NATURAL SCIENCES TRIPOS Part II

Monday 4 June 2007 09:00am – 12:00pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

The paper is divided into Section **A** *and Section* **B**. *Candidates may attempt Parts from* **ALL** *questions in Section* **A** *and from not more than* **THREE** *questions in Section* **B**.

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A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

Question 1X Theory of Relativity

(i) Calculate all eight Christoffel symbols for the metric

$$ds^2 = dz^2 + z^2 d\phi^2$$

where $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$.

Calculate the component $\mathrm{R}_{z\phi z\phi}$ of the Riemann tensor and give a geometrical interpretation of your result.

(ii) Consider the motion of two freely falling particles on nearby trajectories,

$x^\mu(\tau)$,

$x^\mu(\tau) + \delta x^\mu(\tau)$

where τ is an affine parameter. Show that

$$\frac{\mathrm{D}^2 \delta x^\mu}{\mathrm{D}\tau^2} + \mathrm{R}^\mu_{\lambda\sigma\rho} \frac{dx^\sigma}{d\tau} \frac{dx^\lambda}{d\tau} \delta x^\rho = 0$$

and give a physical interpretation of your result.

[**CONTINUED...**

Question 2Y Astrophysical Fluid Dynamics

(i) Show that the equation of motion for a fluid of density ρ and pressure p can be written in the form

$$\frac{\partial \underline{v}}{\partial t} + \underline{\nabla}\left(\frac{1}{2}v^2\right) - \underline{v} \wedge \underline{\omega} = -\frac{1}{\rho}\underline{\nabla} p,$$

where $\underline{\omega} = \underline{\nabla} \wedge \underline{v}$.

Hence show that for a barotropic fluid ($p(\rho)$) undergoing a steady flow, the quantity

$$H = \frac{1}{2}v^2 + \int \frac{dp}{\rho}$$

is constant along streamlines.

Give a physical explanation of this result.

(ii) Consider the quasi one-dimensional flow of gas through a pipe of cross-sectional area $A(z)$. Assume that the flow is steady, irrotational and adiabatic. Show that the flow velocity, $v(z)$, satisfies

$$(v^2(z) - c_s^2)\frac{d\ln v}{dz} = c_s^2 \frac{d\ln A}{dz},$$

where c_s is the adiabatic sound speed. Hence show that the flow velocity is almost constant in a highly supersonic flow.

What quantity is nearly constant in a highly subsonic flow?

[**TURN OVER**

A highly collimated, steady and highly supersonic jet propagates from a galactic nucleus. Assuming that the gas within the jet is adiabatic, with equation of state $p \propto \rho^{5/3}$, show that the cross sectional area of the jet varies as $A \propto p^{-3/5}$.

If the jet is confined by an external medium with pressure $p_{ext} \propto r^{-2}$ (where r is radial distance in a spherical coordinate system centred on the nucleus), determine how the angle subtended by the jet, for an observer situated at $r = 0$, varies with radius, r.

[**CONTINUED...**

Question 3Y Statistical Physics

(i) The pressure, P, entropy, S, internal energy, U, and Helmholtz Free Energy F of a fixed mass of gas can be considered as functions of the variables volume, V, and temperature, T. Prove the following thermodynamic identities:

$$\textbf{(a)} \quad P = -\left.\frac{\partial \mathrm{F}}{\partial \mathrm{V}}\right|_T$$

$$\textbf{(b)} \quad \left.\frac{\partial S}{\partial \mathrm{V}}\right|_T = \left.\frac{\partial P}{\partial T}\right|_V$$

$$\textbf{(c)} \quad \left.\frac{\partial U}{\partial V}\right|_T = T\left.\frac{\partial P}{\partial T}\right|_V - P$$

(ii) Show that the Helmholtz Free Energy ($\mathrm{F} = \mathrm{U} - \mathrm{TS}$) for a system in thermal equilibrium at temperature T is given by

$$\mathrm{F} = -\mathrm{N}k\mathrm{T}\ln Z$$

where Z is the partition function, k is Boltzmann's constant and N is the number of particles.

In a model of a non-ideal gas, the energy per particle is given by

$$\mathrm{E} = \frac{1}{2}mv^2 + \Phi(V)$$

where m, v are the particle mass and velocity and Φ is the interatomic potential as a function of a system volume, V, such that

$$\begin{aligned} \Phi(V) &= \infty \ (V < a) \\ \Phi(V) &= 0 \ (V \geq a) \end{aligned}$$

[**TURN OVER**

Using the results given in **(i)** or otherwise, derive the equation of state of the gas. Discuss how the temperature of such a gas would change following a free adiabatic expansion.

[You may assume without proof that the number of ways of distributing $\mathrm{N} = \Sigma n_i$ particles into states with energy E_i and degeneracy g_i is given by $\Omega = \frac{\mathrm{N}!\Pi_i g_i{}^{n_i}}{\Pi_i n_i!}$ and also that the particle distribution follows the Boltzmann distribution, so $n_i \propto g_i\ \exp(-\mathrm{E}_i/kT)$.]

End of Section A

[**CONTINUED...**

SECTION B

Question 4Z Structure and Evolution of Stars

(i) For a star in hydrostatic equilibrium, use the equations of stellar structure to derive the behaviour of the mass, $m(r)$, and the pressure $\mathrm{P}(r)$, close to the centre of the star, $r = 0$.

(ii) Describe the principles of homology scaling and their application to stars.

Using homology arguments, derive the mass-luminosity relation for radiative stars on the main sequence with:

(a) low mass, opacity $\kappa = \kappa_o \rho \mathrm{T}^{-3}$ and equation of state $\mathrm{P} = \frac{\Re}{\mu}\rho\mathrm{T}$, where T is the temperature, ρ is the density, κ_o is a constant, P is the pressure, $\Re$ is the gas constant and the mean molecular weight, μ, is constant throughout the star.

(b) intermediate mass, opacity $\kappa = \kappa_o$ and equation of state $\mathrm{P} = \frac{\Re}{\mu}\rho\mathrm{T}$.

(c) high mass, opacity $\kappa = \kappa_o$ but where radiation pressure exceeds the gas pressure and $\mathrm{P} \simeq \frac{1}{3}a\mathrm{T}^4$, where a is the radiation constant.

Why would the application of these homology scalings not be valid for main sequence stars of extremely low mass?

[**TURN OVER**

Question 5X Physical Cosmology

(i) The Friedman-Robertson-Walker (FRW) metric is

$$\mathrm{ds}^2 = c^2\mathrm{dt}^2 - \mathrm{R}^2(t)\left[\frac{\mathrm{dr}^2}{1-\kappa r^2} + \mathrm{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right]$$

where $\mathrm{R}(t)$ is the scale factor and κ is the curvature.

Consider two galaxies A and B. Suppose galaxy A has a redshift $z_A = 1$ and galaxy B has a redshift $z_B = 5$ as viewed from Earth. What is the redshift of B as viewed by an observer on A?

(ii) Consider a source S that is separated by a coordinate distance r from an observer O in a FRW cosmology. If the source has luminosity L, show that the flux observed at O is

$$F = \frac{\mathrm{L}}{4\pi\mathrm{R}_0^2 r^2(1+z)^2}$$

where $\mathrm{R}_0 = \mathrm{R}(t_0)$ is the scale factor at the present time. Hence, derive the luminosity distance d_L of the source.

Suppose further the source has a proper length l perpendicular to the line of sight. Show that the angle subtended by the source as recorded by the observer O is

$$\gamma = \frac{l(1+z)}{\mathrm{R}_0 r}.$$

Hence, derive the angular distance d_A of the source.

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For a spatially flat ($\kappa = 0$) FRW Universe with $\Omega_0 = 1$, show that

$$\begin{aligned} d_L &= \frac{2c}{\mathrm{H}_0}(1+z)\left[1-(1+z)^{-1/2}\right] \\ d_A &= \frac{2c}{\mathrm{H}_0}\frac{1}{(1+z)}\left[1-(1+z)^{-1/2}\right] \end{aligned}$$

where $\mathrm{H}_0 = \dot{\mathrm{R}}_0/\mathrm{R}_0$.

Plot d_A and d_L as a function of z. Explain why d_A has a maximum and find its location.

[**TURN OVER**

Question 6Y Stellar Dynamics and Structure of Galaxies

(i) State how the period of a Keplerian ellipse depends on the semi-major axis, eccentricity and total system mass.

A rocket is launched tangentially to the Earth's surface from a point on the equator so that it just has enough energy to reach a geo-stationary satellite which is located at zero longitude. From what longitude should the rocket be launched if it is to reach the satellite in the minimum time?

(ii) Show the (Jaffe) potential generated by the spherical density distribution

$$\rho(r) = \frac{M}{4\pi} \frac{r_j}{r^2 (r + r_j)^2}$$

is

$$\Phi(r) = \frac{\mathrm{G}M}{r_j} \ln \left\{ \frac{r}{r + r_j} \right\}$$

with M and r_j constants.

Verify that the total mass is M.

Calculate and sketch how the circular speed, V_c, varies as a function of r, clearly indicating its asymptotic behaviour at small and large r. Discuss whether the Jaffe potential is a good description for spiral galaxies with measured rotation curves.

State one advantage and one disadvantage of using gas rather than stars in order to trace the rotation curve of external galaxies. Explain what measurements are necessary in order to measure the rotation curve of the Milky Way.

[**CONTINUED...**

Question 7Z Topics

(i) In a classical nova system hydrogen is accreted onto the surface of a white dwarf of mass $1M_\odot$. A surface layer becomes hot enough for nuclear fusion to take place when the mass of the layer is $10^{-4}\mathrm{M}_\odot$. The observed peak luminosity is

$$L_{max} \approx L_{edd} \approx 1.3 \times 10^{31} W$$

where L_{edd} is the Eddington limiting luminosity. 100 days after maximum luminosity the nova has faded by a factor of 10^3. If all the hydrogen is converted to helium and all the energy released is radiated away, how long could the nova outburst last? Compare your answer with the observed outburst duration and comment on your result. What does the fact that the peak luminosity is equal to the Eddington limit suggest about the dynamics of the system? [The efficiency of conversion of rest mass energy into radiation is 0.7% for fusion of Hydrogen].

(ii) The luminosity of a white dwarf of mass M_{WD} is given by

$$L_{wd} = CT^{7/2}$$

where $C = 0.1022(M_{WD}/M_\odot)[\mathrm{W}^\circ\mathrm{K}^{-7/2}]$ and T is the interior temperature.

Show that the cooling timescale is

$$\tau_{cool} = \frac{3}{2}k\frac{M_{WD}}{Am_H CT^{5/2}}$$

[TURN OVER

where A is the number of nucleons per nucleus in the interior and $m_H (\approx m_P)$ is the mass of a hydrogen atom. For a carbon white dwarf with $L_{WD} = 0.03 L_\odot$ and $M_{WD} = 1 M_\odot$ calculate the internal temperature and τ_{cool}. How does this timescale compare to the actual time it takes the white dwarf to cool?

Show that the internal temperature and the luminosity will vary with time t according to

$$T = T_0 \left(1 + \frac{5}{2}\frac{t}{\tau_0}\right)^{-2/5}$$

and

$$L_{WD} = L_0 \left(1 + \frac{5}{2}\frac{t}{\tau_0}\right)^{-7/5}$$

where T_0, L_0 and τ_0 are the values of T, L_{WD} and τ_{cool} at $t = 0$.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Wednesday 6 June 2007 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

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SECTION A

Question 1X Theory of Relativity

(i) Define 4-momentum in special relativity. Apply conservation of 4-momentum to show that a free electron cannot absorb a photon.

(ii) The collision of a proton p and an antiproton $\bar{p}$ can result in the production of a W^+/W^- pair. Show that the total energy $\mathrm{E}_{tot} = \mathrm{E}_p + \mathrm{E}_{\bar{p}}$ required to produce a $\mathrm{W}^+/\mathrm{W}^-$ pair is minimal when $\mathrm{E}_p = \mathrm{E}_{\bar{p}}$. Compare this with the required energy in a collision where one of the protons is at rest. The velocity of the protons can be written as $v = c(1-\varepsilon)$ with $\varepsilon \ll 1$. Calculate ε for both cases.

[rest masses : $m_p c^2 = 938\mathrm{MeV}, m_w c^2 = 80.4\mathrm{GeV}$]

[**CONTINUED...**

Question 2Y Astrophysical Fluid Dynamics

(i) Show that the Lagrangian derivative $\frac{\mathrm{D}f}{\mathrm{D}t}$ of a quantity f (i.e. the rate of change of the quantity along the path of the fluid flow) is related to the Eulerian derivative $\partial f/\partial t$ by

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\underline{v}.\underline{\nabla})f$$

where $\underline{v}$ is the velocity of the fluid.

Hence derive the Euler equation for fluid flow

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\underline{\nabla})\underline{v} = -\frac{1}{\rho}\underline{\nabla}p - \underline{\nabla}\phi$$

where ρ is the fluid density, p is its pressure and ϕ is the gravitational potential.

(ii) Consider a fluid body rotating with constant angular velocity $\underline{\Omega}$. The rate of change of a vector $\underline{\mathrm{A}}$ in the inertial (i.e. non-rotating) frame, $\left(\frac{\mathrm{D}\underline{\mathrm{A}}}{\mathrm{D}t}\right)_{inertial}$ is related to the rate of change in the rotating frame $\left(\frac{\mathrm{D}\underline{\mathrm{A}}}{\mathrm{D}t}\right)_{body}$ by

$$\left(\frac{\mathrm{D}\underline{A}}{\mathrm{D}t}\right)_{inertial} = \left(\frac{\mathrm{D}\underline{A}}{\mathrm{D}t}\right)_{body} + \underline{\Omega} \wedge \underline{A}.$$

Show that the acceleration of a fluid element in the inertial frame is related to the acceleration in the rotating frame by

$$\begin{aligned}\left(\frac{\mathrm{D}^2\underline{r}}{\mathrm{D}t^2}\right)_{inertial} = \left(\frac{\mathrm{D}^2\underline{r}}{\mathrm{D}t^2}\right)_{body} &+ 2\underline{\Omega} \wedge \left(\frac{\mathrm{D}\underline{r}}{\mathrm{D}t}\right)_{body} \\ &+ \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r}).\end{aligned}$$

[**TURN OVER**

Hence show that the Euler equation for the fluid velocity $\underline{v}$ in the rotating frame is

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\underline{\nabla})\underline{v} + \underline{\Omega} \wedge (\underline{\Omega} \wedge \underline{r}) = -\frac{1}{\rho}\underline{\nabla} p - \underline{\nabla}\phi - 2\underline{\Omega} \wedge \underline{v} \qquad (*)$$

For slowly changing circulation in a thin planetary atmosphere, it is a good approximation to neglect the terms on the left hand side of **(*)** and to also neglect velocity components normal to the planet surface. Assume that the planet is spherical and rotating at constant angular velocity $\underline{\Omega}$, and consider a point $\underline{r}$ on the surface of the planet at polar angle θ to the spin axis. By transforming to a local Cartesian coordinate system $\underline{\hat{x}}, \underline{\hat{y}}, \underline{\hat{z}}$ (where $\underline{\hat{z}}$ is normal to the planet surface and $\hat{x}$ and $\hat{y}$ are respectively along lines of latitude and longitude) show that the horizontal components of the velocity satisfy

$$\begin{aligned} v_x &= -\frac{1}{2\Omega\cos\theta\rho}\frac{\partial p}{\partial y}, \\ v_y &= \frac{1}{2\Omega\cos\theta\rho}\frac{\partial p}{\partial x}. \end{aligned}$$

Show that the velocity can be written as

$$\underline{v} = -\frac{1}{2\Omega\cos\theta\rho}(\underline{\hat{z}} \wedge \underline{\nabla} p).$$

Hence show that the atmosphere flows along lines of constant pressure.

[**CONTINUED...**

Question 3Y Statistical Physics

(i) The temperature of a unit mass of monatomic gas is gradually increased until radiation pressure makes a significant contribution to the total pressure. Calculate the value of β ($= \mathrm{P}_{gas}/\mathrm{P}_{total}$) at which the heat capacity at constant volume attains a value that is twice its value at $\beta = 1$.

(ii) The gas described in **(i)** is contained in a vessel at fixed pressure, whose volume is used to measure empirical temperature, T_{emp}. This empirical temperature scale is calibrated such that $\mathrm{T}_{emp} \propto \mathrm{V}$ (the volume of the vessel) and also such that the T_{emp} coincides with the Kelvin (ideal gas) scale, T, when $\beta = 1$.

Sketch how V depends on T and hence explain why $T_{emp} > \mathrm{T}$ at high temperature.

The apparatus is set up with $\beta = \beta_0$ and the temperature raised by 1 Kelvin. Determine the value of β_0 if the corresponding change in empirical temperature differs from one degree by 10%.

End of Section A

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SECTION B

Question 4Z Structure and Evolution of Stars

(i) The atmosphere of a star consists of material with density $\rho, (\mathrm{kgm}^{-3})$, and opacity, $\kappa, (\mathrm{m}^2\mathrm{kg}^{-1})$. Write down expressions for the mean free path of a photon and for the optical depth of the atmosphere in terms of ρ and κ.

How is the perceived 'surface' of a stellar atmosphere related to the optical depth?

An image of the Sun taken in visible light shows a clear systematic effect, called limb darkening, whereby the apparent brightness decreases from a maximum at the centre of the Solar disc to a minimum at the edge. Explain carefully, using a sketch if appropriate, the physical origin of the limb darkening.

(ii) A double-lined spectroscopic binary system has a period P = 10yrs and shows peak to peak wavelength shifts of 1.2Å and 0.2Å measured using the Hα line (rest wavelength, 6563Å). The stars are on circular orbits and the orbits are viewed edge-on.

(a) Calculate the orbital velocities and the masses of the two binary components.

The binary is observed to have a bolometric magnitude m = 10.00 out of eclipse and shows a primary eclipse with minimum brightness m = 10.50 and a secondary eclipse with minimum brightness m = 10.01. Both eclipses last for a maximum duration of 5 days and both eclipses show a flat-bottomed central minimum of duration 4 days.

(b) Calculate the radii of the two stars.

(c) If the low mass star has an effective temperature $T_{eff} = 10,000$K calculate the effective temperature of the high mass star. [You may neglect the effects of limb darkening].

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Question 5X Physical Cosmology

(i) The deceleration parameter is

$$q_0 = -\frac{\ddot{\mathrm{R}}_0}{\mathrm{R}_0 \mathrm{H}_0^2},$$

where R_0 is the scale factor today, and $\mathrm{H}_0 = \dot{\mathrm{R}}_0/\mathrm{R}_0$ is the Hubble constant.

Show that in a matter-dominated FRW Universe with $\wedge = 0$

$$\Omega_{M,0} = 2q_0,$$

where $\Omega_{M,0}$ is the ratio of matter today to that required for a flat Universe.

(ii) Show further that in a matter dominated FRW Universe

$$\left(\frac{\dot{\mathrm{R}}}{\mathrm{R}_0}\right)^2 = \mathrm{H}_0^2\left[1 - 2q_0 + 2q_0\frac{\mathrm{R}_0}{\mathrm{R}}\right].$$

If $q_0 = \frac{1}{2}$, derive the relationship between scale-factor and time

$$\mathrm{H}_0 t = \frac{2}{3}\left(\frac{\mathrm{R}}{\mathrm{R}_0}\right)^{3/2}.$$

If $q_0 < \frac{1}{2}$, show by using the substitution

$$\frac{\mathrm{R}}{\mathrm{R}_0} = \frac{q_0}{2q_0 - 1}(1 - \cos\theta)$$

that

$$H_0 t = q_0(2q_0 - 1)^{-3/2}(\theta - \sin\theta).$$

[TURN OVER

If $q_0 > \frac{1}{2}$, find an appropriate substitution and show

$$H_0 t = q_0(1 - 2q_0)^{-3/2}(\sinh\Psi - \Psi).$$

where Ψ is a parameter.

Plot the evolution of the scale factor with time in these three cases.

[**CONTINUED...**

Question 6Y Stellar Dynamics and Structure of Galaxies

(i) The relative trajectory of two stars with masses m_1 and m_2 is described by the polar equation

$$r = \frac{a(1-e^2)}{1+e\ \cos\phi}.$$

Use the concept of reduced mass to show that the angular momentum of the system is

$$J = \frac{m_1 m_2 a^2 (1-e^2)^2 \dot{\phi}}{(m_1+m_2)(1+e\cos\phi)^2}.$$

Hence show that in order for J, and the total energy, E, to be conserved around the orbit, J and E are related to a, e and the component masses via:

$$\begin{aligned} J^2 &= \frac{Gm_1^2 m_2^2}{(m_1+m_2)} a(1-e^2) \\ E &= -\frac{Gm_1 m_2}{2a}. \end{aligned}$$

Show also that in the case of a hyperbolic orbit, the angle θ, through which each star is deflected following the gravitational encounter, is given by $\sin\frac{1}{2}\theta = \frac{1}{e}$.

(ii) A star cluster consists of stars of mass m with typical relative velocity v. Use the results given in **(i)** to show that - in the case of encounters that are highly hyperbolic - there is a maximum impact parameter, b_{max}, for encounters that will cause a deflection $\geq \beta$. Show that in the limit that β is small

$$b_{max} = \frac{f\mathrm{G}m}{v^2\beta}$$

where f is a factor of order unity. Hence if the number density of stars in the cluster is n, show that the timescale on which a star is expected to experience a deflection $> \beta$ is approximately $\tau \simeq \frac{v^3\beta^2}{nG^2m^2f^2}$.

[**TURN OVER**

A black hole, mass M_{bh}, situated at the centre of a cluster, swallows any stars that stray within a radius $R_{bh}(\ll \frac{GM_{bh}}{v^2})$. At radius r in the cluster, stars are said to be "in the loss cone" if – neglecting the effect of encounters with other stars – they are on orbits that will take them within R_{bh}. Show (assuming all stars have speed v) that the "loss cone" consists of stars whose velocity vectors lie within an angle θ_{max} of the inward pointing radius vector, and show that $\theta_{max} = \sqrt{\frac{2GM_{bh}R_{bh}}{rv}}$.

A star in the loss cone can avoid being swallowed by the hole if it is deflected out of the loss cone by a close stellar encounter before it reaches R_{bh}. Explain whether you expect a star in the loss cone to be swallowed in the following case:

$$\begin{array}{lll} M_{bh} = 10^3 M_\odot, & R_{bh} = 3 \times 10^6 \mathrm{m}, & m = 1M_\odot \\ v = 10\mathrm{kms}^{-1}, & n = 10^3 \mathrm{pc}^{-3}, & r = 1\mathrm{pc}. \end{array}$$

[**CONTINUED...**

Question 7Z Topics

(i) The observed rate of Gamma ray bursts (GRBs) at low redshifts is $0.44\ G\mathrm{pc}^{-3}\mathrm{yr}^{-1}$. The space density of typical galaxies is $3 \times 10^{-3}\mathrm{Mpc}^{-3}$. Assuming that a GRB within 2kpc of the Earth is dangerous (i.e. causes mass extinctions) and that our galaxy is a uniform disc of 30kpc diameter, estimate the rate of dangerous GRBs. How many will the Earth have experienced in its lifetime? Would a calculation based on more realistic assumptions give a higher or lower estimate? How would the GRB cause mass extinctions?

(ii) The angular deviation of photons passing a distance of closest approach x from a mass M is

$$\phi = \frac{4GM}{xc^2}$$

Show that

$$\theta^2 - \beta\theta - \frac{4GM}{c^2}\left(\frac{S-L}{SL}\right) = 0$$

where S is the distance of the source, L is the distance of the lensing mass, θ is the observed angular separation of the source and the lens and β is what the observed angle between the source and lens would have been if the photons were not deviated by the lens. This quadratic equation in θ will generally have two solutions θ_1 and θ_2.

Derive an expression for the angular radius of the Einstein ring

$$\theta_E = \theta_1 = \theta_2$$

for the special case where the two solutions are the same.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Thursday 7 June 2007 1:30pm – 4:30pm

ASTROPHYSICS - PAPER 3

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SECTION A

Question 1X Theory of Relativity

(i) A three-hyperboloid is a three-dimensional hypersurface in four-dimensional Euclidean space specified by coordinates x, y, z, w satisfying the equation

$$x^2 + y^2 + z^2 - w^2 = -a^2$$

where a is a constant.

By transforming to the coordinates (χ, θ, ϕ) where

$$\begin{aligned} x &= a \ \mathrm{sinh}\chi \ \mathrm{sin}\theta \ \mathrm{cos}\phi \\ y &= a \ \mathrm{sinh}\chi \ \mathrm{sin}\theta \ \mathrm{sin}\phi \\ z &= a \ \mathrm{sinh}\chi \ \mathrm{cos}\theta \\ w &= a \ \mathrm{cosh}\chi \end{aligned}$$

show that the metric on the three-hyperboloid is given by

$$ds^2 = g_{\mu v} dx^{\mu} dx^{v} =$$

$$a^2 d\chi^2 + a^2 \mathrm{sinh}^2\chi d\theta^2 + a^2 \mathrm{sinh}^2\chi \mathrm{sin}^2\ \theta d\phi^2$$

(ii) For the metric in part **(i)**, the diagonal part of the Ricci tensor satisfies $\mathrm{R}_{\mu\nu} = -2k\mathrm{g}_{\mu\nu}$.

Verify this equation for the diagonal component $\mathrm{R}_{\chi\chi}$ and determine the constant k.

[**CONTINUED...**

Question 2Y Astrophysical Fluid Dynamics

(i) Consider the gravitational collapse of a spherical gas cloud, initially with uniform density ρ_0 and radius R_0, assuming that the cloud is initially at rest and the internal pressure is negligible. Explain briefly why the cloud remains with uniform density, $\rho_c(t)$, throughout its collapse and show that ρ_c is related to the cloud radius $\mathrm{R}(t)$ via

$$\dot{\rho}_c = -\frac{3\dot{\mathrm{R}}}{\mathrm{R}}\rho_c,$$

where dots denote differentiation with respect to time. State a qualitative difference in the nature of the collapse if the internal pressure is not negligible. The density, velocity and gravitational potential at radius r within the cloud are perturbed such that

$$\begin{aligned}
\rho(\underline{r},t) &= \rho_c(t) + \rho_1(t)e^{i\underline{k}.\underline{r}/\mathrm{R}(t)}\\
\underline{v}(\underline{r},t) &= \frac{\dot{\mathrm{R}}}{\mathrm{R}}\underline{r} + \underline{v}_1(t)e^{i\underline{k}.\underline{r}/R(t)}\\
\phi(\underline{r},t) &= \phi_c(\underline{r},t) + \phi_1(t)e^{i\underline{k}.\underline{r}/\mathrm{R}(t)}
\end{aligned}$$

where ϕ_c represents the unperturbed potential and $\underline{k}$ is independent of time. Explain why the quantity $\underline{k}.\underline{r}/\mathrm{R}(t)$ is invariant with time for a particular fluid element. Explain why the unperturbed velocity is $\frac{\dot{\mathrm{R}}}{\mathrm{R}}\underline{r}$. Show that ϕ_1 and ρ_1 are related via

$$-\frac{k^2\phi_1}{\mathrm{R}^2(t)} = 4\pi G\rho_1.$$

[**TURN OVER**

(ii) Consider the collapsing cloud described in **(i)** and, using any of the expressions given in **(i)**, show that the evolution of the density and velocity perturbations (ρ_1 and $\underline{v}_1$) are described by

$$\dot{\rho}_1 + \frac{3\dot{\mathrm{R}}}{\mathrm{R}}\rho_1 + i\underline{v}_1.\underline{k}\frac{\rho_c(t)}{\mathrm{R}(t)} = 0$$

and

$$\dot{\underline{v}}_1 + \frac{\dot{\mathrm{R}}}{\mathrm{R}}\underline{v}_1 = \frac{4\pi G\rho_1}{-ik}\mathrm{R}(t)$$

Assume without proof that these equations imply that the fractional density perturbation $\delta = \rho_1(t)/\rho_c(t)$ evolves according to

$$\ddot{\delta} + 2\frac{\dot{\mathrm{R}}}{\mathrm{R}}\dot{\delta} - 4\pi G\rho_c\delta = 0 \qquad (*)$$

and that at late times in the collapse $\dot{\mathrm{R}} = -\sqrt{\frac{8\pi G\rho_0 R_0^3}{3R}}$. Show that in this case, there are solutions to **(*)** of the form $\delta \propto R^{-a}$ and determine the value of a corresponding to growing perturbations.
By what factor must the cloud collapse before density perturbations of initial amplitude of 1% become non-linear (i.e. $\delta \sim 1$)?

[**CONTINUED...**

Question 3Y Statistical Physics

(i) Particles with mass m, and (positive) charge q are free to move on a planar surface on which an electric field $\underline{\mathrm{E}}$ is uniformly directed towards the origin ($\mathrm{R} = 0$) of a cylindrical coordinate system.

Calculate, in the case that the ensemble is in a state of thermal equilibrium at temperature T, the fraction of particles at radius R' which have sufficient energy to reach radius $\mathrm{R} = \mathrm{R}_0$.

At time $t > t_0$, particles at $\mathrm{R} \geq \mathrm{R}_0$ are 'tagged' (i.e. assigned a permanent label that affects neither their mass or charge). Give a qualitative description of how the fraction of tagged particles at $\mathrm{R} = 0$ evolves with time, providing estimates of relevant timescales where possible.

(ii) Show using kinetic theory that the pressure exerted by a gas with number density n, mean squared particle velocity $\bar{v^2}$ and particle mass m is

$$\mathrm{P} = \frac{1}{3} n m \bar{v^2}.$$

Specify what constraints this places on the particle velocity distribution.

Hence derive an expression for the pressure, P, of a zero temperature non-relativistic Fermi gas with number density n, particle mass m.

A white dwarf, of mass M, radius R and (internal) temperature T is supported by electron degeneracy pressure. Show (by considering the pressure required to support the overlying crust, or otherwise) that at a depth $x(\ll \mathrm{R})$ below the surface, the pressure and number density of electrons are related via

$$x \simeq \frac{\mathrm{R}^2}{\mathrm{GM} m_p} \left(\frac{\mathrm{P}}{n} \right),$$

[**TURN OVER**

where m_p is the mass of a proton. Hence (for the case $\mathrm{M} = 2 \times 10^{30}$kg, $\mathrm{R} = 10^6$m, $\mathrm{T} = 10^4$K) determine the value of x where **(a)** the electrons become degenerate and **(b)** the electrons become relativistic.

End of Section A

[**CONTINUED...**

SECTION B

Question 4Z Structure and Evolution of Stars

(i) Energy generation in a star formed from ^{1}H and ^{4}He occurs via the sequence of reactions

$$
\begin{aligned}
{}^1\mathrm{H} + {}^1\mathrm{H} &\rightarrow {}^2\mathrm{H} + \mathrm{e}^+ + \nu \\
{}^1\mathrm{H} + {}^2\mathrm{H} &\rightarrow {}^3\mathrm{He} + \gamma \\
{}^3\mathrm{He} + {}^3\mathrm{He} &\rightarrow {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H}
\end{aligned}
$$

If the number of reactions between species i and species j per unit volume per unit time is $\mathrm{N}_i\mathrm{N}_j\lambda_{ij}$, where N_i is the number of atoms of species i per unit volume and λ_{ij} is the reaction rate for species i and j, write down the equations governing the rate of change of the number density of each of the species ^{1}H, ^{2}H, ^{3}He and ^{4}He.

Due to the rapidity of the reactions involved, the number of ^{2}H and ^{3}He nuclei is essentially constant. Write down the equations determining the rate of change of ^{1}H and ^{4}He in this case. How do you interpret your answer?

Sketch the ^{3}He abundance versus the stellar radius assuming there has been no convective mixing and that the star is halfway through its life on the main sequence.

(ii) A plot of absolute V magnitude, $\mathrm{M_V}$, versus colour, B-V, is given below.

[TURN OVER

The effective temperature, T_{eff}, is given for several values of B-V at the top of this colour-magnitude diagram. What are the six features labelled A-F?

For stars in each of these areas describe the energy source that gives rise to the star's observed luminosity.

At some point in their evolution, stars with mass less than $2.2\mathrm{M}_{\odot}$ will arrive in region E. What event immediately precedes this? Describe what happens. Which region are these stars in when this event happens? Why do stars with mass greater than $2.2\mathrm{M}_{\odot}$ not experience this event?

[CONTINUED...

Explain how lines of constant stellar radius can be plotted on the diagram and show how they appear. Where are the physically smallest and largest stars? Estimate the radii in units of solar radius $R_\odot$ of the smallest and largest stars on the diagram.

[The Sun has $M_V = 4.8$, $T_{eff} = 5770$K and $B - V = 0.64$]

[**TURN OVER**

Question 5X Physical Cosmology

(i) In big-bang nucleosynthesis, explain why the neutron to proton ratio at neutrino decoupling is given by

$$\frac{n_n}{n_p} = \exp\left(-Q/k\mathrm{T}_d\right)$$

where $Q = (m_n - m_p)c^2$, T_d is the decoupling temperature, m_n is the mass of the neutron and m_p is the mass of the proton.

If $Q = 1.29MeV$ and $k\mathrm{T}_d = 0.8MeV$, estimate n_n/n_p.

(ii) Show that the primordial abundance of helium by mass is

$$Y = 2\left(1 + \frac{n_p}{n_n}\right)^{-1}$$

Roughly estimate Y, assuming the neutron half-life is 615s and deuterium is stabilised at 300s. If the neutron-proton mass difference Q were increased by 10%, how would this affect the helium abundance?

Consider a Universe with the same cosmological parameters as ours, but with the difference that the universal expansion between 1s and 300s was accelerated by an unknown force. How does this affect the helium abundance?

[**CONTINUED...**

Question 6Y Stellar Dynamics and Structure of Galaxies

(i) By taking moments of the collisionless Boltzmann equation, show that the mean stellar velocities v_i and velocity dispersion σ_{ij}^2 satisfy

$$\nu\frac{\partial \bar{v}_j}{\partial t} + \nu\bar{v}_i\frac{\partial \bar{v}_j}{\partial x_i} = -\nu\frac{\partial \phi}{\partial x_j} - \frac{\partial(\nu\sigma_{ij}^2)}{\partial x_i}$$

where ν is the luminosity density, ϕ the gravitational potential.

(ii) A non-rotating spherical stellar system, of mass M and isotropic one dimensional velocity dispersion $\sigma(r)$ has a density profile

$$\begin{aligned} \rho(r) &= \rho_0\left(\frac{r_0}{r}\right)^{\frac{1}{2}} \qquad r \le r_{out} \\ &= 0 \qquad\qquad\qquad r > r_{out} \end{aligned}$$

where ρ_0 and r_0 are constants.

Show that for the case that the cluster is in equilibrium

$$\sigma^2 = \frac{8\pi}{5}\mathrm{G}\rho_0 r_0^{\frac{1}{2}} r^{\frac{1}{2}}(r_{out} - r).$$

Show furthermore that the escape velocity, v_{esc} from radius r is given by

$$v_{esc}^2 = \frac{16\pi}{3}\mathrm{G}\rho_0 r_0^{\frac{1}{2}}\left(r_{out}^{\frac{3}{2}} - \frac{2}{5}r^{\frac{3}{2}}\right).$$

Hence show that the cluster satisfies the viral theorem:

$$2\mathrm{T} + \mathrm{W} = 0$$

where T is the total kinetic energy and W the total gravitational potential energy of the cluster.

[**TURN OVER**

Question 7Z Topics

(i) A binary system consisting of two black holes, each of initial mass $10^4 \mathrm{M}_\odot$ and initial separation 1pc, accretes non-rotating gas. Use the principle of angular momentum conservation to explain why the binary orbit shrinks as the black holes grow in mass. Derive an expression for the separation of the binary when the black hole mass is M.

As the black holes spiral in, they start to lose angular momentum as a result of the emission of gravitational radiation. For holes of mass M and separation a, the separation shrinks due to gravitational radiation on a timescale

$$\tau_{gr} = 1.7 \times 10^{25} \text{ years } \frac{(a/1\mathrm{pc})^4}{(\mathrm{M}/10^4 \mathrm{M}_\odot)^3}.$$

Determine (in the case that the black holes accrete non-rotating gas at a rate of $10^{-3} \mathrm{M}_\odot \mathrm{yr}^{-1}$), the radius at which gravitational radiation takes over as the main mechanism driving the black holes together. What is the mass of the black holes at this point?

(ii) A radio galaxy jet is inclined at an angle of 40° to the line of sight. A component of the jet is seen to have superluminal motion with an apparent velocity $V_{app} = 2.5c$. What is the actual bulk velocity of this jet component?

What angle to the line of sight maximises the apparent jet velocity? Use your value of the actual jet velocity to find V_{app} in this case.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Friday 8 June 2007 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

The paper is divided into Section **A** *and Section* **B**. *Candidates may attempt Parts from* **ALL** *questions in Section* **A** *and from not more than* **THREE** *questions in Section* **B**.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of questions.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 5X** *should be in one bundle and* **2Y, 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

Question 1X Theory of Relativity

(i) In Newton's Theory of gravity, show that the equation of motion of a test particle orbiting a point mass M in the plane $\theta = \pi/2$ can be written as

$$\frac{d^2u}{d\phi^2} + u = \frac{\mathrm{GM}}{h^2}, \qquad h = r^2\frac{d\phi}{dt}$$

where $u = 1/r$, G is the gravitational constant (r, θ, ϕ) are the usual spherical polar coordinates. Show there is a circular orbit at any radius.

In general relativity, the equation of motion becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{\mathrm{GM}}{h^2} + \frac{3\mathrm{GM}u^2}{c^2}, \qquad h = r^2\frac{d\phi}{d\tau} \qquad (*)$$

where τ is proper time. Show that there is no circular orbit with

$$r < \frac{3\mathrm{GM}}{c^2}$$

(ii) A test particle is moving tangentially with velocity $v = 50\mathrm{kms}^{-1}$ at distance $b = 1$ parsec relative to a black hole with mass $M = 10^9 M_\odot$. Starting from equation **(*)** in part **(i)** of the question, show that the test particle will cross the horizon of the black hole despite its angular momentum.

[CONTINUED...

Question 2Y Astrophysical Fluid Dynamics

(i) Consider non-relativistic flow in a highly conducting fluid threaded by a magnetic field $\underline{\mathrm{B}}$. Explain why it is a good approximation to assume that the current density $\underline{j}$ and magnetic force density $\underline{F}_{mag}$ are given by

$$\underline{j} = \frac{1}{\mu_0}\underline{\nabla} \wedge \underline{B},$$

$$\underline{F}_{mag} = \underline{j} \wedge \underline{B}.$$

Show that the magnetic force density can be written as

$$\underline{F}_{mag} = \frac{1}{\mu_0}\left[-\underline{\nabla}\left(\frac{\mathrm{B}^2}{2}\right) + (\underline{\mathrm{B}}.\underline{\nabla})\,\underline{\mathrm{B}}\right]. \qquad (*)$$

Give a physical interpretation of the two terms in **(*)**.

A plasma is threaded by a magnetic field $\underline{\mathrm{B}} = (0, \mathrm{B}_0, 0)$ in cylindrical polar coordinates, $(\hat{e}_{\mathrm{R}}, \hat{e}_\phi, \hat{e}_z)$ where B_0 is a constant. Show that the magnetic force density is

$$\underline{F}_{mag} = -\frac{1}{\mu_0}\frac{\mathrm{B}_0^2}{\mathrm{R}}\hat{e}_{\mathrm{R}}.$$

(ii) A solar prominence can be modelled as a thin isothermal sheet of gas, temperature T, with normal in the x direction, protruding from the sun whose surface normal vector is in the z direction. Assume that the magnetic field within the prominence is $\underline{\mathrm{B}} = (\mathrm{B}_x, \mathrm{B}_y, \mathrm{B}_z(x))$ in Cartesian coordinates (where B_x and B_y are constant) and that all quantities in the prominence are independent of y and z. By using **(*)** from part **(i)** or otherwise, show that in a steady state the pressure and density of the prominence must satisfy

$$\frac{\mathrm{dp}}{\mathrm{d}x} = -\frac{1}{2\mu_0}\frac{\mathrm{dB}_z^2}{\mathrm{d}x},$$

$$\rho g = \frac{1}{\mu_0}\mathrm{B}_x\frac{\mathrm{dB}_z}{\mathrm{d}x},$$

[**TURN OVER**

where g is the gravitational acceleration at the surface of the sun.

Hence show that if $B_z = 0$ at $x = 0$ then $\mathrm{B}_z(x)$ is given by

$$\mathrm{B}_z(x) = \mathrm{B}_0 \tanh\left(\frac{\mathrm{B}_0}{2\mathrm{B}_x}\frac{x\mu g}{\Re \mathrm{T}}\right),$$

where B_0 is a constant, μ the mean molecular weight of the gas and $\Re$ is the gas constant. What is the interpretation of B_0?

Determine the density structure $\rho(x)$ of the prominence.

[**CONTINUED...**

Question 3Y Statistical Physics

(i) Define the Rosseland mean opacity. State an astronomical environment in which this opacity is useful for calculating the transfer of radiation and explain, without detailed calculation, why this is the case.

The opacity of a gas as a function of frequency, v, is given by

$$\kappa_\nu = \kappa_1 \ (\nu_1 < \nu < \nu_2)$$

and

$$\kappa_\nu = \kappa_2 \quad \text{otherwise,}$$

where $\kappa_1 \ll \kappa_2$.

The Planck mean opacity is defined by

$$\kappa_p = \frac{\int \kappa_\nu B_\nu(\mathrm{T}) d\nu}{\int B_\nu(\mathrm{T}) d\nu}.$$

State, without detailed calculation, whether it is the value of κ_1 or κ_2 which mainly determines the value of **(a)** the Rosseland mean opacity and **(b)** the Planck mean opacity. Use the astronomical example that you gave above in order to briefly explain your answer to **(a)**.

(ii) Explain briefly why the coefficient of thermal conductivity is of the form

$$\mathrm{K} = \frac{1}{3} C_V n \bar{v} \lambda$$

defining all the terms (and their dimensions) in the above equation.

[**TURN OVER**

A fixed mass of monatomic gas is uniformly distributed in a sphere of initial radius R_0. Show, in the limit when the gas expands quasistatically and undergoes negligible heat exchange with the surroundings, that its subsequent temperature, T, and radius, R, are related by

$$T = T_0 \left(\frac{R}{R_0}\right)^a \qquad (*)$$

where T_0 is the temperature at $R = R_0$ and a is a constant which you should determine.

Assume now that the gas suffers some heat loss through conduction across a shell (radius R, thickness ΔR, where $\Delta R = \beta R$ and β is a constant $\ll 1$) which separates it from a medium with temperature $\ll T$. Assuming the relationship for T(R) **(*)** above, show that the cooling time of the gas is of the form $t_{cool} \propto R^b$ and determine the value of b.

If the size of the gas sphere doubles in a timescale τ explain why equation **(*)** is only approximately valid for a restricted range of τ, $\tau_1 \leq \tau \leq \tau_2$. Indicate, without detailed calculation, what are the physical criteria that set the values of τ_1 and τ_2.

End of Section A

[**CONTINUED...**

SECTION B

Question 4Z Structure and Evolution of Stars

(i) Derive an expression for the central pressure P_c of a uniform density self-gravitating sphere, with mass M and radius R, in hydrostatic equilibrium. Explain how the scaling relationship $P_c(M, R)$ is changed when the object is not of uniform density.

If the equation of state for the degenerate white dwarf is given by $P = k\rho^{5/3}$, where ρ is density and k is a constant, show that the density and radius of the white dwarf scale with mass according to

$$\begin{aligned} \rho &\propto M^2 \\ R &\propto M^{-1/3} \end{aligned}$$

(ii) A star of mass M loses mass on the red giant branch at a rate given by

$$\frac{dM}{dt} = -\frac{AL^{1.5}}{M}$$

where L is the luminosity of the star and A is a constant. The luminosity is determined by the core mass of the star M_c according to

$$L = BM_c^6$$

where B is a constant, and the growth in the mass of the core due to nuclear burning is given by

$$E\frac{dM_c}{dt} = L$$

where E is the nuclear energy produced per unit mass.

[**TURN OVER**

(a) Assuming that mass loss begins at a suitably low value of core mass, show that the evolution should terminate at a final mass $M_c = M \equiv M_f$ given by

$$M_f^2 + 0.5AB^{0.5}EM_f^4 = M_0^2$$

where M_0 is the initial mass of the star.

With masses and luminosity in solar units and time in years, the values of the constants are $A \simeq 10^{-12}, B \simeq 9 \times 10^4, E \simeq 7 \times 10^{10}$.

(b) Determine the initial masses of stars that produce remnants of $0.5\mathrm{M}_\odot$ and $1\mathrm{M}_\odot$.

Do your answers seem reasonable given your knowledge of stellar evolution and of the masses of white dwarfs?

[**CONTINUED...**

Question 5X Physical Cosmology

(i) The density ρ_i and pressure p_i of a fluid are related by the equation of state

$$p_i = w_i \rho_i c^2.$$

Consider a Friedman-Robertson-Walker Universe made up of two such components. Show that

$$\frac{\Omega_i}{\Omega_j} \propto \mathrm{R}^{-3(w_i - w_j)},$$

where R is the scale factor and Ω_i and Ω_j are fractional contributions to the critical density.

What are the implications of this result for the concordance cosmology in which the fractional contribution of matter, Ω_m, and cosmological constant, $\Omega_\wedge$, to the critical density are $\Omega_m = 0.3$ and $\Omega_\wedge = 0.7$ respectively?

(ii) The cross section for scattering of free electrons by photons is

$$\sigma_e = 6.7 \times 10^{-29} m^2.$$

Assuming an electron density at the present epoch of $0.2 m^{-3}$, estimate **(a)** the density of electrons at the epoch of decoupling ($z = 1000$), **(b)** the mean free path and time between photon-electron collisions just before decoupling.

What does this imply about the opacity of the Universe at early times?

Suppose at an epoch after recombination, an early generation of stars ionized the neutral gas so that the mean number density of free electrons is

$$n_e = 1.1 \times 10^{-5} \Omega_{\mathrm{IGM}}\ \mathrm{h}^2 \mathrm{cm}^{-3},$$

[**TURN OVER**

where h $= H_0/100$ kms^{-1}Mpc^{-1} and $\Omega_{\rm IGM}$ is the fractional contribution to the critical density.

Show that in a matter dominated Friedman-Robertson-Walker Universe, the optical depth for scattering by free electrons is

$$\tau = 0.046 \ \mathrm{h}\Omega_{\rm IGM}\Omega_m^{-1/2}(1+z)^{3/2}.$$

Estimate the redshift at which $\tau \approx 1$, given $h = 0.7$, $\Omega_{\rm IGM} = 0.02$ and $\Omega_m = 0.3$.

[**CONTINUED...**

Question 6Y Stellar Dynamics and Structure of Galaxies

(i) The density profile of a spherical star cluster is Gaussian, i.e.

$$\rho(r) \propto \exp\left(-\frac{r^2}{2r_0^2}\right)$$

Derive an expression for the projected surface density profile $\Sigma(\mathrm{R})$, where R is radius measured on the sky, for the case that $\Sigma(0) = \Sigma_0$.

Within what value of R is half of the mass contained in projection?

State without calculation whether this is greater or smaller than the radius containing half of the mass in three dimensions.

(ii) A star describes a nearly circular orbit at $\mathrm{R} \sim \mathrm{R}_G$ in an axisymmetric potential. The 'guiding centre' of the orbit is defined as a point co-moving with the motion of a star in a circular orbit at $\mathrm{R} = \mathrm{R}_G$. Show that the star's radial offset from the guiding centre can be described as simple harmonic motion with frequency κ where

$$\kappa^2 = \left(\mathrm{R}\frac{d\Omega^2}{d\mathrm{R}} + 4\Omega^2\right)\Big|_{R_G}$$

and Ω is the angular velocity of the guiding centre.

Explain why the frequency of excursions from the guiding centre in the azimuthal direction is also given by κ. Show, in the case that the contribution of the local mass density to the gravitational potential can be neglected, that the frequency of oscillations normal to the orbital plane, ν, is related to κ and Ω via

$$\nu^2 + \kappa^2 = 2\Omega^2.$$

[TURN OVER

Show that the orbit of a star on a nearly circular orbit in the orbital plane of the guiding centre can be described as an ellipse with respect to the guiding centre, with axis ratio

$$\frac{\mathrm{X}}{\mathrm{Y}} = \frac{\kappa}{2\Omega},$$

where X and Y are respectively the maximum excursions in the radial and azimuthal directions.

Show that $\frac{\mathrm{X}}{\mathrm{Y}} \sim 0.7$ for stars in the solar neighbourhood (for which Oort's constants are $\mathrm{A} = 14.5\mathrm{kms}^{-1}\mathrm{kpc}^{-1}$ and $\mathrm{B} = -12\mathrm{kms}^{-1}\mathrm{kpc}^{-1}$). Explain, without detailed calculation, why although individual epicycles are elongated in the azimuthal direction, the velocity ellipsoid of stars in the solar neighbourhood is elongated in the radial direction.

[**CONTINUED...**

Question 7Z Topics

(i) A planet orbits a star, radius R_*, temperature T_*, at distance a. Find approximate expressions for the temperature of the side of the planet closer to the star, T_{close} in each of the following cases: **(a)** the incident flux is re-radiated from the entire surface of the planet, which has uniform temperature and **(b)** the incident flux is re-radiated entirely from the side of the planet facing the star. Provide numerical values for T_{close} for each of **(a)** and **(b)** in the case that $R_* = 7 \times 10^8$m, $T_* = 5700$K and $a = 0.1$ $A.U.$

Suggest what factors would determine whether **(a)** or **(b)** is more likely.

A suitably positioned observer sees a primary eclipse (when the planet passes in front of the star) and a secondary eclipse (when the star passes in front of the planet). State, without detailed calculation, how the depths of the primary and secondary eclipses (measured relative to the total system flux near the time of the eclipse) compare for the two cases.

(ii) A dust grain of radius $r = 10\mu$m and density 1000kgm^{-3} is located at a distance of $a = 1A.U.$ from the Sun. Estimate the force due to solar radiation acting on the grain and compare this with the gravitational force exerted by the Sun. For what radius of grain would the two forces be equal and how does this critical dust grain size depend on the grain's distance from the Sun?

A grain of radius r is irradiated by starlight on one side with intensity $I(W\text{m}^{-2})$. If this incident flux is balanced by heat flow due to conduction through the grain, show that the difference in temperature between the two sides of the grain is approximately

$$\Delta T \sim \frac{Ir}{K}$$

where K is the conductivity.

[**TURN OVER**

The grain is immersed in gas with pressure P and temperature T, equal to the mean temperature of the grain. Equal numbers of molecules per second bombard the grain from both sides, but those that evaporate from the hotter side impart a greater recoil velocity to the grain. Show that the magnitude of this ("photophoresis") force on the grain is (omitting factors of order unity):

$$\mathrm{F} \sim \frac{\mathrm{PI}r^3}{KT}.$$

Is the grain propelled towards or away from the star? Explain how the relative importance of photophoresis and radiation pressure depends on the grain size.

END OF PAPER