NATURAL SCIENCES TRIPOS Part II

Saturday 29 May 2010 9.00 am to 12.00 noon

EXPERIMENTAL AND THEORETICAL PHYSICS (2)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (2)

Candidates offering the **whole** *of this paper should attempt a total of* **six** *questions, three from Section A* **and** *three from Section B. The questions to be attempted are* **A1**, **A2** *and* **one** *other question from Section A and* **B1**, **B2** *and* **one** *other question from Section B.*

Candidates offering **half** *of this paper should attempt a total of* **three** *questions,* **either** *three from Section A* **or** *three from Section B. The questions to be attempted are* **A1**, **A2** *and* **one** *other question from Section A or* **B1**, **B2** *and* **one** *other question from Section B. These candidates will leave after* **90 minutes**.

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **7** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

A separate Answer Book should be used for each section.

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Metric graph paper
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

ADVANCED QUANTUM PHYSICS

A1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) Write down the operator corresponding to the spin component in the direction $\widehat{\boldsymbol{n}} = (1/\sqrt{2})\,(1, 0, 1)$ for a spin-half particle, and calculate its eigenvalues and non-normalised eigenvectors. [4]

$\left[\textit{The Pauli spin matrices } \sigma_x,\ \sigma_y,\ \sigma_z \textit{ are } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\right]$

(b) For a wavefunction of the form $\psi = Axf(r)$, where $r^2 = x^2 + y^2 + z^2$, show that the uncertainty in the angular momentum component L_x is zero. [4]

(c) A two-dimensional harmonic oscillator is described by a potential of the form

$$V = \frac{1}{2}m\omega^2\left[(x^2 + y^2) + \alpha(x - y)^2\right],$$

where α is a positive constant. Find the ground-state energy of the oscillator. [4]

A2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) the addition of angular momenta in quantum mechanics;

(b) Fermi's golden rule;

(c) space and spin wavefunctions for identical particles.

A3 *Attempt* **either** *this question* **or** *question A4.*

A system is described by a Hamiltonian $\widehat{H}_0$ possessing a complete set of non-degenerate eigenstates $|n^0\rangle$ of energy E_n^0. Show that, when a small perturbing potential $\widehat{H}_1$ is applied to the system, the first-order contribution to the change in energy of the n^{th} eigenstate is $E_n^1 = \langle n^0|\widehat{H}_1|n^0\rangle$. (Note that the superscripts are used to label the order of each term.) [6]

Show also that the amplitude of the state $|m^0\rangle$ in the first-order perturbation expansion of the wave function for the nth state ($m \neq n$) is

$$\langle m^0|n^1\rangle = \frac{\langle m^0|\widehat{H}_1|n^0\rangle}{E_n^0 - E_m^0}. \qquad [2]$$

Hence show that the second-order contribution to the change in energy of the n^{th} state is

$$E_n^2 = \sum_{m\neq n} \frac{\left|\langle m^0|\widehat{H}_1|n^0\rangle\right|^2}{E_n^0 - E_m^0}. \qquad [5]$$

A particle moving in a simple-harmonic potential of frequency ω is described by the Hamiltonian

$$\widehat{H}_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

where

$$[a, a^\dagger] = 1.$$

A small perturbation is applied to the system, giving the total Hamiltonian $\widehat{H} = \widehat{H}_0 + \widehat{H}_1$, where

$$\widehat{H}_1 = \beta\left[(a^\dagger)^3 + 3(a^\dagger)^2 a + 3a^\dagger a^2 + a^3\right].$$

Find the energy of the perturbed ground state up to and including terms of second order in β. [8]

Describe briefly circumstances in which perturbation theory in the form described above may break down or need modification. [4]

[*The m^{th} excited state of a simple-harmonic oscillator is given by*

$$|m\rangle = \frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle,$$

where $|0\rangle$ is the ground state.]

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A4 *Attempt* **either** *this question* **or** *question A3.*

State Hund's rules and explain the underlying physical principles upon which they are based. [7]

The neutral zirconium atom has electronic configuration

$$[\mathrm{Kr}](5\mathrm{s})^2(4\mathrm{d})^2.$$

Determine the spectroscopic terms for the possible multiplets and predict which is the ground state, assuming that LS coupling holds approximately in zirconium. [5]

A sample of zirconium is placed in a magnetic field of 1 T. Show that it can absorb microwaves with a wavelength of approximately 32 mm. [6]

State the selection rules that apply to the total angular momentum quantum numbers J and m_J. Determine the spectrum for transitions in zirconium between a $^1\mathrm{D}_2$ state and a $^3\mathrm{F}_2$ state in the presence of a weak magnetic field B, expressing the line separations in terms of $\mu_{\mathrm{B}}B$, where μ_{B} is the Bohr magneton. [7]

[*You may assume the formula for the Landé g-factor:*

$$g = \frac{3J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$
]

SECTION B

OPTICS AND ELECTRODYNAMICS

B1 *Attempt* **all** *parts of this question. Answers should be concise, and relevant formulae may be assumed without proof.*

(a) The power gain of a Hertzian electric dipole is given by $G = (3/2)\sin^2\theta$. What is the maximum effective area in μm^2 of such a dipole when illuminated by light of wavelength 500 nm? [4]

(b) A 1D photonic crystal is fabricated from equal volumes of two materials of refractive index $n_1 = 1.4$ and $n_2 = 1.6$, with a periodicity 200 nm. Sketch the dispersion relation for the crystal and find the approximate wavelength at the bandgap for normal incidence light. [4]

(c) For an electron, the phase difference between two paths enclosing a magnetic flux Φ is $\Delta = e\Phi/\hbar$. Indicate the origin of this equation and estimate the magnetic field strength B required to convert constructive to destructive electron interference around a circular conducting loop of radius 1 μm. [4]

B2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) what spatial and spectral coherence reveal about a light source;

(b) how antennas can be made directional;

(c) the properties of synchrotron radiation.

(TURN OVER

B3 *Attempt* **either** *this question* **or** *question B4.*

Define the *four-vector potential* and *four-vector current* and explain how they can be used to write Maxwell's equations in Lorentz-covariant form. [5]

Using the four-vector transformation given by

$$\begin{pmatrix} a_0' \\ a_1' \\ a_2' \\ a_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

derive the following transformation relations for electric and magnetic fields from an inertial frame S to an inertial frame S' moving relative to S with velocity u in the x-direction:

$$\begin{aligned} E_x' &= E_x, & B_x' &= B_x \\ E_y' &= \gamma(E_y - uB_z), & B_y' &= \gamma(B_y + uE_z/c^2) \\ E_z' &= \gamma(E_z + uB_y), & B_z' &= \gamma(B_z - uE_y/c^2). \end{aligned}$$ [6]

A large thin conducting plate lying in the x–y plane carries a charge σ per unit area and a current density in the y-direction of magnitude J_y (per unit width), both measured in the rest frame of the plate. Find the electric field outside the plate in its rest frame, and show that the electrostatic potential in the region $z > 0$ is given by

$$\phi = -\frac{\sigma z}{2\epsilon_0}.$$ [2]

Given that the magnetic vector potential above the plate is $\boldsymbol{A} = (0, -\mu_0 J_y z/2, 0)$, deduce the magnetic field measured in the rest frame of the plate. [2]

By explicitly transforming the fields from the rest frame, calculate the electric and magnetic fields measured in a frame moving relative to the plate with relativistic velocity u in the x-direction. [4]

Hence find expressions for the charge and current densities measured in the moving frame. Account qualitatively for how the charge and current densities in the moving frame differ from those measured in the rest frame. [6]

B4 *Attempt* **either** *this question* **or** *question B3.*

Explain the use of Jones vectors and Jones matrices to describe the propagation of polarised light through optical components. [4]

Show that the Jones matrix for a linear polariser mounted with its transmitting axis at an angle θ to the transverse x-axis of a propagating light beam is

$$\underline{\underline{\boldsymbol{J}}}(\theta) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix},$$

and use this result to prove that no light can pass through crossed polarisers. [5]

An optical system consists of N linear polarisers mounted sequentially, with the transmitting axis of the n^{th} polariser oriented at an angle $n\theta$ ($n = 1, 2, \ldots, N$) to the x-axis. By evaluating the matrix $\underline{\underline{\boldsymbol{R}}}(\theta)\cdot\underline{\underline{\boldsymbol{J}}}(\theta)$, or otherwise, show that

$$\begin{pmatrix} j_{x'} \\ j_{y'} \end{pmatrix} = \cos^N\theta \begin{pmatrix} 1 & \tan\theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix},$$

where (j_x, j_y) is a Jones vector describing the incident light beam in a coordinate basis defined by the x-axis, and $(j_{x'}, j_{y'})$ is the Jones vector for the transmitted light in a coordinate basis defined such that the x'-axis is oriented along the transmitting axis of the final polariser. [6]

For the case that the incident light beam is linearly polarised along the x-direction and emerges with its plane of polarisation rotated through 90°, find the fractional transmitted intensity for a system containing $N = 20$ polarisers and show that the number of polarisers needed to reduce the intensity loss below 1% is given approximately by

$$N \approx \frac{(10\pi)^2}{4}.$$ [8]

Explain why it is impracticable to construct such a low-loss polarisation rotation element from conventional linear polarisers. [2]

[*The matrix which rotates the orientation of axes through an angle θ is*

$$\underline{\underline{\boldsymbol{R}}}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$
]

END OF PAPER