{2\pi m_i c^2 kT}{h^2}\right)^{3/2} \exp\left(\frac{\mu_i - m_i c^2}{kT}\right)$$

where g_i, μ_i and m_i are respectively the number of spin states, chemical potential and mass of particle i, and k is the Boltzmann constant. Show that

$$\frac{n_e n_p}{n_H} \approx \left(\frac{2\pi kT m_e c^2}{h^2}\right)^{3/2} \exp\left(-\frac{B_H}{kT}\right),$$

where $B_H = 13.6\text{eV} \equiv (158000\text{K})/k$ is the binding energy of hydrogen.

Show that the ionization fraction $X_e = n_e/(n_e + n_H)$ is approximately

$$\frac{X_e^2}{1 - X_e} \approx \left((1 - Y_{\text{He}})\frac{\rho_B}{m_p}\right)^{-1} \left(\frac{2\pi kT m_e c^2}{h^2}\right)^{3/2} \exp\left(-\frac{B_H}{kT}\right), \qquad (*)$$

where ρ_{B} is the mean baryon density and Y_{He} is the helium abundance by mass.

Why does hydrogen recombination occur at a significantly lower temperature than B_H/k?

Explain physically why (*) is a relatively poor description of the recombination process.

(ii) The optical depth to Thomson scattering along a line of sight back to redshift z is

$$\tau_{\text{opt}}(z) = \int_{t_(z)}^{t_0} \sigma_T n_e c \, dt,$$

where t_0 is the time at the present day, n_e is the mean density of free electrons and σ_T is the Thomson cross-section. Assume that the Universe is described by an Einstein-de Sitter model with baryon density parameter $\Omega_B(0)$ at the present day. Show that if the intergalactic medium is fully reionized at redshift z, and remains fully ionized until the present day,

[TURN OVER

$$\tau_{\rm opt}(z) \approx \frac{H_0 \sigma_T c}{4\pi G m_p}\left(1 - \frac{Y_{\rm He}}{2}\right)\Omega_B(0)[(1+z)^{3/2} - 1],$$

where H_0 is the Hubble constant at the present day.

Estimate the redshift of reionization required to produce an optical depth of $\tau_{\rm opt} = 0.1$ and comment on whether quasars could have produced the UV photons necessary to reionize the Universe at this redshift.

Discuss the effects of reionization on fluctuations in the cosmic microwave background radiation.

Can observational signatures of late time reionization and the recombination process described in part (i) be used to constrain the helium abundance $Y_{\rm He}$?

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Question 6Z Stellar Dynamics and Structure of Galaxies

(i) Suppose a collisionless, steady-state, stellar system with density ρ is moving in a gravitational potential ϕ. The Jeans equations take the form

$$\rho\langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i}+\frac{\partial(\rho\sigma_{ij}^2)}{\partial x_i}=-\rho\frac{\partial\phi}{\partial x_j},$$

where angled brackets denote averages over the distribution function and the velocity dispersion tensor $\sigma_{ij}^2=\langle(v_i-\langle v_i\rangle)(v_j-\langle v_j\rangle)\rangle$. Derive the virial theorem in the form

$$2T_{ij}+W_{ij}=0,$$

where T_{ij} is the total kinetic energy tensor

$$T_{ij}=\frac{1}{2}\int\rho\sigma_{ij}^2d^3x+\frac{1}{2}\int\rho\langle v_i\rangle\langle v_j\rangle d^3x,$$

and W_{ij} is the potential energy tensor

$$W_{ij}=\int\rho x_i\frac{\partial\phi}{\partial x_j}d^3x.$$

Demonstrate that W_{ij} is a symmetric tensor, that is, $W_{ij}=W_{ji}$.

(ii) For a spherical stellar system, show that the trace of the potential energy tensor W is

$$W=-4\pi G\int_0^\infty r\rho(r)M(r)dr,$$

where $M(r)$ is the mass enclosed within spherical polar radius r.

Show further that the surface brightness $I(R)$ as a function of projected radius R from the centre and luminosity density $j(r)$ of a spherical galaxy are related by

$$I(R)=2\int_R^\infty\frac{j(r)rdr}{\sqrt{r^2-R^2}}.$$

This is Abel's integral equation, whose inversion may be assumed to be

$$j(r)=-\frac{1}{\pi}\int_r^\infty\frac{dI}{dR}\frac{dR}{\sqrt{R^2-r^2}}.$$

Now, let us define the strip brightness $S(x)$ such that $S(x)dx$ is the luminosity in a strip of width dx passing a radius x from the projected centre of a spherical galaxy. Show that

[TURN OVER

$$S(x) = 2\int_x^\infty \frac{I(R)RdR}{\sqrt{R^2 - x^2}}.$$

Hence show that the strip brightness and the luminosity density are related by

$$j(x) = -\frac{1}{2\pi x}\frac{dS}{dx}.$$

If the mass-to-light ratio is Υ , show that the mass enclosed within radius r is

$$M(r) = -2\Upsilon \int_0^r x\frac{dS}{dx}dx.$$

Finally, show that the gravitational potential energy may be written as

$$W = -2G\Upsilon^2 \int_0^\infty [S(x)]^2 dx.$$

[CONTINUED...

Question 7X Topics in Astrophysics

(i) The Sun's luminosity is produced by the conversion of hydrogen to helium. In the process, 0.7% of the hydrogen's rest mass energy is released. Estimate the number of hydrogen nuclei that are converted in each second.

If the nuclear reactions involved in the conversion release $\sim$3 neutrinos per hydrogen nucleus consumed, and if the neutrino collision cross-section is negligibly small, estimate the total number of neutrinos that pass through a typical elephant during its lifetime stating your assumptions clearly.

Would an elephant make a good neutrino detector?

(ii) Consider a thin spherical shell of radius r which contains particles. Each particle moves a typical distance $\lambda \quad (\ll r)$ in a random direction in time $\triangle t$. Show that, on average, the distance of a particle from the centre of the shell increases by approximately λ^2/r in time $\triangle t$.

Show that the average time for a photon to escape from the centre of the Sun is $\sim NR_\odot/c$ where $R_\odot$ is the solar radius and N is the number of mean free paths between the centre and the surface of the Sun.

The timescale on which the Sun radiates away its internal energy is $t_{\rm KH} \sim (GM_\odot^2)/(R_\odot L_\odot)$ where $M_\odot$ is the mass of the Sun and $L_\odot$ is the luminosity of the Sun. The ratio of radiation pressure to gas pressure in the Sun is ~ 0.01. Estimate the average time taken by a photon to leak out from the centre of the Sun and hence estimate the average photon mean free path.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Friday 5 June 2009 09:00am – 12:00pm

ASTROPHYSICS - PAPER 5

Before you begin read these instructions carefully.

Candidates may attempt not more than 6 questions.

Each question is divided into Part (i) *and Part* (ii)*, which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i)*. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1Y and 5Y** *should be in one bundle and* **2Z and 6Z** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS
Script Paper
Blue Cover Sheets
Yellow Master Cover Sheets
1 Rough Work Pad
Tags

SPECIAL REQUIREMENTS
Formulae Booklet
Approved Calculators Allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1Y Theory of Relativity

(i) Consider the hyperboloid

$$-z^2 + x^2 + y^2 = \mathcal{R}^2,$$

embedded in three-dimensional Minkowski space with metric

$$ds^2 = dz^2 - dx^2 - dy^2.$$

By transforming to the variables χ and t, where

$$x = \mathcal{R}\,\frac{\cos\chi}{\cos t}, \qquad y = \mathcal{R}\,\frac{\sin\chi}{\cos t},$$

show that the metric of the hyperboloid takes the form

$$ds^2 = \frac{\mathcal{R}^2}{\cos^2 t}\,[dt^2 - d\chi^2]. \qquad (*)$$

Draw a diagram of the hyperboloid in the Cartesian coordinate system (x, y, z).

Sketch the path of a photon emitted at $t = 0$, $\chi = 0$.

(ii) For the metric $(*)$ of part (i), show that the one independent component of the curvature tensor is

$$R^0{}_{101} = \sec^2\mathrm{t}, \qquad (\mathrm{x}^0 = \mathrm{t}, \mathrm{x}^1 = \chi),$$

and hence

$$R_{0101} = \frac{\mathcal{R}^2}{\cos^4\mathrm{t}}.$$

Show that this is consistent with the formula

$$R_{\sigma\mu\lambda\nu} = \frac{1}{\mathcal{R}^2}(g_{\sigma\nu}g_{\lambda\mu} - g_{\sigma\lambda}g_{\mu\nu}). \qquad (**)$$

For a space-time of N dimensions and a curvature tensor given by $(**)$, show that the field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0,$$

[CONTINUED...

are satisfied if

$$\Lambda = \frac{(N-1)(N-2)}{2\mathcal{R}^2}.$$

Give a physical explanation of this result.

$$\left[\begin{array}{l} \text{You may assume that} \\ R^{\rho}{}_{\sigma\mu\nu} = \partial\Gamma^{\rho}_{\nu\sigma}/\partial x^{\mu} - \partial\Gamma^{\rho}_{\sigma\mu}/\partial x^{\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \\ R_{\mu\nu} = R^{\lambda}{}_{\mu\nu\lambda}. \end{array}\right]$$

[**TURN OVER**

Question 2Z Astrophysical Fluid Dynamics

(i) Define the terms 'particle path' and 'streakline'.

Under what circumstances do the particle paths and the streaklines coincide.

A planet which orbits a star on a circular orbit with speed v emits a retrograde jet of ballistic particles with speed u ($\ll v$) with respect to the planet and tangential to the planet's orbit. On a single diagram, sketch a particle path and a streakline in a frame co-rotating with the planet.

On a second diagram, sketch a particle path and a streakline in the inertial frame of the star.

(ii) The Lane-Emden equation for the hydrostatic equilibrium of a self-gravitating polytropic sphere with polytropic index $n = 1$ is

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\Theta}{d\zeta}\right) = -\Theta,$$

where Θ and ζ are related to the density ρ, potential ψ and radius r via

$$\zeta = \left(\frac{4\pi G\rho_c}{\psi_{\mathrm{b}} - \psi_{\mathrm{c}}}\right)^{1/2} r, \qquad \rho = \rho_{\mathrm{c}}\Theta.$$

Here, the suffices 'c' and 'b' denote values at the centre and at the outer boundary respectively. Show that

$$\Theta = \frac{\sin\zeta}{\zeta},$$

justifying your choice of boundary conditions.

The inner regions with $\zeta < \zeta_{\mathrm{p}}$ are removed and replaced with a rocky planet with the same mass M_{p} and radius R_{p} as the removed gas. The overlying polytropic gas constitutes the atmosphere of the planet and has mass M_{a} and radius R_{a}. Write down expressions for

$$\frac{R_{\mathrm{a}}}{R_{\mathrm{p}}}, \qquad \text{and} \qquad \frac{M_{\mathrm{a}}}{M_{\mathrm{p}}},$$

in terms of ζ_{p}.

Show that in the limit $M_{\mathrm{a}} \ll M_{\mathrm{p}}$,

[CONTINUED...

$$\frac{R_\mathrm{a}}{R_\mathrm{p}} \approx 1 + \frac{1}{\pi}\sqrt{\frac{2M_\mathrm{a}}{M_\mathrm{p}}}.$$

How does the density of the atmosphere decline with height above the planet's surface?

[TURN OVER

Question 3X Statistical Physics

(i) A box of volume V contains particles with number density n and root mean square speed $\bar{c}$. Show that the number of particles striking the sides of the box per unit area per unit time is

$$F = \frac{1}{4} n \bar{c}.$$

A small hole of area A is made in the box at time $t = 0$, when the number density is n_0. Assuming that there is a vacuum outside the box, find $N(t)$, the total number of particles in the box as a function of time for $t > 0$.

(ii) Consider an ideal, classical, monatomic gas in the presence of a uniform gravitational field in the z direction of a Cartesian coordinate system (x, y, z). Assume that the gas is in a large cubic box. Compute the internal energy of the gas.

What is the probability that an atom is located at a height between z and $z + dz$?

A planet has a radius $R = 6000$ km and an atmosphere of molecular nitrogen with temperature $T = 300$ K. What is the minimum planetary mass required for this atmosphere to be retained?

[CONTINUED...

Question 4X Structure and Evolution of Stars

(i) Describe the dominant energy transport mechanism close to the centre, at the half mass radius, and just below the photosphere, for each of the following stars

(a) A $0.1M_\odot$ main sequence star

(b) A $1M_\odot$ main sequence star

(c) A $2M_\odot$ main sequence star

(d) A $0.5M_\odot$ white dwarf

(ii) In the outer layers of a star the opacity κ is described by a power law in pressure P and temperature T,

$$\kappa = \kappa_0 P^{(\alpha-1)} T^{(4-\beta)},$$

where κ_0, α and β are constants. In these layers there is no energy generation and energy transport is radiative. The mass contained in these layers is negligible compared to the mass of the star. Show that

$$\frac{dT}{dP} = AP^{(\alpha-1)}T^{(1-\beta)}$$

and find the constant A in terms of the star's radius, mass and effective temperature.

Give appropriate boundary conditions for this equation and hence determine $T(P)$ in the outer layers of the star.

The adiabatic temperature gradient is

$$\frac{P}{T}\frac{dT}{dP} = \frac{2}{5}.$$

Show that $\alpha < \frac{2}{5}\beta$.

[TURN OVER

Question 5Y Cosmology

(i) In linear perturbation theory of a Friedmann-Robertson-Walker universe, a Fourier mode $\delta_{\mathbf{k}}$ of the (pressureless) matter overdensity obeys the equation,

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{R}}{R}\dot{\delta}_{\mathbf{k}} - 4\pi G\overline{\rho}_m \delta_{\mathbf{k}} = 0, \qquad (*)$$

where dots denote differentiation with respect to time, $R(t)$, is the scale factor, $\overline{\rho}_m$ is the mean matter density and $\mathbf{k}$ is the wavenumber, which is assumed to be much greater than the Hubble radius ct. Assume that the universe is described by the Einstein-de Sitter model and that the scale factor R satisfies,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}(\overline{\rho}_m + \overline{\rho}_R),$$

where $\overline{\rho}_R$ is the mean density of radiation. Assume further that the radiation is uniform and that the matter interacts with the radiation only via gravity. Show that (*) can be written as

$$\frac{d^2\delta_{\mathbf{k}}}{d\eta^2} + \frac{(2+3\eta)}{2\eta(1+\eta)}\frac{d\delta_{\mathbf{k}}}{d\eta} - \frac{3}{2}\frac{\delta_{\mathbf{k}}}{\eta(1+\eta)} = 0, \qquad (**)$$

where $\eta = \overline{\rho}_m/\overline{\rho}_R$.

Show that the growing mode solution to (**) is

$$\delta_{\mathbf{k}} \propto 1 + \frac{3}{2}\eta.$$

Describe briefly the implications of this result for the growth of fluctuations in a universe dominated by weakly interacting dark matter at the present day.

(ii) The photon entropy per baryon is

$$S = \frac{4aT^3}{3kn_B},$$

where a is the radiation constant, k is the Boltzmann constant, n_B is the number density of baryons and T is the radiation temperature. Show that adiabatic perturbations in the baryons and radiation must satisfy

[CONTINUED...

$$\frac{3}{4}\delta_\gamma = \delta_B, \qquad \delta_\gamma = \frac{(\rho_\gamma - \overline{\rho}_\gamma)}{\overline{\rho}_\gamma}, \qquad \delta_B = \frac{(\rho_B - \overline{\rho}_B)}{\overline{\rho}_B},$$

where ρ_γ and ρ_B are the densities of radiation and baryons respectively and $\overline{\rho}_\gamma$ and $\overline{\rho}_B$ are their mean densities.

Prior to recombination, baryons and electrons are tightly coupled by Thomson scattering. Show that the adiabatic sound speed of the radiation-baryon fluid is

$$c_s = \frac{c}{\sqrt{3}} \left(\frac{3}{4} \frac{\overline{\rho}_B}{\overline{\rho}_\gamma} + 1 \right)^{-1/2}.$$

Estimate the Jeans mass

$$M_J = \frac{4\pi}{3} \overline{\rho}_B \lambda_J^3, \qquad \lambda_J = c_s \left(\frac{\pi}{G\overline{\rho}_B} \right)^{1/2},$$

at the time of recombination $T_{\rm rec} \approx 3000$K for a universe with baryon density parameter $\Omega_B = 0.06$ at the present day.

How does this mass compare with the masses of typical galaxies and rich clusters of galaxies?

[TURN OVER

Question 6Z Stellar Dynamics and Structure of Galaxies

(i) The relaxation time of a stellar system of N stars is

$$T_{\rm relax} \approx 0.1 \frac{N}{\log N} T_{\rm cross},$$

where $T_{\rm cross}$ is the crossing time. Estimate the relaxation time for a galaxy.

Hence, explain why the motion of stars in galaxies can be described by the *collisionless* Boltzmann equation and derive the Jeans theorem.

Suppose the phase space distribution function f depends on binding energy E alone. Show that the velocity dispersion tensor is isotropic.

Now suppose that the phase space distribution function of a spherical system depends on the binding energy E and the modulus of the angular momentum vector L via

$$f(E, L) = L^{-2\beta} g(E),$$

where β is a constant and g is an arbitrary function. Demonstrate that the ratio of the radial velocity dispersion to the tangential velocity dispersion is constant.

(ii) A galaxy with finite total mass M has the potential

$$\phi = -\frac{GM}{r + a},$$

where a is a length-scale and r is the spherical polar radius. Find the rotation curve of the model. Using Poisson's equation, find the mass density of the model.

By introducing spherical polar coordinates in velocity space, verify that the model has a distribution function

$$f(E, L) = \frac{CE^2}{L},$$

where C is a constant to be identified.

What is the ratio of the radial velocity dispersion to the tangential velocity dispersion in the model?

[CONTINUED...

Question 7X Topics in Astrophysics

(i) A point mass M is at a distance d from the Earth whilst a background star is at a distance $2d$. The small angle between these two objects at the Earth is β. A light ray from the star passes to within a distance h from the point mass and is gravitationally deflected by an angle

$$\alpha = \frac{4GM}{c^2 h}.$$

Show that an observer on Earth generally sees two images of the star, which are at apparent angular distances θ_1 and θ_2 from the point mass, where θ_1 and θ_2 are the roots of the equation

$$\theta^2 - \beta\theta - \frac{2GM}{c^2 d} = 0.$$

(ii) For the situation outlined in part (i) sketch a graph which shows how the angular distances θ_1 and θ_2 vary with β/ϵ where $\epsilon^2 = 2GM/c^2 d$.

What does an observer on Earth see when $\beta = 0$?

Now consider the lensing object to be extended rather than a point mass. Strong lensing (i.e. significant image distortion and brightness amplification) occurs when $\beta \leq \epsilon$. Show that for this to occur, the surface mass density Σ of the lensing object must be such that

$$\Sigma \geq \frac{c^2}{2\pi G d}.$$

Which of the following objects can produce strong lensing:

a) an extra-solar Earth-like planet with $d = 1$ kpc

b) a globular cluster with mass $\sim 10^5 M_\odot$, radius ~ 20 pc and $d = 10$ kpc.

END OF PAPER