UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 5 June 2017 13.30pm – 16.30pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (i) *and Part* (ii), *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i). *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 2X and 7X** *should be in one bundle and* **3Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) Derive the addition formulae for the x, y and z velocity components according to Special Relativity.

In a static medium, the speed of light is reduced to c/n, where $n \geq 1$ is the refractive index. Now consider a medium which is moving at speed u relative to an observer. Show that the speed of a light beam propagating through the medium measured by the observer is

$$c_{\text{eff}} \approx \frac{c}{n} + u\left(1 - \frac{1}{n^2}\right), \qquad \text{where} \quad \frac{u}{c} \ll 1.$$

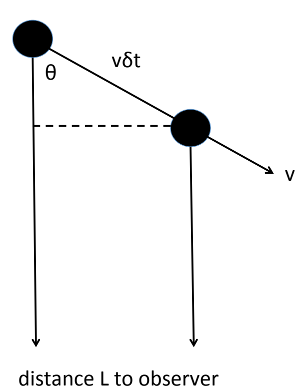
How does this result differ from that obtained according to Newtonian theory?

(ii) Consider a compact object in our Galaxy located at a large distance L away from us. The object emits a blob of material at angle θ to our line-of-sight with an ejection speed v close to the speed of light as shown in the Figure. Radiation from the blob is detected by an observer at time t_1 as measured in our rest frame. At a later time, the blob has moved a distance $v\delta t$ and emits radiation which is detected by the observer at time t_2. Show that the apparent speed of the blob on the sky is

$$v_{\text{app}} = \frac{v \sin\theta}{[1 - (v/c)\cos\theta]}.$$

Find the angle that maximises v_{app} and find the value of v_{app} at this angle.

Explain why the apparent 'superluminal' speed of the blob is compatible with Special Relativity.



Question 2X - Astrophysical Fluid Dynamics

(i) Define streamlines, particle paths and streaklines. For which flows do they coincide? Explain why.

A planetoid is moving through a uniform interstellar medium with a temperature $T_{\rm ISM} = 100\,{\rm K}$. The planetoid's speed is $0.1\,{\rm km\,s^{-1}}$. Sketch the streamlines in the rest frame of the planetoid. Draw a similar sketch assuming the planetoid's speed is $100\,{\rm km\,s^{-1}}$ and comment on the possible differences in the streamlines.

(ii) Consider a small adiabatic perturbation of density ρ', pressure p' and velocity $\boldsymbol{u'}$ in a uniform stationary fluid characterized by a constant shear viscosity η. Derive the following wave-like equation for the perturbation

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{4}{3}\eta\frac{1}{\rho_0}\boldsymbol{\nabla}^2\frac{\partial \rho'}{\partial t} = c_{\rm s}^2\boldsymbol{\nabla}^2\rho', \qquad (*)$$

where ρ_0 is the unperturbed density and $c_{\rm s}$ is the sound speed of the fluid.

Hence derive the dispersion relation between the wave number k and the angular frequency ω.

Qualitatively discuss whether exponentially dampened solutions exist and, if so, their physical meaning.

Using (*), or otherwise, derive an order of magnitude estimate for the condition on the wavelength λ for viscous effects to be small.

$$\left[\begin{array}{l} \text{The Navier} - \text{Stokes equation is :} \\ \qquad \rho\partial\boldsymbol{u}/\partial t + (\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \eta\left[\boldsymbol{\nabla}^2\boldsymbol{u} + \tfrac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{u})\right], \\ \text{where } \boldsymbol{u} \text{ is the fluid velocity, } \rho \text{ the density and } p \text{ the pressure.} \end{array}\right]$$

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) State the Cosmological Principle and discuss whether it is compatible with observations.

Show that if the Cosmological Principle applies in an expanding Universe the Hubble law $v(r) = Hr$ holds, where v is the relative speed between points separated by distance r and H is the Hubble parameter.

(ii) Assume $\Omega_{\mathrm{k},0}$, $\Omega_{\mathrm{m},0}$, and $\Omega_{\Lambda,0}$ are the present-day contributions to the critical density by curvature, matter, and the cosmological constant, respectively. Starting from the Friedmann equations, show that in a spatially flat Universe with $\Omega_{\mathrm{k},0} = 0$, $\Omega_{\mathrm{m},0} + \Omega_{\Lambda,0} = 1$, and $\Omega_{\Lambda,0} \neq 0$, the age of the Universe can be written as

$$t(z) = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln \left[\frac{1+\cos\theta}{\sin\theta}\right] ,$$

where z is the cosmological redshift, H_0 is the present-day value of the Hubble parameter, and

$$\tan\theta = \left(\frac{\Omega_{\mathrm{m},0}}{\Omega_{\Lambda,0}}\right)^{1/2} (1+z)^{3/2} .$$

Show that the present age of the Universe is then given by

$$t_0 = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln \left[\frac{1+\Omega_{\Lambda,0}^{1/2}}{(1-\Omega_{\Lambda,0})^{1/2}}\right] .$$

Explain why observations of supernovae help reconcile flat Universe models with estimates of the ages of globular clusters?

Question 4Z - Structure and Evolution of Stars

(i) An optical spectrum of a supernova in the Milky Way, observed some days after maximum brightness, shows the hydrogen Balmer Hα line (rest wavelength 6562.8 Å) with a P-Cygni profile. The emission peaks at 6565 Å and extends by several hundred Angstroms to longer wavelengths. An absorption feature is seen at shorter wavelengths with a maximum depth at 6303 Å. Estimate the speed of the expanding ejecta.

Where is the hydrogen responsible for the observed emission at $\sim$6565 Å and $\sim$6800 Å located?

The initial evolution of a supernova is well approximated by free expansion at constant speed until the mass of material swept up from the interstellar medium is equal to the mass of the supernova ejecta. Assume that the ejecta mass is $5\,\mathrm{M}_\odot$ and that the number density of the hydrogen in the interstellar medium is 1 atom per cm^3. Estimate the radius and the age at the end of the free expansion phase.

What type of star was the supernova precursor?

(ii) Describe the evolution of a star like the Sun from its arrival on the main sequence through to the end of its life. Include estimates of relevant time-scales, physical processes in the central regions and the location of the star on a T_{eff}-L diagram, where T_{eff} is the effective temperature and L is the luminosity. Include a sketch of the evolutionary track on the Hertzsprung-Russell diagram with key events and phases indicated.

TURN OVER...

Question 5Z - Statistical Physics

(i) Explain what is meant by the microcanonical ensemble for a quantum system.

Sketch the derivation of the probability distribution for the canonical ensemble from the microcanonical ensemble.

Under what physical conditions should each type of ensemble be used?

(ii) A paramagnetic solid contains atoms with magnetic moment $\boldsymbol{\mu} = \mu_0 \mathbf{J}$ where μ_0 is a positive constant and $\mathbf{J}$ is the intrinsic angular momentum of the atom. In an applied magnetic field $\mathbf{B}$, the energy of an atom is $-\boldsymbol{\mu} \cdot \mathbf{B}$. Each magnetic atom has total angular momentum quantum number J. The possible values of J_z are $J_z = m$ where m is an integer with $-J \leq m \leq J$. A constant magnetic field B is applied in the z-direction. Write down the partition function $Z_1(T, B)$ for a single atom.

Show that the average magnetic moment of the atom is given by

$$\langle \mu_z \rangle = \frac{1}{\beta} \left(\frac{\partial \log Z_1}{\partial B} \right)_T ,$$

where $\beta = 1/(k_{\mathrm{B}} T)$.

Evaluate Z_1 and hence prove that

$$\langle \mu_z \rangle = \mu_0 J B_J(x),$$

where $x = \beta \mu_0 B$ and

$$B_J(x) = \frac{1}{J} \left\{ \left(J + \frac{1}{2} \right) \coth \left[\left(J + \frac{1}{2} \right) x \right] - \frac{1}{2} \coth \left(\frac{x}{2} \right) \right\} .$$

Determine the behaviour of $B_J(x)$ for $x \gg 1$ and $x \ll 1$ and sketch $B_J(x)$ for a few different values of J on the same graph.

The total magnetization is $M_z = N \langle \mu_z \rangle$ where N is the number of atoms. The magnetic susceptibility is defined by

$$\chi = \left(\frac{\partial M_z}{\partial B} \right)_T .$$

Show that the solid obeys Curie's law $\chi \propto T^{-1}$ when $x \ll 1$.

Comment briefly on the behaviour of M_z for $x \gg 1$.

Question 6Z - Principles of Quantum Mechanics

(i) The position and momentum operators of a harmonic oscillator can be written as

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\, a + a^\dagger \,), \qquad \hat{p} = \left(\frac{\hbar m\omega}{2}\right)^{1/2} i(\, a^\dagger - a\,),$$

where m is the mass, ω is the frequency and the Hamiltonian is

$$H \;=\; \frac{1}{2m}\hat{p}^2 \;+\; \frac{1}{2}m\omega^2\hat{x}^2.$$

Write down the commutation relations for a and $a^\dagger$ and the Hamiltonian in terms of a and $a^\dagger$.

Determine the energy levels of the oscillator.

Assuming a unique ground state, explain how all other energy eigenstates can be constructed from the ground state.

(ii) Consider a modified Hamiltonian

$$H' \;=\; H \;+\; \lambda\hbar\omega\,(\, a^2 + a^{\dagger\,2}\,),$$

where λ is a dimensionless parameter and the remaining quantities are defined in Part (i). Calculate the modified energy levels to second order in λ, quoting any standard formulae which you require.

Show that the modified Hamiltonian can be written as

$$H' \;=\; \frac{1}{2m}(1-2\lambda)\hat{p}^2 \;+\; \frac{1}{2}m\omega^2(1+2\lambda)\hat{x}^2.$$

Calculate the modified energies exactly.

Assuming $|\lambda| < \frac{1}{2}$, show that the results are compatible with those obtained from perturbation theory.

TURN OVER...

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) By considering a particle's angular momentum and energy, describe qualitatively the general properties of bound orbits in spherical potentials.

Show that the equation governing the evolution of a small radial perturbation ε of the circular orbit with angular frequency Ω under spherical force F in the epicyclic approximation is

$$\ddot{\varepsilon} + \left(3\Omega^2 - \frac{dF(r)}{dr}\right)\varepsilon = 0.$$

(ii) Consider a flattened potential

$$\Phi = \Phi\left(\sqrt{x^2 + y^2 + \frac{z^2}{q^2}}\right),$$

where $0 < q \leq 1$ describes a constant flattening in the z-direction. A circular orbit in the $z = 0$ plane at a distance R_0 from the origin with the angular frequency Ω is perturbed slightly in the z direction. Deduce the frequency of the vertical motion.

Write down the L_x, L_y and L_z components of the angular momentum of this orbit and describe the orbital plane behaviour.

Find the frequency of the orbital plane precession of this orbit if $q \approx 1$.

Question 8Y - Physics of Astrophysics

(i) The energy of a binary system is pumped by distant encounters with field stars in the Galaxy at a rate that is independent of the binary's semi-major axis. Consider the case that binaries are created continuously at a constant rate with semi-major axis a_0 and that all have equal mass components of mass M. Show that in a steady state, the probability $p(a)$ of a binary having semi-major axis in the range a to $a + \mathrm{d}a$ is $p(a) \propto a^{-\beta}\mathrm{d}a$, where β is a constant that you should determine.

Explain why one expects binaries to be disrupted at finite semi-major axis due to a process that does not depend on encounters with individual field stars.

[You may assume that the energy of a binary with semi-major axis a and components of mass M is $E = -GM^2/2a$.]

(ii) A massive black hole in the nucleus of a galaxy is embedded in a stellar cluster. A star in the outer regions of the cluster is orbited by a planet with the mass and orbital radius of Jupiter. Assuming that the orbits of the stars in the outer cluster are approximately isotropic, estimate the probability that the planet is stripped by tidal interaction with the black hole within the time it takes the star to cross the cluster.

Assume that the cluster is uniform in density. Assume further that a destructive encounter occurs if the distance of closest approach of two stars is less than the planet's orbital radius. Estimate the probability that the star-planet system undergoes such a destructive encounter with another star.

[Assume the following numerical values for your calculation: mass of black hole $= 10^6 M_\odot$; typical stellar mass $= 1 M_\odot$; number of stars in cluster $= 1000$; radius of cluster $= 0.1$ pc; orbital radius of Jupiter $= 5$ au; mass of Jupiter $= 0.001 M_\odot$.]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 6 June 2017 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 2

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (i) *and Part* (ii), *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i). *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 2X and 7X** *should be in one bundle and* **3Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS
Script Paper (lined on one side)
Blue Cover Sheets
Yellow Master Cover Sheets
1 Rough Work Pad
Tags

SPECIAL REQUIREMENTS
Astrophysics Formulae Booklet
Approved Calculators Allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) A Lorentz transformation from a frame S with coordinate system x^i to another frame S' with coordinates $x'^{\,i}$,

$$x'^{\,i} = \Lambda^i_j x^j,$$

leaves the coordinate interval,

$$ds^2 = \eta_{ij} dx^i dx^j = c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

invariant. Show that the transformation coefficients Λ^i_j must satisfy

$$\Lambda^i_s \Lambda^j_t \eta_{ij} = \eta_{st}. \qquad (*)$$

Show that (*) is satistfied if

$$\Lambda^0_0 = \gamma, \quad \Lambda^\alpha_0 = \Lambda^0_\alpha = \gamma \frac{v^\alpha}{c}, \quad \Lambda^\alpha_\beta = \delta_{\alpha\beta} + \frac{v^\alpha v^\beta}{v^2}(\gamma - 1), \qquad \alpha = 1, 2, 3,$$

where v^α is the velocity of the frame S relative to S' and $\gamma = (1 - v^2/c^2)^{-1/2}$.

Comment on the form of Λ^α_β.

(ii) The energy-momentum tensor of a perfect fluid at rest in reference frame S is

$$\begin{aligned} T^{00} &= \rho c^2, \\ T^{\alpha 0} &= T^{0\alpha} = 0, \\ T^{\alpha\beta} &= p\delta_{\alpha\beta}, \end{aligned}$$

where ρc^2 is the proper energy density and p is the pressure of the fluid. Suppose that the fluid is moving with velocity $\mathbf{v}$ in the 'laboratory' frame S'. Using the results of Part (i) show that the components of the energy-momentum tensor in the frame S' are given by

$$\begin{aligned} T'^{\,00} &= \gamma^2(\rho c^2 + p v^2/c^2), \\ T'^{\,\alpha 0} &= \gamma^2(\rho c^2 + p) v^\alpha / c, \\ T'^{\,\alpha\beta} &= \gamma^2(\rho c^2 + p) v^\alpha v^\beta / c^2 + p\delta_{\alpha\beta}, \end{aligned}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

Briefly indicate how these equations, together with conservation of fluid particle number, can be used to derive the equations of relativistic hydrodynamics.

Question 2X - Astrophysical Fluid Dynamics

(i) Consider the flow of a fluid with pressure p, density ρ and speed v along a pipe parallel to the z-direction. Write down the expression for the stress tensor of this fluid and explain the reasoning behind each term.

A spiral galaxy of radius R experiences a face-on wind as it moves through the intracluster medium with speed v. Determine how much mass in the intracluster medium of density $\rho_{\rm ICM}$ is swept up.

Hence calculate the ram pressure exerted on the galaxy.

Assume that the galaxy moves at a speed of $1000\,{\rm km\,s^{-1}}$ and that the self-gravitational force per unit area of the disk is $10^{-11}\,{\rm g\,cm^{-1}\,s^{-2}}$. Assume further that the density $\rho_{\rm ICM}$ at distance r from the centre of the cluster is well described by

$$\rho_{\rm ICM}(r) = \rho_{\rm ICM,0}\left(\frac{r+r_0}{r_{\rm s}}\right)^{-1}\left(1+\frac{r}{r_{\rm s}}\right)^{-2},$$

where $\rho_{\rm ICM,0} = 10^{-26}\,{\rm g\,cm^{-3}}$, $r_{\rm s} = 50\,{\rm kpc}$, $r_0 = 5\,{\rm kpc}$ and the intracluster gas extends to $1\,{\rm Mpc}$. Estimate whether the intracluster medium is able to unbind the galactic material by ram pressure at $r = 1\,{\rm kpc}$ and $r = 1\,{\rm Mpc}$.

(ii) Consider an oblique adiabatic shock propagating through the intracluster medium. In the rest frame of the shock the pre-shock gas velocity makes an angle θ with respect to the shock front, while the post-shock gas flows at an angle δ with respect to the direction of motion of pre-shock gas. From the fluid equations derive the Rankine-Hugoniot conditions for this oblique shock.

Determine the minimum angle $\theta_{\rm min}$ for which the gas is shock heated and explain why it cannot be zero. How does $\theta_{\rm min}$ compare to the angle which defines the Mach cone?

Using the relation

$$M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)},$$

or otherwise, where M_1 and M_2 are the Mach numbers of the pre-shock and post-shock gas, respectively, and γ is the adiabatic index, express the ratio of pre- and post-shock gas densities as a function of M_1 and θ alone.

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) Show that with a non-zero cosmological constant Λ and zero pressure, the deceleration and density parameters are related by

$$q = \frac{1}{2}\Omega_{\rm m} - \frac{\Lambda c^2}{3H^2},$$

where $\Omega_{\rm m}$ is the contribution of the matter density to the critical density and H is the Hubble parameter.

Calculate q for a universe with zero cosmological constant dominated by radiation for which $p = (1/3)\rho c^2$, where p is the pressure and ρ is the density.

(ii) Explain what astronomers mean by the term *'Standard Candle'*. Give examples used in cosmology.

Discuss briefly the main difficulties in the measurement of the Hubble constant in the nearby Universe.

Describe how the *'Cosmic Distance Ladder'* method is used to measure relative cosmological distances.

Discuss the methods used to establish an absolute distance scale.

Suppose that a series of four different standard candles are used to step out along the cosmic distance ladder from our Galaxy to distances far enough to accurately measure the true expansion rate. Each standard candle has an uncertainty of $\Delta M = \pm 0.2$ magnitudes in the calibration of its absolute magnitude. Calculate the resulting uncertainty in the measurement of the Hubble constant.

Question 4Z - Structure and Evolution of Stars

(i) What is the dominant source of opacity for ultraviolet and optical photons in the photosphere of the Sun?

An optical image of the Sun shows a decrease in the brightness from the centre of the Sun towards the edge, an effect termed "limb darkening". Explain the physical basis for this phenomenon.

Images of the Sun are obtained through two narrow filters, each 2 nm wide. One is sensitive to radiation with wavelengths 655 nm to 657 nm, including the hydrogen Balmer Hα transition. The second is centred at 654 nm, immediately adjacent to the first. Assume that the change in flux from a blackbody due to the small wavelength difference can be neglected. How will the two images differ?

(ii) The evolution of a white dwarf can be modelled by considering an isothermal, electron-degenerate, core with density ρ and temperature T_c, containing almost all the star's mass M surrounded by a thin, non-degenerate, outer layer. The pressure in the core is $P = K_1(\rho/\mu_e)^{5/3}$, where $1/\mu_e$ is the average number of free electrons per nucleon and K_1 is a constant. Material in the outer layer behaves as an ideal gas and is in radiative equilibrium. The opacity κ obeys Kramer's law $\kappa = \kappa_0 \rho T^{-7/2}$. Assume that there is a sharp boundary at $r_b < r$ between the degenerate core and the outer layer, with the mass $m(r > r_b) \simeq M$. Calculate the relation between the luminosity L, M and T_c.

The energy source of the white dwarf results from the thermal motions of ions in the degenerate core. By considering the rate of thermal energy depletion of the core, calculate the rate of change of the luminosity as a function of core temperature.

What do you conclude about the cooling rate of a white dwarf?

For a white dwarf with a degenerate core of carbon and oxygen and an atmosphere composed primarily of hydrogen, describe the key features of the optical spectrum once the star has cooled to the point where the surface temperature is $\sim 10\,000\,\text{K}$?

TURN OVER...

Question 5Z - Statistical Physics

(i) The entropy of a thermodynamic ensemble is defined by

$$S = -k_{\mathrm{B}} \sum_n p(n) \log p(n),$$

where k_{B} is Boltzmann's constant. Explain what is meant by $p(n)$. Write down an expression for $p(n)$ in the grand canonical ensemble and hence show that the entropy S is related to the partition function $\mathcal{Z}(T, \mu, V)$ by

$$S = k_{\mathrm{B}} \left[\frac{\partial}{\partial T} (T \log \mathcal{Z}) \right]_{\mu, V},$$

where T is the temperature, V is the volume, and μ is the chemical potential.

(ii) Consider a gas of N non-interacting fermions with single-particle energy levels ϵ_i. Show that the grand canonical partition function $\mathcal{Z}$ is given by

$$\log \mathcal{Z} = \sum_i \log \left(1 + e^{-(\epsilon_i - \mu)/(k_{\mathrm{B}} T)} \right),$$

using the notation of Part (i).

Assume that the energy levels are continuous with density of states $g(\epsilon) = AV\epsilon^a$, where A and a are positive constants. Show that

$$\log \mathcal{Z} = VT^b f(\mu/T),$$

and give expressions for the constant b and the function f.

The gas undergoes a reversible adiabatic change. By considering the entropy per particle S/N, show that μ/T remains constant.

Deduce that VT^c and pV^d remain constant in this process, where c and d are constants whose values you should determine.

Question 6Z - Principles of Quantum Mechanics

(i) A particle moving in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$, respectively, whose eigenstates obey

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\,, \qquad \langle x|x'\rangle = \delta(x-x')\,, \qquad \langle p|p'\rangle = \delta(p-p')\,.$$

Given a state $|\psi\rangle$ determine the corresponding position and momentum space wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ and show how each of these can be expressed in terms of the other.

(ii) Using the Dirac formalism compute $\langle x|\hat{p}|x'\rangle$ and $\langle p|\hat{x}|p'\rangle$.

The Hamiltonian for the particle described in Part (i) is

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x})\,,$$

where m is the mass of the particle. Calculate $\langle x|H|x'\rangle$ and express $\langle x|\hat{p}|\psi\rangle$ and $\langle x|H|\psi\rangle$ in terms of the position space wavefunction $\psi(x)$.

Compute the momentum space Hamiltonian for the harmonic oscillator with potential

$$V(\hat{x}) = m\omega^2\hat{x}^2/2\,,$$

where the frequency ω is constant.

TURN OVER...

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) Dynamical friction is the net deceleration experienced by a massive body with mass M moving with speed v through a sea of less massive particles with mass density ρ. The change in speed of the massive body is

$$\frac{\mathrm{d}v}{\mathrm{d}t} \propto \frac{G^2 \rho M \ln \Lambda}{v^2}, \qquad (*)$$

where $\ln \Lambda \approx b_{\max}/b_{90}$, b_{90} is the impact parameter yielding a deflection of $90°$, $b_{\max}$ is the maximal impact parameter considered and the massive body is assumed to be a point mass. Describe astrophysical phenomena where dynamical friction plays a key role.

Consider a spherical object with the Plummer-law density profile

$$\rho_M(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}},$$

where M is the mass of the object and a is its scale radius. The object moves with speed v through a sea of point mass particles of mass m and number density n. Using the impulse approximation, calculate the velocity kicks a particle with impact parameter b acquires in the directions parallel and perpendicular to its motion as a result of the interaction with the Plummer sphere.

$$\left[\text{Note that} \int \frac{x^2 dx}{(1+x^2)^{5/2}} = \frac{1}{3}\frac{x^3}{(1+x^2)^{3/2}}, \qquad \int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{\sqrt{1+x^2}}.\right]$$

(ii) Using conservation of energy and momentum for an interaction with a single particle at impact parameter b and integrating over all impact parameters, deduce the deceleration that the Plummer sphere of Part (i) experiences as a result of interaction with many particles, assuming that the Plummer sphere remains stationary during an interaction with a particle.

How does the resulting deceleration formula compare to the dynamical friction formula (*) for a point mass?

Assume that the particles interacting with the Plummer sphere represent Cold Dark Matter. The Cold Dark Matter is predicted to be arranged in a hierarchy of clumps with a mass function

$$\frac{\mathrm{d}n}{\mathrm{d}m} \propto m^{-2.1}.$$

Approximating the clumps as point masses, comment on which mass range makes the largest contribution to dynamical friction.

$$\left[\text{Note that} \int \frac{x^3}{(1+x^2)^2} dx = \frac{1}{2}\left(\frac{1}{1+x^2} + \ln(1+x^2)\right).\right]$$

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) A group of alien explorers leaves their home planet X (located close to the centre of a spherical dwarf galaxy) in a rocket travelling at 1000 km s^{-1}. On arrival at each star they explore its planetary system and then head off in a randomly chosen direction until they encounter another star within 0.1 pc of their path, at which point they visit this star and repeat the procedure. The first star that they visit beyond 100 pc from X has a planetary system containing habitable planet Y. Estimate the expected time interval between leaving planet X and arriving at planet Y on the assumption that the explorers spend 10 years exploring each star they visit on the way.

If Y is the tenth habitable planet that the aliens have visited, estimate the fraction of stars in this galaxy that have habitable planets.

[Assume that the galaxy has uniform stellar number density of 10 stars pc^{-3} within 100 pc of its core.]

(ii) Two identical supernovae explode simultaneously at points $x = \pm R_0, y = 0, z = 0$ in a uniform medium of density 10^{-21} kg m^{-3} where $R_0 = 3$ pc. Each supernova produces a spherical blast wave comprising a thin shell of swept up interstellar medium whose radius (for $t < t_0$) is given by

$$R = R_0 \left(\frac{t}{t_0} \right)^{2/5}. \qquad (*)$$

Evaluate t_0 in the case that one third of the total energy deposited in the surrounding medium by the supernova, $E_{\rm SN} = 10^{44}$J, is contained in the kinetic energy of the expanding shell.

At $t > t_0$ the two shells start to collide in the $x = 0$ plane. You may assume that every point on the expanding shell expands according to (*) until it reaches the $x = 0$ plane, at which point the component of the velocity in the x direction of the colliding material is set to zero. If all the energy dissipated in the collision is radiated away, show that the luminosity of the collision can be written in the form

$$L_{\rm coll} = A \left(\frac{E_{\rm SN}}{t_0} \right) \left(\frac{t}{t_0} \right)^{-11/5},$$

where A is a constant that you do not need to evaluate.

Each of the original supernovae had an apparent magnitude of 10 as measured on Earth and involved the radiation of 1% of $E_{\rm SN}$ over a timescale $t_{\rm SN} = 1$ day. Estimate the required limiting magnitude of a telescope that would be able to detect the signature of the shell-shell collision.

[You may assume where required that $A \sim 1$]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 8 June 2017 09:00am – 12:00pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part (i) *and Part* (ii)*, which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part (ii) *of each question carrying twice as many marks as Part* (i)*. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 2X and 7X** *should be in one bundle and* **3Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) Consider the spherical static metric

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (*)$$

Write down the non-zero components of the affine connection Γ^i_{jk}.

Show that the component R_{00} of the Ricci tensor is given by

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB},$$

where primes denote derivatives with respect to the radial coordinate r.

[You may assume that $R_{ij} = \partial\Gamma^s_{is}/\partial x^j - \partial\Gamma^t_{ij}/\partial x^t + \Gamma^s_{it}\Gamma^t_{sj} - \Gamma^t_{ij}\Gamma^s_{ts}$.]

(ii) For the metric (*) of Part (i), the remaining components of the Ricci tensor are

$$\begin{aligned} R_{11} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \\ R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \\ R_{33} &= R_{22}\,\sin^2\theta, \end{aligned}$$

where primes denote derivatives with respect to the radial coordinate r. Consider a perfect fluid with energy-momentum tensor

$$T_{ij} = (\rho + p/c^2)u_i u_j - p g_{ij},$$

where $\rho(r)$ and $p(r)$ are the proper mass density and pressure of the fluid, respectively. Show that the Einstein equations

$$R_{ij} = -\kappa\left(T_{ij} - \frac{1}{2}T g_{ij}\right), \qquad \kappa = \frac{8\pi G}{c^4},$$

require

$$\begin{aligned} u_i &= c\sqrt{A}\,\delta^0_i, \\ R_{00} &= -\frac{1}{2}\kappa(\rho c^2 + 3p)A, \\ R_{11} &= -\frac{1}{2}\kappa(\rho c^2 - p)B, \\ R_{22} &= -\frac{1}{2}\kappa(\rho c^2 - p)r^2, \\ R_{33} &= R_{22}\sin^2\theta. \end{aligned}$$

By eliminating the pressure p from these equations show that

$$B(r) = \left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1},$$

where

$$m(r) = 4\pi \int_0^r \rho(s) s^2 ds.$$

Give a physical interpretation of this solution.

TURN OVER...

Question 2X - Astrophysical Fluid Dynamics

(i) The mass M and radius r of a polytrope of index n are related by $M \propto r^{\frac{3-n}{1-n}}$. For stars similar to the Sun it is observed that $M \propto r$. Explain which of the assumptions adopted to derive $M \propto r^{\frac{3-n}{1-n}}$ do not generally hold in such stars and why.

Consider a binary stellar system where a more massive companion fills its Roche lobe and delivers a substantial fraction of its outer envelope material to a smaller Sun-like star. Sketch the time sequence of the radius and mass of the smaller star on the $\log r$-$\log M$ plane for the case where the Roche lobe overflow takes 1 Myr and 100 Myrs, respectively, and briefly explain your sketch.

(ii) Consider a star in hydrostatic equilibrium with mass M and radius R. The total pressure is the sum of thermal and radiation pressure. The star's mean pressure is

$$\bar{P} = \frac{1}{V}\int_0^R 4\pi r^2 P(r)dr\,,$$

where V is its volume. Show that $\bar{P} = E_{\rm g}/3V$, where $E_{\rm g}$ is the gravitational binding energy of the star.

Assume that thermal and radiation pressure are equal. Hence find an expression for the total pressure as a function of density alone.

Calculate the corresponding stellar mass $M_{\rm equ}$ in Solar masses. You may assume that $E_{\rm g} \approx GM_{\rm equ}^2/R$ and that the star is composed of fully ionized hydrogen.

Using the virial theorem, roughly estimate the typical sound speed of a star with mass equal to $M_{\rm equ}$.

Assume that the stellar radius is $R_{\rm equ} = 30\,R_\odot$. Determine whether the changes in the radiation pressure occurring on a timescale of months can be effectively communicated with the rest of the star.

Question 3Y - Introduction to Cosmology

(i) Discuss the physical processes leading to the production of light elements during Big Bang Nucleosynthesis.

(ii) Sketch a diagram of the relative abundances of the elements produced in Big Bang Nucleosynthesis (BBN) as a function of $\eta = n_{\mathrm{b}}/n_\gamma$, where n_{b} and n_γ are the number densities of baryons and photons, respectively.

Explain the main reason for the behaviour of each element abundance as a function of η.

During BBN, an energy of 28.2 MeV is released for every ^{4}He nucleus formed. Estimate the increase in radiation temperature ΔT at the end of BBN, assuming that all of the ^{4}He was formed when the radiation temperature was $T = 10^9\,\mathrm{K}$. Is the temperature increase significant?

How does the ^{4}He abundance change if neutrino-like particles exist in addition to the three neutrino species of the Standard Model of particle physics.

TURN OVER...

Question 4Z - Structure and Evolution of Stars

(i) The dominant source of energy in the cores of stars with masses less than $2\,\mathrm{M}_\odot$ is the fusion of hydrogen to helium via the proton-proton chain (p-p chain). The process can occur via three sequences of reactions, designated p-p I, p-p II and p-p III, although the p-p I reaction is the primary route. The p-p I chain proceeds via three reactions as follows.

$$ {}^1_1\mathbf{H} + {}^1_1\mathbf{H} \rightarrow {}^2_1\mathbf{H} + e^+ + \nu \ , $$

$$ {}^2_1\mathbf{H} + {}^1_1\mathbf{H} \rightarrow {}^3_2\mathbf{He} + \gamma \ , $$

$$ {}^3_2\mathbf{He} + {}^3_2\mathbf{He} \rightarrow {}^4_2\mathbf{He} + {}^1_1\mathbf{H} + {}^1_1\mathbf{H} \ . $$

In the p-p II and p-p III chains, following the creation of ${}^3_2\mathbf{He}$ through the second reaction above, the fusion proceeds via the creation of ${}^7_4\mathbf{Be}$ as

$$ {}^3_2\mathbf{He} + {}^4_2\mathbf{He} \rightarrow {}^7_4\mathbf{Be} + \gamma \ . $$

Subsequent interactions involve no further creation or destruction of ${}^3_2\mathbf{He}$. Consider two species X and Y with atomic masses i and j, respectively, and their abundances denoted by ${}^i\mathbf{X}$ and ${}^j\mathbf{Y}$, respectively. The reaction rate for a reaction between the two species is given by $r_{ij} = \lambda_{ij}\,{}^i\mathbf{X}\,{}^j\mathbf{Y}$, where λ_{ij} is the reaction cross-section. Show that the equilibrium abundance of ${}^3_2\mathbf{He}$ when all three of the p-p chains are operating is given by

$$ {}^3_2\mathbf{He} = \frac{1}{2\lambda_{33}}\left[\sqrt{\lambda_{34}^2({}^4_2\mathbf{He})^2 + 2\lambda_{11}\lambda_{33}({}^1_1\mathbf{H})^2} - \lambda_{34}\,{}^4_2\mathbf{He}\right]. $$

You may assume that the lifetime of a deuterium nucleus in a stellar core is extremely short and that deuterium thus rapidly achieves equilibrium abundance.

(ii) A star of one solar mass and solar composition spends 10^{10} years on the main sequence, with an average luminosity of $1\,\mathrm{L}_\odot$. How much energy is produced during the main-sequence phase?

What is the mass of the helium core at the end of the main-sequence lifetime?

Following the main-sequence phase the star becomes a red giant with energy derived from a hydrogen-burning shell surrounding the core. The initial luminosity at the bottom of the red giant branch is $L_0 \simeq 1\,\mathrm{L}_\odot$. The luminosity on the red giant branch is a strong function of the helium core mass, with $L = KM_{\mathrm{core}}^8$ where K is a constant. The star remains a red giant until the core mass reaches $0.45\,\mathrm{M}_\odot$, when helium burning in the core begins. Calculate the lifetime of the star on the red giant branch.

Following further short-lived evolutionary phases the star will end its life as a white dwarf with a degenerate carbon core of mass $M = 0.57\,\mathrm{M}_\odot$. What fraction of the star's energy over its lifetime is produced while on the main sequence?

[The energy per nucleon produced from fusing helium to carbon is approximately 9 per cent of the amount produced by fusing hydrogen to helium.]

TURN OVER...

Question 5Z - Statistical Physics

(i) Describe the *Carnot cycle* using plots in the pressure-volume (p-V) and temperature-entropy (T-S) planes.

In which steps of the cycle is heat absorbed or emitted by the gas?

In which steps is work done on or by the gas?

(ii) An ideal monatomic gas undergoes a reversible cycle described by a triangle in the pressure-volume (p-V) plane with vertices at the points A, B, C with coordinates (p_0, V_0), $(2p_0, V_0)$ and $(p_0, 2V_0)$, respectively. The cycle is traversed in the order $ABCA$. Write down the equation of state and an expression for the internal energy of the gas.

Derive an expression relating TdS to dp and dV. Use your expression to calculate the heat supplied to, or emitted by, the gas along AB and CA.

Show that heat is supplied to the gas along part of the line BC and heat is emitted by the gas along the other part of this line.

Calculate the efficiency $\eta = W/Q$ where W is the total work done by the cycle and Q is the total heat supplied.

Question 6Z - Principles of Quantum Mechanics

(i) A quantum mechanical system consists of two identical non-interacting particles with wavefunctions $\psi_i(x)$ and energies E_i, $i = 1, 2, ...$, where $E_1 < E_2 < ...$ Show how the two lowest energy levels of the two-particle system are constructed and discuss their degeneracy when the particles have (a) spin 0, and (b) spin 1/2.

(ii) The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show how the Pauli matrices can be used to describe the spin operator $\mathbf{S}$ for a particle of spin 1/2.

An electron is at rest in the presence of a magnetic field $\mathbf{B} = (B, 0, 0)$ and experiences an interaction of $-\mu\boldsymbol{\sigma} \cdot \mathbf{B}$. At time $t = 0$ the state of the electron is the eigenstate of the 3-component of $\mathbf{S}$, s_3, with eigenvalue $\hbar/2$. Calculate the probability that at a later time t the electron will be measured to be in the eigenstate of s_3 with eigenvalue $\hbar/2$.

TURN OVER...

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) For a tracer population with a number density $\nu(r)$ and a radial velocity dispersion $\sigma^2(r)$ moving in a spherical gravitational potential $\phi(r)$ the Jeans equation is

$$\frac{\mathrm{d}}{\mathrm{dr}}\left(\nu\sigma_r^2\right) + \frac{2\beta}{r}\nu\sigma_r^2 = -\nu\frac{\mathrm{d}\phi}{\mathrm{d}r}, \qquad (*)$$

where $\beta = 1 - \frac{\sigma_\theta^2+\sigma_\phi^2}{2\sigma_r^2}$ is the anisotropy parameter. Assume that the tracer population shows no rotation and its density and the radial velocity dispersion are power-laws, $\nu \propto r^{-\gamma}$ and $\sigma_r \propto r^{\alpha}$. Deduce the link between velocity dispersion in the r and θ directions, σ_r^2 and σ_θ^2, and the circular velocity V_c^2.

At large distances from the Sun, the Galactic stellar halo is measured to have isotropic velocity dispersion, a density profile described by an index $\gamma = 4$ and a constant radial velocity dispersion of 110 kms^{-1}. What can you deduce about the behaviour of the density of the dark matter halo between the Solar radius and the Galactic outskirts?

At a certain radius in the outer halo of the Galaxy the radial velocity dispersion starts to decrease from a constant value. What inference can you make about the change in behaviour of the Milky Way's stellar and dark matter halos?

(ii) Re-write the Jeans equation (*) of Part (i) using the integrating factor $Q(r)$ satisfying

$$\frac{\mathrm{d}\ln Q(r)}{\mathrm{d}r} = \frac{2\beta(r)}{r}.$$

Prove that the local three-dimensional velocity dispersion is

$$\sigma^2 = \sigma_r^2\frac{d\ln(r^3Q^{-1})}{d\ln r}.$$

Show that the average three-dimensional velocity dispersion of tracer particles within a sphere with radius r_{out} is

$$\langle\sigma^2\rangle = \frac{4\pi}{N_{\mathrm{tot}}}\int_0^{r_{\mathrm{out}}} Q\nu\sigma_r^2 d(r^3Q^{-1}),$$

where N_{tot} is the total number of tracer particles.

Hence show that the spherical Jeans equation is consistent with the scalar virial theorem.

Question 8Y - Physics of Astrophysics

(i) A gravitational wave was detected at location A 7 ms after it was detected at location B, where A and B are two points on the Earth's surface separated by 3000 km. Assuming that gravitational waves travel at the speed of light, describe the constraints that can be placed on the location of the source of gravitational wave emission.

It is proposed to add a further detector at position C on the Earth's surface. Explain how this could be used to further constrain the location of future gravitational wave events.

Explain whether you would choose a site for C which lies along the line AB.

(ii) A cloud of gas, of radius 0.1 pc and located 10 pc from a starburst region is ionised by radiation from the starburst and produces a luminosity in the Hα line of 2×10^{28} W. Assume that the cloud density is 10^{-19} kg m^{-3} and use the data given below to discuss whether you expect the main de-excitation mechanism from the $n = 3$ to $n = 2$ state to be radiative or collisional.

Estimate the total number of ionising photons per second produced by the starburst if on average it takes around 10 ionisations before an Hα photon is generated.

A simplified prescription for the rate of ionising photons produced by a star of mass m_* is given by

$$\Phi_{\rm ion} = 10^{49} \left(\frac{m_*}{100 M_\odot} \right)^{1.5} {\rm s}^{-1},$$

for $m_* > 17 M_\odot$ and zero otherwise. Assume that the stars in the starburst are distributed according to a Salpeter IMF (i.e. the number of stars per linear mass interval scales as $m_*^{-2.35}$ for m_* in the range $0.1 M_\odot$ to $100 M_\odot$). Assume further that the average stellar lifetime for stars more massive than $17 M_\odot$ is 3 Myr. Estimate the total star formation rate in the starburst.

[The Hα line of hydrogen corresponds to the $n = 3$ to $n = 2$ transition, where the electronic energy levels are given by $E_n = -13.6 eV/n^2$. The rate of radiative de-excitation from $n = 3$ to $n = 2$ is 4×10^7 s^{-1} and the cross-section for collisional de-excitation of the $n = 3$ state is 10^{-18} m^2. You may assume where required that the gas temperature is 10^4 K.]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 9 June 2017 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 2X and 7X** *should be in one bundle and* **3Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) Consider a five-dimensional 'warped' spacetime with metric,

$$ds^2 = g_{IJ}dx^I dx^J = \mathrm{e}^{-2A(w)}\eta_{ij}dx^i dx^j - dw^2, \quad \eta_{ij} = (1, -1, -1, -1). \qquad (*)$$

Show that the non-zero components of the five-dimensional affine connections,

$$\Gamma^P_{MN} = \frac{1}{2}g^{PR}\left(\frac{\partial g_{NR}}{\partial x^M} + \frac{\partial g_{RM}}{\partial x^N} - \frac{\partial g_{MN}}{\partial x^R}\right),$$

are

$$\Gamma^4_{ij} = -A'\mathrm{e}^{-2A}\eta_{ij}, \qquad \Gamma^i_{j4} = -A'\delta_{ij},$$

where primes denote derivatives with respect to the coordinate $x^4 = w$, lower case latin indices run from 0-3 and upper case latin indices run from 0-4.

If $A = |w|/L$, where L is a constant, show by integrating over the fifth dimension w, that

$$\int \sqrt{|g|}\, d^4x\, dw = \frac{L}{2}\int d^4x.$$

(ii) Adopting the same notation as in Part (i), show that for the metric (*) the non-zero components of the Ricci tensor are

$$R_{ij} = (A'' - 4A'^2)\mathrm{e}^{-2A}\eta_{ij}, \quad R_{44} = 4(A'^2 - A'').$$

Show that the five-dimensional Einstein equations

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R = -\kappa T_{MN},$$

where κ is a constant, require

$$G_{44} = -6A'^2 = \Lambda,$$

if $T_{MN} = \Lambda g_{MN}/\kappa$, where Λ is a constant.

Show that if we require a solution that is invariant under the transformation $w \to -w$, the cosmological constant Λ must be negative and that the metric (*) takes the form

$$ds^2 = \mathrm{e}^{-2|w|/L}\eta_{ij}dx^i dx^j - dw^2,$$

where $L = (-\Lambda/6)^{-1/2}$.

Give a physical interpretation of this solution.

[You may assume that $R_{MN} = \partial\Gamma^P_{MP}/\partial x^N - \partial\Gamma^P_{MN}/\partial x^P + \Gamma^Q_{MP}\Gamma^P_{QN} - \Gamma^Q_{MN}\Gamma^P_{QP}$.]

Question 2X - Astrophysical Fluid Dynamics

(i) Demonstrate explicitly that the Helmholtz equation for inviscid barotropic flows implies that the vorticity flux moves with the fluid.

In the case of a non-barotropic flow, derive a generalized Helmholtz equation and describe the physical meaning of all terms contributing to the time evolution of vorticity.

(ii) From the momentum equation derive the Bernoulli equation for a steady, self-gravitating, barotropic flow. Explain the physical meaning of the different terms in the equation.

Show that the Bernoulli equation still holds in the case of ideal magnetohydrodynamics (MHD) when the magnetic field $\boldsymbol{B}$ is parallel to the streamlines.

Consider now an ideal MHD flow where the fluid velocity $\boldsymbol{u}$ is perpendicular to the magnetic field $\boldsymbol{B} = (0, 0, B_{\mathrm{z}})$ and the flow is independent of the z-coordinate. Show that the time evolution of the magnetic field is given by

$$\frac{\partial B_{\mathrm{z}}}{\partial t} = -\nabla \cdot (B_{\mathrm{z}} \boldsymbol{u}) \, .$$

Show that for this flow the magnetic field contribution in the momentum equation can be viewed as a pressure term and discuss qualitatively how this affects the physical interpretation of the Bernoulli equation.

TURN OVER...

Question 3Y - Introduction to Cosmology

(i) Discuss briefly the evidence for the existence of dark matter in spiral galaxies and galaxy clusters.

How does the typical mass-to-light ratio in the B-band in these two classes of objects compare with the average for the Solar neighbourhood?

The discs of spiral galaxies typically show an exponential light distribution, with surface density $\Sigma(r)$ given by

$$\Sigma(r) = \Sigma_0 \, e^{-r/r_{\rm s}}, \qquad (*)$$

where r_s is the exponential scale length and Σ_0 is the central surface density. Assume a constant mass-to-light ratio $\mathcal{M}/\mathcal{L}$ independent of radius, and hence calculate the total stellar mass of the stellar disk.

(ii) Starting from the Robertson-Walker metric show that proper distance l and redshift z are related by

$$\frac{{\rm d}l}{{\rm d}z} = \frac{c}{H(z)(1+z)},$$

where $H(z)$ is the Hubble parameter.

Assume a population of disc galaxies with comoving number density $n_0 = 10^{-2}$ Mpc^{-3} exists at $z = 3$ that have gaseous discs with an exponential surface mass density profile as in (*) of Part (i). Assume further that the hydrogen in the disc is neutral and that the neutral hydrogen causes Lyα absorption in a background quasar at $z > 3$. Calculate the incidence rate ${\rm d}\mathcal{N}/{\rm d}z$ of absorbers with neutral hydrogen column density $N_{\rm HI} > 2 \times 10^{20}{\rm cm}^{-2}$. You may neglect the variation of column density with viewing angle.

[Assume that the present-day contributions to the critical density by curvature, matter and the cosmological constant are $\Omega_{\rm k,0}$ =0, $\Omega_{\rm m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$. Assume further that $r_{\rm s} = 4$ kpc and $\Sigma_0 = 4.5\,{\rm g\,cm^{-2}}$.]

Question 4Z - Structure and Evolution of Stars

(i) The energy source of a cloud of mass M collapsing to form a protostar with radius $R_{\rm ps}$ is gravitational potential energy. The initial cloud radius is $\gg R_{\rm ps}$ and the gas is composed of hydrogen and helium with mass fractions X and Y respectively. Any small fraction of metals may be ignored. The hydrogen is initially in molecular form. Assume that half the gravitational potential energy released ionizes the gas with no significant energy loss due to radiation. Show that the protostar radius is related to the cloud mass and the mass fraction of hydrogen by the relation

$$\frac{R_{\rm ps}}{R_\odot} = \frac{K}{1-0.2X}\frac{M}{M_\odot},$$

where K is a constant to be determined.

[The energy required to disassociate a hydrogen molecule is 4.5 eV and the energy required to ionize a hydrogen and helium atom is 13.6 eV and 79.0 eV, respectively]

(ii) Once a protostar of luminosity L and mass M between 0.5 and 10 $M_\odot$ has reached the bottom of the Hayashi track on the Hertzsprung-Russell diagram, the internal temperature rises and the opacity decreases. Energy transport is no longer due to convection and the star achieves radiative equilibrium. As the star continues to shrink in size, the effective temperature $T_{\rm eff}$ increases. The star follows a so-called "Henyey track", migrating across the Hertzsprung-Russell diagram towards a location on the zero-age main sequence determined by its mass. Assume that the opacity per unit mass is given by Kramer's opacity law with $\kappa \propto \rho T^{-3.5}$, where ρ is the density. The rate of change in the structure of the star is sufficiently slow that it can be assumed that the virial theorem applies. Calculate the logarithmic slope of the Henyey track in the $T_{\rm eff} - L$ plane.

For very massive stars, material becomes completely ionised and the dominant opacity is due to electron scattering, i.e., $\kappa =$ constant. How does the logarithmic slope in the $T_{\rm eff} - L$ plane change?

What is the relevant timescale for the evolution of stars on the Hayashi and Henyey tracks?

Where does a star spend the majority of time in the $T_{\rm eff} - L$ plane prior to arriving on the zero-age main sequence?

TURN OVER...

Question 5Z - Statistical Physics

(i) The van der Waals equation of state is

$$p = \frac{k_{\mathrm{B}}T}{v-b} - \frac{a}{v^2} \ ,$$

where p is the pressure, $v = V/N$ is the volume divided by the number of particles, T is the temperature, k_{B} is Boltzmann's constant, and a, b are positive constants. Explain what is meant by the critical point and determine the values p_c, v_c, T_c corresponding to this point.

(ii) Prove that the Gibbs free energy $G = E + pV - TS$ satisfies $G = \mu N$, where E is the internal energy, S is the entropy, μ is the chemical potential, and the remaining quantities are as defined in Part (i).

Derive an expression for $(\partial\mu/\partial p)_{T,N}$ and use it to explain the Maxwell construction for determining the pressure at which the gas and liquid phases can coexist at a given temperature.

By defining $\bar{p} = p/p_c$, $\bar{v} = v/v_c$ and $\bar{T} = T/T_c$, derive the law of corresponding states

$$\bar{p} = \frac{8\bar{T}}{3\bar{v}-1} - \frac{3}{\bar{v}^2}.$$

To investigate the behaviour near the critical point, let $\bar{T} = 1+t$ and $\bar{v} = 1+\phi$, where t and ϕ are small. Expand $\bar{p}$ to cubic order in ϕ and hence show that

$$\left(\frac{\partial \bar{p}}{\partial \phi}\right)_t = -\frac{9}{2}\phi^2 + \mathcal{O}(\phi^3) + t\left[-6 + \mathcal{O}(\phi)\right].$$

At fixed small t, let $\phi_l(t)$ and $\phi_g(t)$ be the values of ϕ corresponding to the liquid and gas phases on the co-existence curve. By changing the integration variable from p to ϕ, use the Maxwell construction to show that $\phi_l(t) = -\phi_g(t)$.

Deduce that as the critical point is approached along the co-existence curve

$$\bar{v}_{\mathrm{gas}} - \bar{v}_{\mathrm{liquid}} \sim (T_c - T)^{1/2}.$$

Question 6Z - Principles of Quantum Mechanics

(i) The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + \lambda V(t)\,,$$

where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for eigenstates in the interaction picture.

(ii) Let $|n\rangle$ and $|m\rangle$ be eigenstates of H_0 in Part (i) with distinct eigenvalues E_n and E_m, respectively. Show that if the system is initially in state $|n\rangle$ then the probability of measuring it to be in state $|m\rangle$ after a time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt' \langle m|V(t')|n\rangle e^{i(E_m-E_n)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\;.$$

Deduce that if $V(t) = e^{-\mu t/\hbar}W$, where W is a time-independent operator and μ is a positive constant, then the probability for such a transition to have occurred after a very long time is approximately

$$\frac{\lambda^2}{\mu^2+(E_m-E_n)^2}|\langle m|W|n\rangle|^2\;.$$

TURN OVER...

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) A polytrope is a form of the distribution function f which is a power-law of relative energy $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$ such that

$$f = \begin{cases} F\mathcal{E}^{n-\frac{3}{2}}, & \mathcal{E} > 0, \\ 0 \quad\quad , & \mathcal{E} \leq 0, \end{cases}$$

where F is a constant and $\Psi = -\Phi + \Phi_0$ is the relative potential. Using the substitution $v^2 = 2\Psi \cos^2\theta$, show that the density ρ of the polytrope varies as

$$\rho \propto \Psi^n.$$

Assuming that $\rho(r) = C\Psi^n$, where C is a constant, derive the Lane-Emden equation

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi}{ds}\right) = \begin{cases} -\psi^n, & \psi > 0, \\ \;\;0 \;\;, & \psi \leq 0, \end{cases} \qquad (*)$$

where $s = r/b$, $b = (4\pi G\Psi_0^{n-1}C)^{-\frac{1}{2}}$ and $\psi = \Psi/\Psi_0$ with $\Psi_0 = \Psi(0)$.

(ii) Show that the solution of the Lane-Emden equation (*) of Part (i) for the case $n = 1$ is

$$\psi = \begin{cases} (\sin s)/s, & s < \pi, \\ \pi/s - 1, & s \geq \pi. \end{cases}$$

Find the total mass of the polytrope using the definitions of the variables s and ψ of Part (i).

For a polytrope with an arbitrary index n, how does the velocity dispersion depend on the relative potential ψ?

Question 8Y - Physics of Astrophysics

(i) A young solar mass star is observed to undergo sinusoidal variations in its radial velocity with amplitude 1.2 km s $^{-1}$ and a period of 9 days. The star is surrounded by a large scale protoplanetary disc whose major and minor axes on the sky are 2.1 and 1.7 arcseconds, respectively. Assume that the radial velocity variations are caused by a planet whose orbit is co-planar with the large-scale disc. Determine the mass and orbital radius of the planet.

What can you deduce about the eccentricity of the planet from the information given in the question?

(ii) A rocky planetesimal falls inwards on a radial orbit through a gaseous protoplanet. The planetesimal is accelerated by the local gravitational attraction of the planet $g(r)$ and decelerated by hydrodynamical drag $a_{\rm drag}(r)$, where r is the distance of the planetesimal from the centre of the planet. Show that if the local value of the planetesimal's terminal velocity $v_t(r)$ is much less than the local free-fall velocity $v_{\rm ff}(r)$ then the planetesimal will move a distance that is a small fraction of r before it attains terminal velocity.

Derive an expression for $v_t(r)$ in the case that

$$a_{\rm drag} = \frac{3\rho_{\rm pl}(r)\dot{r}^2}{8\rho_{\rm rock}b},$$

where $\rho_{\rm pl}(r)$ is the local density of the planet, $\dot{r}$ is the planetesimal's velocity, $\rho_{\rm rock}$ is the mass density of the planetesimal and b is the radius of the planetesimal.

Explain what conditions need to be met in order that $\dot{r}$ is close to $v_{\rm t}(r)$ at all radii.

Now assume that the drag also causes the planetesimal to lose mass, and hence change its radius at a rate given by

$$\dot{b} = A\rho_{\rm pl}(r)\dot{r}^3,$$

where A is a constant that depends on the property of the planetesimal rock. Show that if the planet can be approximated as a uniform sphere of radius $R_{\rm pl}$, the radius of a planetesimal of initial size b_0 is given by

$$b = b_0 \exp(B(r^2 - R_{\rm pl}^2))$$

TURN OVER...

where $B = 16\pi\, G\, \rho_{\rm rock}\, A\, \rho_{\rm pl}/9$.

By what factor does a planetesimal of initial size 100 m reduce in size when falling to the centre of a uniform density planet of mass 10^{27} kg and radius 6×10^{8} m?

[You may assume that $\rho_{\rm rock} = 5000$ kg m^{-3} and $A = 5 \times 10^{-13}$ m^3 J $^{-1}$.]

END OF PAPER