Lecture 17

COSMOLOGY

CMB anisotropies
spherical harmonics $Y_{lm}(\theta, \phi)$

$l = 0$

$l = 1$ dipole

$l = 2$ quadrupole

$l = 3$ octapole
Fluctuations in the CMB

We can get a good idea of the evolution of fluctuations prior to recombination by making the following assumptions:

- **Radiation and baryons are tightly coupled by Thomson scattering.** We will neglect weakly interacting dark matter.

- **Neglect gravity.** This is accurate for fluctuations with scales much smaller than the Hubble radius $\lambda < ct$, since the dynamics is dominated by pressure not gravity.

- **Neglect the expansion of the Universe.** This is not a particularly good approximation because recombination is quite an extended process taking $\Delta z \sim 200$ at $z \sim 1000$. 
Let us write the photon distribution function as

\[ f(x, q, t) = f_0 + f_1, \]

where \( f_1 \) is a perturbation on the black-body function \( f_0 \), and \( q \) is the comoving photon momentum. We can define the perturbation to the radiation brightness as

\[ \Delta(x, q, t) = f_1 \left( \frac{T_0}{\frac{4}{\partial T_0}} \right)^{-1}. \]

Then since Thomson scattering is independent of photon energy, the Boltzmann equation for the perturbation \( \Delta \) is

\[ \frac{\partial \Delta}{\partial t} + \frac{\gamma_i}{R} \frac{\partial \Delta}{\partial x^i} = \sigma_T n_e [\Delta_0 + 4 \gamma_i v_b^i - \Delta], \]

where the \( \gamma^i \) are the direction cosines of \( \hat{q} \), \( \sigma_T \) is the Thomson cross-section, \( n_e \) is the free electron density, \( v_b \) is the matter
velocity, and $\Delta_0$ is the isotropic part of $\Delta$:

$$\Delta_0 = \frac{1}{4\pi} \int \Delta d\Omega.$$ 

Fourier transforming the Boltzmann equation:

$$\frac{d\Delta}{dt} + \frac{ik\mu}{R} \Delta = \sigma_T n_e [\Delta_0 + 4\mu v_b - \Delta], \quad \mu = \hat{k}.\hat{q}. \quad (1)$$

The equation of motion for the matter (neglecting expansion) is:

$$\frac{dv_b}{dt} = \sigma_T n_e \frac{\bar{\rho}_\gamma}{\bar{\rho}_b} \left[ \Delta_1 - \frac{4}{3} v_b \right], \quad (2)$$

where

$$\Delta_1 = \frac{1}{2} \int_{-1}^{1} \Delta \mu d\mu,$$

is the photon energy flux. (This equation tells that it is difficult for baryons to move relative to the radiation if they are strongly
coupled to the photons). Finally, mass conservation gives the familiar equation of continuity

$$\frac{d\delta_b}{dt} = -\frac{ikv_b}{R},$$

(3)

where $\delta_b$ is the baryon overdensity ($\delta_b = (\rho_b - \bar{\rho}_b)/\bar{\rho}_b$). Equations (1) - (3) allow us to calculate the evolution of the fluctuations:

Prior to recombination, the baryon component is highly ionised and the mean-free time for Thomson scattering

$$t_c = \frac{1}{\sigma_T n_e},$$

is much smaller than the expansion rate, $t_c \ll t$. This is what we mean when we say that the matter and radiation are *tightly coupled*. The equations (1) and (2) are therefore very 'stiff',
and so to first order, the terms in square brackets must be close to zero:

$$\Delta = \Delta_0 + 4\mu v.$$ 

Now define the quantity $X = \Delta_0 + 4\mu v$ and insert back into (1) and (2). Then to second order in $t_c$

$$\Delta = X - t_c \left[ \dot{X} + i\frac{\mu k X}{R} \right] + t_c^2 \left[ \ddot{X} + 2i\frac{\mu k \dot{X}}{R} - \frac{\mu^2 k^2 X}{R^2} \right] + \mathcal{O}(t_c^3),$$

i.e. for imperfect coupling (non-zero $t_c$) the radiation develops quadrupolar, octopolar and higher order perturbations. This allows us to get a closed set of equations for $\Delta_0$ and $v$, valid to first order in $t_c$:

$$\dot{\Delta}_0 = \frac{-ik}{R} \left[ \frac{4}{3} v - t_c \left( \frac{4}{3} \dot{v} + \frac{ik \Delta_0}{3R} \right) \right],$$

$$\ddot{v} = \frac{\bar{\rho}_\gamma}{\bar{\rho}_m} \left[ -\frac{4}{3} \ddot{v} - \frac{ik \Delta_0}{3R} + t_c \left( \frac{4}{3} \dddot{v} + \frac{2ik \dot{\Delta}_0}{3R} - \frac{4k^2 v}{5R^2} \right) \right].$$
These equations have solutions of the form:

\[
\left\{ \begin{array}{c}
v \\
\Delta
\end{array} \right\} \propto \exp(-\Gamma t),
\]

with

\[
\Gamma = \pm \frac{ik}{R\sqrt{3B}} - \frac{k^2 t_c}{6R^2} \left(1 - \frac{6}{5B} + \frac{1}{B^2}\right), \tag{\star}
\]

where

\[
B = 1 + \frac{3\bar{\rho}_b}{4\bar{\rho}_\gamma}.
\]

The first term in (\star) describes acoustic oscillations with adiabatic sound speed

\[
c_s = \frac{c}{\sqrt{3}} \left(1 + \frac{3\bar{\rho}_b}{4\bar{\rho}_\gamma}\right)^{-1/2}.
\]

In the tightly coupled regime, baryons and photons act as a single
fluid, so the inertia of the baryons reduces the sound speed below the relativistic value \( c/\sqrt{3} \).

The second term in (*) describes \textit{damping} of small-scale fluctuations by \textit{photon diffusion}.

A photon within a perturbation will random walk. The mean time between collisions is \( t_c \), so the number of collisions in time \( t \) is \( N = t/t_c \). Therefore photons will diffuse over a length \( \sqrt{N}ct_c = c\sqrt{tt_c} \) carrying the matter with them, in agreement with (*).

Note at recombination \( z_{rec} \approx 1000, \ B \approx 1.65, \) and the characteristic damping scale is \( k_d/R_0 \sim (15 \text{ Mpc})^{-1} \).
\( \log R(t)/R_0 \)

\( k = 0.1 \text{ Mpc}^{-1} \)

\( k = 1.0 \text{ Mpc}^{-1} \)
CMB fluctuations on large angular scales

On large scales, we need to perform a full general relativistic analysis. However, the result is easy to understand. The temperature anisotropies measure the potential fluctuations (via gravitational redshift) on the last scattering surface the *Sachs-Wolfe effect*:

\[
\frac{\Delta T}{T} \propto \frac{(\delta \phi)_\lambda}{c^2} \sim \frac{G\delta M}{\lambda} \propto \delta \rho \lambda^2 \propto \lambda^{(1-n_s)/2}.
\]

Where the last expression follows for fluctuations with power spectrum \( P(k) \propto k^{n_s} \).
In practice, we compute the *CMB power spectrum*, $C_\ell$, on the sky by expanding in spherical harmonics:

$$\frac{\Delta T}{T} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

$$C_\ell = \langle |a_{\ell m}|^2 \rangle. \quad (1)$$

If $\Omega_K = \Omega_\Lambda = 0$, then the potential fluctuations are independent of time from the last scattering surface to the present day and we then find

$$C_\ell \propto \frac{\Gamma\left(\ell + \left(\frac{n_s - 1}{2}\right)\right)}{\Gamma\left(\ell + \left(\frac{5-n_s}{2}\right)\right)} \propto \ell^{(n_s-3)} \text{ if } \ell \gg 1.$$

Scale-invariant fluctuations therefore lead to $\ell^2 C_\ell \approx \text{constant}$ at multipoles less than $\sim 100$. 
\[ D_l = \frac{l(l+1)C_l}{2\pi}[\mu K^2] \]

- Sachs-Wolfe effect
- Angle subtended by maximum distance travelled by a sound wave
- Acoustic peaks
- Damping tail
acoustic peaks

\[ k_m r_s(z_{rec}) = m\pi, \quad m = 1, 2, 3, \ldots \]  \hfill (9.30)

with maxima for odd values of \( m \) and minima for even values of \( m \). In (9.30), \( r_s(z_{rec}) \) is the sound horizon at recombination,

\[
r_s = \int c_s d\tau = \int \frac{c}{\sqrt{3}} \left( 1 + \frac{3\rho_b}{4\rho_\gamma} \right)^{-1/2} d\tau
\]

\[
= \frac{c}{\sqrt{3}} \frac{1}{H_0 \Omega_m^{1/2}} \int_0^{1/(1+z_{rec})} \frac{da}{(a + a_{equ})^{1/2}(1 + Ba)^{1/2}}
\]  \hfill (9.31a)

\[
\]  \hfill (9.31b)

using the expression (9.19) for the sound speed and

\[
a_{equ}^{-1} = (1 + z_{equ}) = 24185(\Omega_m h^2), \quad B = 30496(\Omega_b h^2).
\]  \hfill (9.32)