NATURAL SCIENCES TRIPOS Part II

Monday 2 June 2008 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of all questions.*

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It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

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Question 1X Theory of Relativity

(i) Write down the 4-momentum of a photon travelling at an angle θ to the x-axis in the $x - y$ plane of flat space.

How fast would you have to drive towards a red traffic light for the light to appear green?

What wavelength does the traffic light appear to emit as you drive past at that speed?

What wavelength do you measure as you drive away from the traffic light at that speed?

$$\left[\begin{array}{c} \text{You may assume that the wavelengths of red and green light are} \\ \lambda_{\text{red}} \approx 700 \text{ nm}, \text{ and } \lambda_{\text{green}} \approx 500 \text{ nm}. \end{array} \right]$$

(ii) Consider a particle in motion with Cartesian components of velocity and acceleration (u_1, u_2, u_3) and (a_1, a_2, a_3) in the inertial frame S. The inertial frame S' is moving at speed $v < c$ along the x-axis of S and the axes of both frames are aligned. By considering the Lorentz transformation between two frames, derive the velocity and acceleration (u'_1, u'_2, u'_3), (a'_1, a'_2, a'_3) in frame S' in terms of (u_1, u_2, u_3), (a_1, a_2, a_3) and v.

Why are the velocity components transverse to the direction of motion of frame S' affected by the transformation?

A photon is travelling in the $x - y$ plane of frame S at an angle θ to the y-axis. What is the angle of the photon's trajectory relative to the y'-axis in frame S'?

[**CONTINUED...**

Question 2Y Astrophysical Fluid Dynamics

(i) Distinguish between the streamlines and particle paths in a fluid flow.

Under what conditions do streamlines and particle paths coincide?

A two dimensional fluid flow has velocity field described by

$$\begin{aligned} v_x &= -\frac{Ay}{(x^2+y^2)}, \\ v_y &= \frac{Ax}{(x^2+y^2)}, \end{aligned}$$

where A is a constant. Sketch the streamlines.

Compute the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \boldsymbol{v}$.

A two dimensional fluid flow is described by

$$\begin{aligned} v_x &= By, \\ v_y &= 0, \end{aligned}$$

where B is a constant. Sketch the streamlines.

Compute the vorticity.

(ii) Consider a small line element $d\boldsymbol{\ell}(\mathbf{r}, t)$ joining two fluid elements at positions $\mathbf{r}$ and $\mathbf{r} + d\boldsymbol{\ell}$. Show that the time evolution of $d\boldsymbol{\ell}$ is given by

$$\frac{Dd\boldsymbol{\ell}}{Dt} = (d\boldsymbol{\ell}.\boldsymbol{\nabla})\,\mathbf{v}. \qquad (*)$$

Show that in the absence of viscosity and external forces the vorticity of a barotropic fluid (i.e. in which the pressure is a function of density) obeys the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \wedge (\mathbf{v} \wedge \boldsymbol{\omega}).$$

[TURN OVER

Using the equation of continuity, or otherwise, show that the specific vorticity, $\boldsymbol{\omega}/\rho$, obeys the same equation as **(∗)**,

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}.\boldsymbol{\nabla}\right)\mathbf{v}.$$

Hence show that the circulation around a closed curve Γ contained within the fluid,

$$C = \oint_{\Gamma} \mathbf{v}.d\mathbf{r},$$

is constant as the curve Γ moves with the fluid.

[You may assume that $\boldsymbol{\nabla} \wedge (\mathbf{v} \wedge \boldsymbol{\omega}) = (\boldsymbol{\omega}.\boldsymbol{\nabla}).\mathbf{v} - (\mathbf{v}.\boldsymbol{\nabla})\boldsymbol{\omega} - \boldsymbol{\omega}(\boldsymbol{\nabla}.\mathbf{v})$]

[**CONTINUED...**

Question 3Z Structure and Evolution of Stars

(i) Planck's distribution for the energy per unit time per unit area per unit solid angle per unit frequency ν at temperature T is

$$dI(\nu, T) = \frac{2h}{c^2} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1},$$

where h is Planck's constant, c the speed of light, and k is Boltzmann's constant. From this distribution show that:

(a) the total radiant energy per unit surface area is

$$F = \sigma T^4,$$

for some constant σ (the Stefan-Boltzmann constant, which you need not determine);

(b) the peak energy emission at temperature T occurs at the wavelength $\lambda_{\max}$, where

$$\lambda_{\max} T = A,$$

where A is a constant (which you need not determine).

(ii) The following stars have effective temperature T_{eff}, luminosities L and masses M as listed:

Star	$T_{\text{eff}}(K)$	$L\ (L_\odot)$	$M(M_\odot)$
SpicaA	25400	13400	10.9
Vega	9600	51	2.6
Sun	5770	1	1
Aldebaran	4100	150	2.5
Sirius B	24800	0.024	0.98

[**TURN OVER**

Using the results of Part (i), with $A = 2.898\ 10^{-3}$ m.K:

(a) Calculate the wavelengths of the peak light emission for each of the stars, and state in which wavelength band (x-ray, optical, *etc.*) they fall.

(b) Determine the radii, R, for each of the stars, and sketch where they lie on a HR diagram.

(c) What phase of stellar evolution is each star in?

For stars with uniform composition which are supported by gas pressure use homology arguments to show that the central temperature $T_c \propto M/R$.

Why is the mass-radius relation $M \propto R$ appropriate for some stars? For which of the stars above is the relation approximately true? What are the conditions in the stars where it is not?

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Question 4X Physical Cosmology

(i) The number density of particles of mass m in equilibrium in the early Universe is given by the integral

$$n = \frac{4\pi g_s}{h^3}\int_0^\infty \frac{p^2 dp}{\exp[(E(p)-\mu)/kT] \pm 1}, \qquad \begin{cases} + \text{ fermions,} \\ - \text{ bosons,} \end{cases}$$

where $E(p) = c\sqrt{p^2 + m^2c^2}$, μ is the chemical potential, and g_s is the spin degeneracy. Give a physical interpretation of the chemical potential μ.

What can you say about the chemical potentials of photons, particles and their antiparticles?

Assuming that $\mu \ll mc^2$ and that particles remain in equilibrium when they become non-relativistic, show that their number density can be approximated as

$$n \approx g_s \left(\frac{2\pi m kT}{h^2}\right)^{3/2} \mathrm{e}^{(\mu - mc^2)/kT}.$$

$$\left[\text{ You may assume that } \int_0^\infty dx\ \mathrm{e}^{-\sigma^2 x^2} = \tfrac{\sqrt{\pi}}{2\sigma}. \right]$$

(ii) At around $t = 100$ seconds, deuterium D forms through the nuclear fusion of non-relativistic protons p and neutrons n via the interaction $p + n \leftrightarrow D$. In equilibrium, what is the relationship between the chemical potentials of the three species?

Show that the ratio of their number densities can be expressed as

$$\frac{n_D}{n_n n_p} \approx \left(\frac{\pi m_p kT}{h^2}\right)^{-3/2} \mathrm{e}^{B_D/kT},$$

where the deuterium binding energy is $B_D = (m_n + m_p - m_D)c^2$ and you may take $g_D = 4$.

Now consider the fractional densities $X_a = n_a/n_B$, where n_B is the baryon number of the Universe. Relate the ratio $X_D/(X_n X_p)$ to the baryon-to-photon ratio, η, of the Universe.

Why does deuterium form only at temperatures much lower than given by $kT \approx B_D$?

[You may assume that the photon density is $n_\gamma = \frac{16\pi\zeta(3)}{h^3c^3}(kT)^3$, where $\zeta(3) = 1.202$.]

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Question 5Z Stellar Dynamics and Structure of Galaxies

(i) The radial density distribution $\rho(r)$ of a Hernquist sphere is

$$\rho(r) = \frac{Ma}{2\pi r}\frac{1}{(r+a)^3},$$

where M and a are constants. Show that the potential for this density distribution is

$$\Phi(r) = -\frac{GM}{r+a}.$$

Determine the circular velocity $v_c(r)$ at radius r, and show that M is the total mass of the sphere.

(ii) The logarithmic potential has the form

$$\Phi(R,z) \;\; = \;\; \frac{1}{2}v_0^2 \ln(R^2+z^2) + \text{constant},$$

where R and z the usual cylindrical coordinates and v_0 is a constant. Show that the circular velocity at any radius R_0 is v_0.

Derive the epicyclic frequency, κ, and vertical frequency, ν, for nearly circular orbits at radius R_0 and $z = 0$ in this potential.

[**CONTINUED...**

Question 6Z Topics

(i) An area of the sky is imaged on to a noise-free detector. The light from a star falls in a small patch which counts a total of $Q + B$ photons in an exposure time T, where Q photons come from the star and B photons come from the background sky. A similar patch receives light from the sky only and is used to subtract the sky background so that Q can be estimated.

Assuming that the errors in the two measurements are equal to the square root of the number of photons, and that measurement errors for a sum add in quadrature, show that the signal-to-noise ratio Z of the estimate of Q is given by

$$Z = \frac{Q}{\sqrt{Q + 2B}}.$$

Show also that to attain a signal-to-noise ratio, Z, the required exposure time T is given by

$$T = \frac{Z^2(R_Q + 2R_B)}{R_Q^2},$$

where R_Q and R_B are the photon arrival rates for the star and the sky respectively.

(ii) A spectrometer on an 8 metre telescope records the spectrum of a star with magnitude $V = 22$. The spectral resolution is 0.1nm and a signal-to-noise ratio of Z is obtained in an exposure time of 18000 seconds at 550 nm. Only 10% of the photons available in the telescope's aperture are recorded. Using the results in part **(i)**:

(a) calculate the photon detection rate from the star for a single spectral resolution element;

(b) ignoring detector noise and assuming the sky is completely dark, calculate the signal-to-noise ratio Z.

In practice, the star's light is collected by an optical fibre of 2 arcsec diameter and the sky spectrum is obtained with a similar fibre. Also, each measurement has a root-mean-square error of 50 photons due to detector noise.

[TURN OVER

(c) Calculate the photon detection rate in the sky spectrum for a single spectral element.

(d) Determine the signal-to-noise ratio, Z, allowing for the detector noise and that the sky spectrum has to be subtracted from the stellar spectrum.

What is the dominant source of error in the measured spectrum?

[A star with $V = 0$ delivers 1.02×10^7 photons $\mathrm{m}^{-2}\mathrm{s}^{-1}$ in a 0.1nm wavelength interval. The sky has a brightness of $V = 21.5$ per square arcsec.]

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Wednesday 4 June 2008 09:00am – 12.00pm

ASTROPHYSICS - PAPER 3

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Question 1X Theory of Relativity

(i) A two dimensional space is described by the metric

$$ds^2 = F(z)d\phi^2 + dz^2, \qquad \begin{cases} 0 < z < \infty, \\ 0 \le \phi < 2\pi. \end{cases} \qquad (*)$$

Show that the only non-vanishing components of the affine connection are

$$\Gamma^{\phi}_{\phi z} = \Gamma^{\phi}_{z\phi} = \frac{F'(z)}{2F(z)}, \quad \Gamma^{z}_{\phi\phi} = -\frac{F'(z)}{2},$$

where $F'(z) = dF/dz$.

Show further that the curvature tensor is described fully by

$$R_{z\phi z\phi} = \frac{1}{2}\Big(F''(z) - \frac{(F'(z))^2}{2F(z)}\Big).$$

(ii) Find a non-trivial function $F(z) \neq$ constant for which the metric (*) of part (i) describes locally flat space.

For this choice of $F(z)$, consider the parallel transport of a vector (A^{ϕ}, A^{z}) with components $(0, 1)$ at point $(\phi_0, z_0) = (0, 5)$. What are its components after parallel transport along a curve with $z =$ constant to the point $(\phi_1, z_1) = (\pi, 5)$.

Is the curve $z(\phi) =$ constant a geodesic?

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Question 2Y Astrophysical Fluid Dynamics

(i) A supernova explodes depositing energy E into the surrounding interstellar medium, which can be assumed to be a perfect gas of uniform density ρ_0 and negligible pressure. Using dimensional arguments, show that the radius of the shock that is formed by the blast propagates as

$$R(t) = \xi_0 \left(\frac{E}{\rho_o}\right)^{1/5} t^{2/5}, \qquad (*)$$

where ξ_0 is a dimensionless constant of order unity.

Estimate the radius and velocity of the supernova remnant 500 years after the explosion.

[Assume that $E = 10^{43} J$ and $\rho_0 = 10^{-21}$kg m^{-3}.]

(ii) As a simple model of a blast wave propagating into a uniform cold interstellar medium of density ρ_0, assume that the mass swept up by the strong shock is confined to a thin shell of density ρ_1 and pressure p_1. Assume that the pressure interior to the shell is uniform and equal to αp_1 where α is a constant. Assume further that the velocity and pressure of the shocked gas are given by

$$\begin{aligned} v &= \frac{2V_0}{\gamma+1}, \\ p_1 &= \frac{2\rho_0 V_0^2}{\gamma+1}, \end{aligned}$$

where $V_0 = dR/dt$ is the velocity of the shock and γ is the adiabatic index of the interstellar medium.

Show, using momentum balance, that the radius of the shell evolves as

$$\frac{dR}{dt} = CR^{3(\alpha-1)},$$

where C is a constant.

[TURN OVER

The internal energy per unit volume of an ideal gas with adiabatic index γ and pressure p is $p/(\gamma-1)$. Use energy conservation to determine the value of α.

Hence show that the constant ξ_0 of **(∗)** of part **(i)** is

$$\xi_0 \simeq \left[\frac{75}{16\pi}\frac{(\gamma-1)(\gamma+1)^2}{(3\gamma-1)}\right]^{1/5} \qquad \mathbf{(**)}$$

For a gas with $\gamma = 5/3$, how does the approximation **(∗∗)** compare with the exact answer $\xi_0 = 1.17$?

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Question 3Y Statistical Physics

(i) Derive the heat capacities per oscillator in the following cases:

(a) a one-dimensional quantum harmonic oscillator of frequency ω in the limit $T \to 0$;

(b) a one-dimensional quantum harmonic oscillator of frequency ω in the limit $T \to \infty$;

(c) a classical particle oscillating in a one-dimensional potential well of the form $V(x) = Ax^n$, where n is an even integer.

Explain whether these results can be derived by invoking energy equipartition.

(ii) Prove that energy fluctuations in a canonical distribution are given by

$$\left\langle (E - \langle E \rangle)^2 \right\rangle \;=\; k_B T^2 C_V,$$

where T is the absolute temperature, $C_V = \frac{\partial \langle E \rangle}{\partial T}|_V$ is the heat capacity at constant volume, and k_B is Boltzmann's constant.

In a similar manner prove the following relation:

$$\left\langle (E - \langle E \rangle)^3 \right\rangle \;=\; k_B^2 \left[T^4 \left. \frac{\partial C_V}{\partial T} \right|_V \;+\; 2T^3 C_V \right].$$

Show that for an ideal gas of N monatomic molecules where $\langle E \rangle = \frac{3}{2} N k_B T$, these equations can be reduced to

$$\frac{1}{\langle E \rangle^2} \left\langle (E - \langle E \rangle)^2 \right\rangle \;=\; \frac{2}{3N} \quad \text{and} \quad \frac{1}{\langle E \rangle^3} \left\langle (E - \langle E \rangle)^3 \right\rangle \;=\; \frac{8}{9N^2}.$$

[**TURN OVER**

Question 4Z Structure and Evolution of Stars

(i)

(a) Show that in equilibrium, a non-rotating star satisfies

$$\frac{d}{dr}\left[p + \frac{Gm^2}{8\pi r^4}\right] < 0, \qquad (*)$$

where $p = p(r)$ is the pressure at radius r, $m(r) = \int_0^r 4\pi r'^2 \rho \, dr'$, and $\rho(r)$ is the density.

(b) Use the relation (*) to deduce a lower limit for the central pressure p_c in a star as a function of the stellar mass M and outer radius R, if $p(R) = 0$.

(ii) The density, ρ, within a star varies linearly with radius r from the centre to the surface at radius R

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right).$$

(a) Derive an expression for the total mass of the star and for $\rho_c/\bar{\rho}$, where $\bar{\rho}$ is the mean density.

(b) Derive the pressure $p(r)$ and thus the central pressure p_c if $p(R) = 0$.

(c) Assuming that gas pressure dominates, derive the temperature profile $T(r)$ and show that $T(R) = 0$.

(d) $K(r)$ is defined such that $p(r) = K(r)\rho(r)^\gamma$, show that $K(R) = 0$ and that $dK/dr > 0$ at $r = 0$.

(e) Hence state which, if any, region of the star is convectively unstable.

[TURN OVER

Question 5X Physical Cosmology

(i) Discuss how measurements of the angular power spectrum of the CMB fluctuations constrain the Universe to be nearly spatially flat.

(ii) Show that the line element of the spherically symmetric ($k = 1$) Friedmann-Robertson-Walker Universe can be written in the form

$$ds^2 = R^2(\eta)\left[d\eta^2 - d\chi^2 - \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

where the new dimensionless time η is defined by $d\eta = c\ dt/R$, and for an appropriate choice of a new radial coordinate χ.

Show that for a Universe with $k = 1,\ \Lambda = 0$, the Friedmann equation is

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi G\rho}{3c^2}R^4 - R^2.$$

Show that, in the matter dominated case for which $\rho = \rho_0(R_0/R)^3$, the solution is

$$\begin{aligned} R(\eta) &= GM(1-\cos\eta)/c^2 \\ t(\eta) &= GM(\eta - \sin\eta)/c^3, \end{aligned}$$

where $M = \frac{4\pi}{3}\rho_0 R_0^3$.

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Question 6Z Stellar Dynamics and Structure of Galaxies

(i) From Newton's second law in a static gravitational potential $\Phi(\mathbf{r})$, $\ddot{\mathbf{r}} = -\nabla\Phi$, derive expressions for the angular momentum about the origin and energy of a particle.

If the potential is spherical, $\Phi = \Phi(r)$, what does the angular momentum equation tell us about the orbits?

(ii) A Plummer sphere has a distribution function in phase space $(\mathbf{r}, \mathbf{v})$ given by

$$f(\boldsymbol{r}, \boldsymbol{v}) = f(r, v) = f(E) = F|E|^{7/2},$$

where $E = \frac{1}{2}v^2 + \Phi(r)$ is the energy per unit mass, $\Phi(r)$ is the potential at radius r, and F is a constant.

Show that the mass density $\rho = C|\Phi|^5$, where C is a constant.

Show that the potential Φ is given by

$$\Phi = \frac{\Phi_0}{\sqrt{1 + \frac{1}{3}\left(\frac{r}{b}\right)^2}},$$

where $b = \left(4\pi G\Phi_0^4 C\right)^{-1/2}$ and $\Phi_0 = \Phi(0)$.

Determine the escape velocity at r, $v_{esc}(r)$.

Set $q = v/v_{esc}$, and show that the probability distribution for the velocities $g(q)$ has the same dependence on q at all radii.

[**CONTINUED...**

Question 7Z Topics

(i) Show that for a star in a circular orbit in the outer parts of a spherical cluster of mass M, radius R, the orbital angular momentum is given by

$$J \propto m M^{1/2} R^{1/2},$$

where m is the star's mass.

A spherically symmetric accretion flow from outside the cluster increases the mass of each star (and hence also the mass of the cluster) by a factor of f_m. Use angular momentum conservation to show how R scales with f_m.

If the cluster mass doubles in this way, by what factor does (a) the mean density and (b) the velocity dispersion increase?

Estimate the increase in mass that would be required for the stellar velocities to become relativistic.

What would you expect to happen before this point is reached?

(ii) According to the unified model of active galactic nuclei (AGN), the central black hole and its associated region of broad line-emitting gas reside within a dusty torus. An observer will, depending on the orientation of the torus with respect to the line of sight, be able to see either the broad line region (and then classify the AGN as a 'Type I Seyfert') or else will be prevented from doing so by the torus (then classifying the AGN as a 'Type II Seyfert'). The ratio of Type I to Type II Seyferts is 1:4. In Seyfert I's, a burst of continuum emission from close to the black hole is followed by a brightening in the broad emission lines after about a week. The width of the broad lines is commonly ascribed to orbital motion of clouds in the black hole's potential. The width of the CIV line at $1549\AA$ is $10\AA$ in the rest frame. It is found, however, that broad emission lines can be detected in Seyfert II's when observed in polarised light. The polarised light is interpreted as being emission from the broad line region which has been scattered into the observer's line of sight. Assume that the line widths in these systems are similar to those in Seyfert I's.

Use the above information to estimate the following:

(a) the typical opening angle for the dust torus;

(b) the distance of the broad line emitting region from the black hole;

[**TURN OVER**

(c) the mass of the black hole;

Furthermore

(d) explain what you can deduce about the orientation of the orbits of the broad line emitting clouds;

(e) explain why, according to this model, Seyfert II AGN have longer radio jets on average.

The dusty torus is located just outside the broad line emitting region and can be modelled as containing spherical silicate grains of radius $1\mu m$ and density 3 gcm^{-3}, whose cross section for the absorption of radiation is equal to their geometrical cross section. Assuming a gas to dust ratio that is typical of the interstellar medium (100:1), determine a lower limit on the torus mass.

Would this limit increase or decrease if the grains were smaller?

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Thursday 5 June 2008 1:30pm – 4:30pm

ASTROPHYSICS - PAPER 4

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Question 1X Theory of Relativity

(i) Write down the formula for the covariant derivative $B_{\alpha\beta;\gamma}$ of the covariant tensor $B_{\alpha\beta}$.

By considering the second covariant derivative $A_{\alpha;\beta;\gamma}$ of a covariant vector A_α, show that

$$A_{\alpha;\beta;\gamma} - A_{\alpha;\gamma;\beta} = A_\delta R^\delta_{\alpha\beta\gamma},$$

where $R^\delta_{\alpha\beta\gamma}$ is the mixed curvature tensor.

What can you conclude about the conditions under which covariant derivatives commute?

(ii) Consider the space-time described by the metric

$$ds^2 = (g_{tt} - g_{\phi\phi}w^2)dt^2 + g_{\phi\phi}(d\phi - w dt)^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2.$$

where w is a constant.

Derive the coordinate velocity $d\phi/dt$ for photons moving in the $\pm\phi$ directions.

Discuss what happens to the coordinate velocity if $g_{tt} = 0$.

By considering the four-velocity of stationary observers, show that they cannot exist in regions with $g_{tt} < 0$.

What does this imply for particles in regions with $g_{tt} < 0$?

Assuming that the space-time is stationary, $\partial g_{\mu\nu}/\partial t = 0$, and that $w = 0$, calculate the redshift of photons emitted and received at fixed spatial coordinates.

[**CONTINUED...**

Question 2Y Astrophysical Fluid Dynamics

(i) Consider a pressureless thin axisymmetric fluid disc of surface density $\Sigma(r)$, located in the plane $z = 0$. The disc is rotating in centrifugal equilibrium around a compact object of mass M. In polar coordinates (r, θ) the disc satisfies the following equations

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma r v_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\Sigma v_\theta\right) = 0,$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = 0,$$

if the self-gravity of the disc can be ignored. Show that these equations require

$$\frac{\partial}{\partial t}\left(r^2 \Sigma \Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma r^3 \Omega v_r\right) = 0, \qquad (*)$$

where $\Omega(r)$ is the angular velocity of the disc.

Give a physical interpretation of **(∗)**.

If the disc has a finite viscosity ν, **(∗)** is modified to

$$\frac{\partial}{\partial t}\left(r^2 \Sigma \Omega\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma r^3 \Omega v_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu \Sigma r^3 \frac{d\Omega}{dr}\right).$$

Describe qualitatively the effects of viscosity on the evolution of the disc.

Estimate a characteristic viscous evolutionary timescale for a disc of radius R and viscosity ν.

(ii) Show that the Keplerian disc of part **(i)** obeys a diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\Sigma\right)\right]. \qquad (**)$$

[**TURN OVER**

Show that if the viscosity is proportional to radius, $\nu = Ar$, equation $(**)$ can be written as

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial z^2}, \qquad (***)$$

where $G = r^{3/2}\Sigma$ and $z = 2r^{1/2}/\sqrt{3A}$.

Verify that $(***)$ is satisfied by the solution

$$G = \exp\left(-\lambda^2 t\right)\sin(\lambda z)$$

for $\lambda = \text{constant}$.

Give a physical interpretation of this solution.

[**CONTINUED...**

Question 3Y Statistical Physics

(i) Prove the Maxwell relation:

$$\left.\frac{\partial S}{\partial V}\right|_T = \left.\frac{\partial p}{\partial T}\right|_V .$$

Hence or otherwise prove that

$$\left.\frac{\partial U}{\partial V}\right|_T = T^2 \frac{\partial}{\partial T}\left.\left(\frac{p}{T}\right)\right|_V .$$

Evaluate this quantity in the case of a perfect (ideal) gas.

Explain how this result:

(a) can be understood in terms of a microscopic model for perfect gas;

(b) can be used to explain why the temperature of a perfect gas is unchanged following a thermally isolated free expansion.

(ii) Derive the following two relations:

$$\left.\begin{aligned} T\,dS &= C_p\,dT - T\left.\frac{\partial V}{\partial T}\right|_p dp, \\ T\,dS &= C_V\,dT + T\left.\frac{\partial p}{\partial T}\right|_V dV. \end{aligned}\right\} \qquad (*)$$

[*You may use any standard Maxwell relation without proving it.*]

Experimentalists very seldom measure C_V directly; they measure C_p and use thermodynamics to extract C_V. Use the results (*) to find a formula for $C_p - C_V$ in terms of the easily measured quantities

$$\alpha = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_p ,$$

(the volume coefficient of expansion) and

[TURN OVER

$$\kappa \;=\; -\frac{1}{V}\,\frac{\partial V}{\partial p}\bigg|_T,$$

(the isothermal compressibility).

[**TURN OVER**

Question 4Z Structure and Evolution of Stars

(i) A spherical star has equation of state $p = K\rho^2$, where p is the pressure, ρ is the density and K is a constant. Show that the density as a function of radius r within the star has the form

$$\rho \propto \sin(Ar)/r,$$

where A is a constant.

Show that the radius of the star is

$$R = \left(\frac{K\pi}{2G}\right)^{1/2}.$$

Comment on the mass-radius relation for such stars.

(ii) A white dwarf contains a degenerate non-relativistic gas with the equation of state $p = K\rho^{5/3}$ where p is the pressure, ρ is the density and K is a constant. Show that the radius R and central density ρ_c scale with mass M according to

$$R \propto M^{-\frac{1}{3}}, \qquad \rho_c \propto M^2.$$

If instead the electrons are relativistic so that $p = K'\rho^{4/3}$, where K' is a constant, show that the mass must have a unique value (the Chandrasekhar limit) independent of the radius or central density.

Explain the significance of this mass in terms of possible end points for stellar evolution.

[**CONTINUED...**

Question 5X Physical Cosmology

(i) Consider small perturbations of a homogeneous and isotropic background universe of zero curvature, consisting of pressureless matter of mean density $\bar{\rho}_c$ and radiation. Plane-wave density perturbations $\delta_{\mathbf{k}}$ in the matter of small wavelength compared to the Hubble radius, ct, obey the evolution equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho}_c\delta_{\mathbf{k}} = 0, \tag{$*$}$$

where dots denote differentiation with respect to time, a is the cosmic scale factor normalised to unity at the present day, and $\mathbf{k}$ is the comoving wavevector of the perturbation. Seek power law solutions of (*), $\delta_{\mathbf{k}} \propto t^{\beta}$ (β = constant), during the matter-dominated epoch to find the approximate solution:

$$\delta_{\mathbf{k}}(t) = A(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{2/3} + B(\mathbf{k})\left(\frac{t}{t_{\rm eq}}\right)^{-1}, \qquad t \gg t_{\rm eq}, \tag{$**$}$$

where A and B are functions of $\mathbf{k}$ only and $t_{\rm eq}$ is the time at which matter and radiation have equal densities.

By considering the behaviour of the scale factor a, show that in the radiation era ($t \ll t_{\rm eq}$) there is no significant growth of matter density perturbations with characteristic scales smaller than the Hubble radius.

(ii) Consider a plane-wave matter perturbation of comoving wavenumber $k = |\mathbf{k}|$ and small amplitude as described in part (i). Show that the time t_H at which this mode crosses the Hubble radius, *i.e.* $ct_H \approx 2\pi a(t_H)/k$, is given by

$$\frac{t_H}{t_0} \approx \begin{cases} (k_0/k)^3, & t_H \gg t_{\rm eq}, \\ (1+z_{eq})^{-1/2}(k_0/k)^2, & t_H \ll t_{\rm eq}, \end{cases}$$

where t_0 is the present age of the universe, $k_0 = 2\pi/(ct_0)$, and the redshift at which matter and radiation have equal densities is given by $(1 + z_{\rm eq}) = (t_0/t_{\rm eq})^{2/3}$.

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Assume that primordial perturbations from inflation are scale-invariant with a constant amplitude $\langle|\delta_{\mathbf{k}}(t_H)|^2\rangle \approx V^{-1}A/k^3$, where A is a constant and V is a large volume, as they cross the Hubble radius. Use the solution (**) or otherwise to show that the power spectrum of matter density perturbations at the present day will be given approximately by

$$P(k) \equiv V\langle|\delta_{\mathbf{k}}(t_0)|^2\rangle \approx \frac{A}{k_0^4} \times \begin{cases} k, & k < k_{\rm eq}, \\ k_{\rm eq}(k_{\rm eq}/k)^3, & k > k_{\rm eq}. \end{cases}$$

Discuss briefly the implications of this result for the formation of the first non-linear objects in a universe with initially scale-invariant perturbations.

[**CONTINUED...**

Question 6Z Stellar Dynamics and Structure of Galaxies

(i) A spherical system has a power law density profile

$$\rho(r) = \rho_0 \left(\frac{a}{r}\right)^{\alpha},$$

where ρ_0, a and α are constants.

Show that the requirement that the mean density over a small region near $r = 0$ be finite implies that $\alpha < 3$.

Show that the projected density $\sigma(R)$ is proportional to $R^{1-\alpha}$.

(ii) The collisionless Boltzmann equation in cylindrical coordinates (R, ϕ, z) is

$$\frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0$$

where dots denote time derivatives, and (v_R, v_ϕ, v_z) are the velocity components in cylindrical coordinates.

For an axisymmetric potential $\Phi(R, z)$, derive the Jeans equation for the first moment of v_z.

Explain how you can use the first moment equation to estimate the mass density of the Galaxy in the solar neighbourhood.

[You may assume that, in these coordinates,

$$\ddot{R} - R^2\dot{\phi} = -\frac{\partial \Phi}{\partial R},$$

$$\frac{d}{dt}(R^2\dot{\phi}) = -\frac{\partial \Phi}{\partial \phi}.$$
]

[**TURN OVER**

Question 7Z Topics

(i) A circular ring of material surrounds SN1987A. Ultraviolet radiation is continuously emitted from the central supernova and it ionises the material in the ring, producing radiation visible from Earth. The ring is viewed at an angle such that it appears as an ellipse with eccentricity $e = 1 - b/a = 0.27$ and the full length of the major axis, a, subtends 1.66 arcsec. The ultraviolet excited radiation is first seen from the whole ring 320 days after the first part of the ring is seen to radiate. What is the radius of the ring?

Use your answer to calculate the distance to SN1987A.

(ii) In a close binary system mass is transferred from one star to another. Assuming that the system conserves angular momentum and mass, show that the orbital period of P of the system obeys

$$\frac{\dot{P}}{P} = \frac{3\dot{M}_1(M_1 - M_2)}{M_1 M_2},$$

where dots denote time derivatives and M_1 and M_2 are the masses of the two stars.

Under what circumstances is the mass transfer rate unstable due to a positive feedback?

Give an example of when this might happen.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Friday 6 June 2008 09:00am – 12:00pm

ASTROPHYSICS - PAPER 5

Before you begin read these instructions carefully.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of all questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 5X** *should be in one bundle and* **2Y, 3Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Question 1X Theory of Relativity

(i) Assume that the space-time around an object with mass M is described by

$$ds^2 = \mathrm{e}^{-2u}c^2dt^2 - \mathrm{e}^{2u}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2), \qquad (*)$$

with $u = \frac{GM}{rc^2}$. Starting from the geodesic equation, show that orbits in this space-time are planar and that the equations of motion can be written as,

$$\frac{d^2t}{dp^2} + \frac{2u}{r}\frac{dt}{dp}\frac{dr}{dp} = 0,$$
$$\frac{d^2r}{dp^2} - \frac{u}{r}\left(\frac{dr}{dp}\right)^2 + \frac{c^2u^2}{r}\mathrm{e}^{-4u}\left(\frac{dt}{dp}\right)^2 + r(u-1)\left(\frac{d\phi}{dp}\right)^2 = 0,$$
$$\theta = \text{constant} = \frac{\pi}{2},$$
$$\frac{d^2\phi}{dp^2} + \frac{2}{r}(1-u)\frac{dr}{dp}\frac{d\phi}{dp} = 0,$$

where p is an affine parameter.

[The only non-zero components of the affine connection of the metric (*) are $\Gamma^r_{rr} = -u/r$, $\Gamma^r_{tt} = (c^2u^2/r)\mathrm{e}^{-4u}$, $\Gamma^r_{\theta\theta} = r(u-1)$, $\Gamma^r_{\phi\phi} = r\sin^2\theta(u-1)$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^t_{rt} = u/r$, $\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = (1-u)/r$, $\Gamma^\phi_{\theta\phi} = \cos\theta/\sin\theta$.]

(ii) Show that for particles orbiting in the metric (*) of part (i)

$$h = \frac{c\dot{\phi}re^{2u}}{u} = \text{constant},$$

where $\dot{\phi} = d\phi/dp$.

Give a physical interpretation of h.

Show that the equation of motion can be written in the form

$$\left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{c^4}{\mathcal{E}},$$

where $c^2d\tau^2 = \mathcal{E}dp^2$, $\mathcal{E}$ is a constant, and $V(r)$ is an effective potential:

$$V(r) = c^2\mathrm{e}^{-2u}\left(1 + \frac{c^2h^2}{\mathcal{E}}\mathrm{e}^{-2u}\right).$$

[**TURN OVER...**

Sketch the effective potential as a function of r.

Show that circular orbits are characterised by

$$h^2 = \text{constant} \times \frac{\mathrm{e}^{2u}}{(u - 2u^2)}.$$

Show that the minimum radius for a stable circular orbit is given by

$$r = \frac{4}{3 - \sqrt{5}} \frac{GM}{c^2}.$$

[**CONTINUED...**

Question 2Y Astrophysical Fluid Dynamics

(i) The equations of ideal magnetohydrodynamics are

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v}) &= 0, \\
\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\, \boldsymbol{v} \right) &= \boldsymbol{j} \wedge \boldsymbol{B} - \boldsymbol{\nabla} p, \\
\boldsymbol{\nabla} \wedge (\boldsymbol{v} \wedge \boldsymbol{B}) &= \frac{\partial \boldsymbol{B}}{\partial t}, \\
\boldsymbol{\nabla} \wedge \boldsymbol{B} &= \mu_0 \boldsymbol{j}, \\
\boldsymbol{\nabla}.\boldsymbol{B} &= 0.
\end{aligned}$$

Show that the equation of motion can be written in form

$$\rho \frac{dv_i}{dt} = -\frac{\partial}{\partial x_j} M_{ij} - \frac{\partial p}{\partial x_i},$$

where

$$M_{ij} = \frac{1}{\mu_0} \left[\frac{B^2}{2} \delta_{ij} - B_i B_j \right]. \qquad (*)$$

Give a physical interpretation of the tensor M_{ij}.

(ii) Starting with the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \wedge (\boldsymbol{v} \wedge \boldsymbol{B}),$$

or otherwise, show that the magnetic energy density in an ideal magnetohydrodynamic fluid satisfies the equation

$$\frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_0} \right) + \frac{\partial}{\partial x_j} \left(v_j \frac{B^2}{2\mu_0} \right) = -M_{ij} \frac{\partial v_i}{\partial x_j},$$

where M_{ij} is the tensor defined in **(∗)** in part (i).

[**TURN OVER**

In the absence of external heating and cooling, show that energy conservation is an ideal magnetised fluid can be expressed as

$$\frac{\partial}{\partial t}\left(E + \frac{B^2}{2\mu_0}\right) = -\frac{\partial}{\partial x_j}\left[\left(E + p + \frac{B^2}{2\mu_0}\right) v_j + v_i M_{ij}\right]$$

where

$$E = \rho\left(\frac{1}{2}v^2 + \varepsilon\right)$$

and ε is the specific internal energy of the fluid.

Comment briefly on how the Rankine-Hugoniot jump conditions at a one-dimensional shock interface in an ideal magnetised fluid with uniform magnetic field perpendicular to the shock differ from those of an unmagnetised fluid.

[**CONTINUED...**

Question 3Y Statistical Physics

(i) Give a short account of Bose-Einstein condensation, including an example of a particle which can manifest this behaviour.

Give an example, with explanation, of a change in physical property associated with Bose-Einstein condensation.

(ii) Show that the Fermi momentum, p_F, of a gas of N non-interacting electrons in volume V is

$$p_F = \left(3\pi^2 \hbar^3 \frac{N}{V} \right)^{1/3} .$$

Consider the electrons to be effectively massless, so that an electron of momentum p has (relativistic) energy cp. Show that the mean energy per electron at zero temperature is $3cp_F/4$.

When a constant external magnetic field of strength B is applied to the electron gas, each electron gets an energy contribution $\pm\mu B$ depending on whether its spin is parallel or antiparallel to the field. Here μ is the magnitude of the magnetic moment of an electron. Assume that the populations of electrons with spins that are aligned and those that are anti-aligned both fill states up to the same Fermi surface. Calculate the total magnetic moment of the electron gas at zero temperature, assuming μB is much less than cp_F.

[**CONTINUED...**

Question 4Z Structure and Evolution of Stars

(i) Explain what is meant by the Kelvin-Helmholtz, or thermal, timescale of a star, t_{KH}.

Show that

$$t_{KH} \sim 3 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} \text{yrs.}$$

Explain how this quantity changes as a pre-main sequence star approaches the main sequence.

Hence estimate the time taken by the Sun to reach the main sequence.

Show that the main sequence lifetime of a star is approximately

$$t_{ms} \sim 10^{10} \left(\frac{M}{M_\odot}\right) \left(\frac{L}{L_\odot}\right)^{-1} \text{yrs.}$$

(ii) Describe the evolution of stars with masses $2M_\odot$ and $25M_\odot$, including estimates of relevant timescales, from the pre-main sequence phase to the end point of stellar evolution in each case.

[**TURN OVER**

Question 5X Physical Cosmology

(i) The Friedmann and Raychaudhuri equations for a homogeneous and isotropic universe are respectively

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \qquad (*)$$

where $a(t)$ is the scale factor, ρ is the mass density, p is the pressure, k is the curvature and dots denote differentiation with respect to time. Suppose that the equation of state of the dominant constituent of the universe is $w = p/(\rho c^2) =$ constant. What are the constraints on w if the universe is accelerating?

If $k = 0$, show that the equations (*) have solutions

$$a(t) \propto t^{\frac{2}{3(1+w)}}, \qquad \rho \propto a^{-3(1+w)}.$$

Give a physical interpretation of these solutions in the limits $w \to 0$ and $w \to -1$.

(ii) Using conformal time τ (defined by $d\tau = dt/a$) show that the equations (*) of part (i) can be rewritten as

$$\frac{kc^2}{\mathcal{H}^2} = \Omega - 1, \qquad 2\frac{d\mathcal{H}}{d\tau} = -(3w+1)(\mathcal{H}^2 + kc^2), \qquad (**)$$

where $\mathcal{H} = a^{-1}da/d\tau$, w is the equation of state parameter defined in part (i), Ω is the density parameter $\rho/\rho_{\rm crit}$, and $\rho_{\rm crit} = 3\mathcal{H}^2/(8\pi G a^2)$.

Use the relations (**) to derive the following evolution equation for Ω:

$$\frac{d\Omega}{d\tau} = (3w+1)\mathcal{H}\Omega(\Omega - 1).$$

For both $w = 0$ and $w = -1$, sketch the evolution of Ω as a function of τ in an expanding universe for the two cases $\Omega > 1$ and $\Omega < 1$ at early times.

Using these results, describe the flatness problem of the standard cosmology and how it can be solved by inflation.

[**TURN OVER**

Question 6Z Stellar Dynamics and Structure of Galaxies

(i) A homogeneous sphere has density ρ_0 for radius $r \leq r_0$, and density zero elsewhere. Determine the equations of motion for a collisionless particle with $r < r_0$, and establish the dynamical time for the system.

Derive an integral equation linking the particle's polar coordinates r and ϕ.

(ii) A star in the disk of the Milky Way has a circular orbit with radius R and velocity V. If the star is at a distance d from the Sun, and has galactic longitude ℓ, show that the radial and transverse components of the velocity of the star with respect to the Sun are

$$v_r = \left(\frac{V}{R} - \frac{V_0}{R_0}\right) R_0 \sin \ell, \quad v_t = \left(\frac{V}{R} - \frac{V_0}{R_0}\right) R_0 \cos \ell - \frac{V}{R} d, \qquad (*)$$

where V_0 and R_0 are the Sun's circular velocity and radius.

How are v_r and v_t measured observationally?

Show that for $d \ll R_0$, the velocity components (*) can be approximated as

$$v_r = A d \sin 2\ell, \qquad v_t = (A \cos 2\ell + B) d,$$

where A and B are the Oort constants.

[**TURN OVER**

Question 7Z Topics

(i) A planet orbits a star in a circular orbit of radius a and is itself orbited by a moon. Within a certain distance (known as the Hill radius) where the orbital period of a satellite about the planet is the same as the planet's orbital period about the star, a moon can have a stable circular orbit around the planet. Derive an expression for the Hill radius in terms of a, the mass of the planet M_p and the mass of the star M_*.

Could Jupiter (mass $10^{-3}M_\odot$, radius 1.4×10^5 km) have a moon if it were at the distance of Mercury (5×10^7km) from the Sun?

(ii) A pulsar is observed over a period of time and its spin period P is seen to be increasing. Assuming that the entire gamma-ray luminosity, L, of the pulsar is driven by the loss of rotational energy, derive an expression for the time derivative $\dot{P}$ in terms of P, L and the pulsar's moment of inertia, I.

If $I = KMR^2$ (where K is a constant, M is the mass of the pulsar and R is its radius) and L_{obs} is the observed gamma-ray flux, derive an expression for the distance of the pulsar, in terms of $P, \dot{P}, K, M, R$ and L_{obs}.

Calculate the pulsar's distance for $M = 2.1M_\odot$, $K = 0.2$, $R = 10$ km, $P = 0.5$s, $\dot{P} = 10^{-15}$ and $L_{obs} = 2.2 \times 10^{-15}\mathrm{Wm}^{-2}$.

Comment on the plausible values M and R might have and calculate the range of distance to the pulsar that this implies.

What mechanism might be responsible for spinning down the pulsar?

END OF PAPER