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Advisory Committee to the
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NATIONAL ACADEMY OF SCIENCES
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APPENDIX 2

HOW MANY PICTURES MUST WE TAKE TO GET A GOOD ONE?

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Let $E(x, y)$ represent the complex field amplitude in the focal plane of a diffraction-limited telescope of focal length F and aperture radius R, which is viewing a point source of light through an intervening turbulent atmosphere. Let $I(x, y) = |E(x, y)|^2$ be the intensity distribution in the image.

Let $M(\xi, \zeta)$ be the normalized lateral coherence function for the light in the entrance pupil, defined in the same manner as was done by Hufnagel and Stanley.[1] With a scaling of the coordinates, M is identical to the average "atmospheric" optical transfer function.

We now consider the special case where the seeing is much worse than the potential diffraction limit of the telescope. This is analogous to saying that $M \approx 0$ when $\xi^2 + \zeta^2 > r_c^2$ for $r_c \ll R$.

When this is the case we may consider that the light which forms the image comes from several effectively uncorrelated sources within the aperture, and by the central limit theorem we may consider the E field in the entrance pupil to be a complex Gaussian random variable.

If this is the case,* then we may use some results obtained by Goodman[2] in connection with light scattering from rough surfaces.

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*To be of nontrivial interest we must exclude the possibility that M is limited only because of simple random tilting of planar (flat) wavefronts. For atmospheric turbulence this will require that $r_c \lesssim 10R$.

First the probability density function for $I(x, y)$ becomes

$$p(I) = \frac{1}{\langle I(x, y) \rangle} \exp\left[-I(x, y)/\langle I(x, y) \rangle \right] \quad , \tag{1}$$

which is just the chi-squared distribution for two degrees of freedom (one for the real part of E and one for the imaginary part).

Goodman also reports that the normalized autocovariance function for I becomes for our case

$$\frac{\langle I(x, y)\, I(x+\Delta x, y+\Delta y) \rangle}{\langle I(x, y) \rangle \langle I(x+\Delta x, y+\Delta y) \rangle} - 1 = \left[\frac{2J_1(2\pi R\rho / F\lambda)}{(2\pi R\rho / F\lambda)} \right]^2 \quad , \tag{2}$$

where $\rho = (\Delta x^2 + \Delta y^2)^{1/2}$.

This implies that the covariance distance is equal to the Airy disk radius of the aperture R. Since the turbulence does not significantly disturb the polarization of the light these formulas hold for both unpolarized and polarized light.

The interpretation of this is that under the stated conditions the point-spread function will be composed of high-contrast "grain" (not unlike the speckle pattern from laser-illuminated diffuse objects), that the grain "size" will be the size of the Airy disk, and that the individual grains will be "uncorrelated."

Examination of short-exposure stellar images from the 82-in.-diameter McDonald Observatory telescope[3] shows these features.

It is of academic (and perhaps practical) interest to compute the probability that any one of these "grains" will contain most of the image flux. When this occurs, the image (for that instant of time) will be nearly diffraction limited. The probability that any single grain will contain over half the total flux is given approximately by

$$P = \sum_j \exp\left[- \frac{\frac{1}{2} \sum_k \langle I_k \rangle}{\langle I_j \rangle} \right] \quad ,$$

where j and k summations are over the uncorrelated "grains."* By replacing the sums by integrations over the blur circle, and assuming the blur to have a Gaussian spatial distribution with a standard deviation r_b, one obtains

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*Even if the "grains" are uncorrelated, they are not independent; the approximation partially ignores the dependence.

$$p \approx \int_0^\infty \frac{2\pi r\, dr}{A} \exp\left\{ - \frac{\frac{1}{2}\int_0^\infty \frac{2\pi r'\, dr'}{A} \exp\left[-\frac{1}{2}(r'/r_b)^2\right]}{\exp\left[-\frac{1}{2}(r/r_b)^2\right]} \right\} ,$$

where A is the area under the normalized autocovariance function given by Eq. (2). If $r_b^2 >> A$, we may keep only the first two terms in the power series expansion for the inside denominator, to yield (after integrating the inside numerator)

$$p \approx \int_0^\infty \frac{2\pi r\, dr}{A} \exp\left\{ (-\pi r_b^2/A) \left[1 + \frac{1}{2}(r/r_b)^2 \right] \right\}$$

which becomes

$$p \approx 2 \exp(-\pi r_b^2/A) .$$

Noting that $A = (F\lambda)^2/\pi R^2$ and (because of the Fourier transform relationship between M and the blur circle) that $r_b \approx (F\lambda)/\pi r_c$, we get finally

$$p \approx 2 \exp\left[-(R/r_c)^2\right] \quad \text{for } R >> r_c .$$

Let us assume that wavefront "samples" become uncorrelated in 0.1 sec of time, and ask what value of R/r_c will give a reasonable probability of getting one sample which yields a diffraction-limited image in a time of 50 sec. For this case p need be approximately 2×10^{-3}, or $R/r_c = \sqrt{7} = 2.6$. This does not satisfy the condition that $r_c < 10R$, but if we reinterpret r_c as a kind of "pseudo coherence radius" corresponding to the pseudo (short-time) transfer function (where wavefront tilts have been removed) then one might just sneak by. In any event, the deviation from effective normality for E will be such that p will be greater than calculated here.

The pseudo coherence radius, r_s, will be a function of both R and r_c. We have estimated that $r_s = R/2.6$ when $r_c \approx R/7$.

To illustrate this with numbers, let the average (long-time) blur be 1 sec of arc, which corresponds to $r_c = 5$ cm. Then if R = 35 cm (30-in. diameter), one has a reasonable chance of getting a picture with nearly 1/7-sec-of-arc resolution if he makes 500 exposures. The model needs refinement, but this preliminary result does not seem too unreasonable. Note that there is an optimum R associated with each r_c and p. If R is less than this optimum, diffraction-limited pictures will be more common but will have more blur from diffraction. If R is greater than optimum, one will not obtain diffraction-limited pictures at all.

REFERENCES

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2. J. W. Goodman, Proc. IEEE 53, 1688 (1965).
3. J. Texereau, Sky and Telescope 25, 347 (1964).