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ABSTRACT:

We have calculated the probability of obtaining near-diffraction-limited images with a system imaging through atmospheric turbulence but with partial wavefront conjugation.

BACKGROUND

Imaging through atmospheric turbulence is a random process, and there is always a finite probability that any one image will be perfect, i.e. limited only by diffraction. In 1966 R. Hufnagel published an estimate of this probability based upon the statistical properties of the speckles formed within a point image. He predicted that, for large diameter systems, the probability of a good exposure would vary as $\exp(-D^2)$. This result was obtained without using the Kolmogoroff spectral properties of the random fields.

In 1978 D. Fried computed the probability of a "lucky exposure" by numerical integration of a chi-square process in a 569-dimensional hyperspace. The statistical parameters used were appropriate for Kolmogoroff turbulence. He obtained the same $\exp(-D^2)$ probability estimated a decade earlier.

In 1981 D. Bensimon et al measured the probability of a lucky exposure and verified these predictions.

INTRODUCTION

In this paper we extend these results to situations for which there is partial wavefront conjugation of the turbulence generated wavefront errors. Specifically, we assume that the wavefront is represented as a sum of Karhunen-Loeve eigenfunctions (as calculated by Fried), and that the first N of these have been perfectly conjugated.

The pupil averaged mean squared wavefront error is thus an infinite sum of chi-square random variables, each of whose variances corresponds to the Karhunen-Loeve eigenvalves starting with the $(N + 1)^{th}$ term. Our goal is to compute the probability that this sum is less than 1 radian-squared (corresponding to a Strehl resolution of approximately $\exp(-1) = 0.37$.

DETAILS & RESULTS

The probability distribution of a sum of random variables is the convolution of the distribution functions for each variable. For our purposes, we approximate the infinite sum with a finite one, composed of appropriately chosen distribution functions.

We note first, that the strength of the eigenvalues in Fried's list can be extrapolated to infinity through a power law relationship of the form.

$$\text{eigenvalue} \sim 0.1709\ (\text{rank})^{-11/6}$$

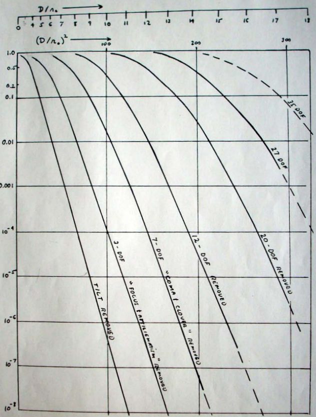
This enables us to compute all the moments of the probability distribution of any sum of variables. We computed the first five moments for the distribution corresponding to the sum from 36 to infinity. We then found that these could be matched quite well by a distribution function that was a convolution of a gaussian (μ = 0.00707, σ = 0.0005766) and a chi-square distribution with 19 degrees of freedom (σ_χ^2 = 0.000179).

We then computed 6 more chi-square distributions with the parameters given in Table 1. The 6th distribution was convolved with the (36-∞) distribution to yield the distribution corresponding to conjugation of 27 degrees-of-freedom. The 5th was then convolved with the previous result to yield the distribution of 20 degrees-of-freedom. And so on to zero degrees of freedom which corresponds to the tilt-only removed situation corresponding to a short exposure.

Each resulting distribution was then integrated to yield the cumulative probability distributions as shown in Figure 1. The lines are dashed where reduced accuracy is expected.

Various checks were performed to insure reasonable accuracy in the numerical computations. In addition, it should be noted that the "tilt removed" curve agrees well with Fried's results down to the 10^{-7} probability level, although there is a slight and gradual difference between the two results beyond that. Convolutions and integrations were performed in steps of 0.0005.

FIGURE 1



PROBABILITY OF LUCKY EXPOSURE FOR VARIOUS NUMBERS OF DEGREES OF FREEDOM REMOVED (CONJUGATED)

TABLE I

CORRESPONDENCE BETWEEN CHI-SQUARE PARAMETERS AND KARHUNEN-LOEVE EIGENVALUES

KARHUNEN-LOÈVE EXPANSION		CHI-SQUARE PARAMETERS*	
RANK	EIGENVALUE	N	σ^2
1 - 2	.023892	3	.023879
3	.023851		
4 - 5	.006641	4	.006618
6 - 7	.006595		
8 - 9	.0027418		
10 - 11	.0020952	4.91	.0023926
12	.0020782		
13 - 14	.0013812		
15 - 16	.0009850	7.6	.0010833
17 - 18	.0009640		
19 - 20	.0007865		
21 - 22	.0005451		
23 - 24	.0004871	7	.0005019
25 - 26	.0004865		
27	.0004759		
28 -29	.0003337		
30 - 31	.0003207	8	.0003035
32 - 33	.0002857		
34 - 35	.0002739		

* When there is a wide spread of eigenvalues within a group, the N and σ^2 values were chosen to preserve the first and second moments of the group.

DISCUSSION OF RESULTS

Even partial wavefront conjugations dramatically improves one's chances of obtaining a near-diffraction-limited quality exposure. Since loss of isoplanatism is one way of preventing perfect conjugation, this implies that there might still be merit in imaging faint objects with nearby bright stellar references.

Within the accuracy limit of the approximations employed there is evidence that the probability of a lucky exposure is asymptotically proportional to $\exp(-D^2)$ in all cases studies, thus supporting the original hypothesis based upon speckle.

These results may also have applications to the evaluations of distributions of positive definite quadratic form of normal random variables for cases where the variances form an infinite converging series. This is a case that has not been amenable to solution by existing approximations reported in mathematics journals.

REFERENCES

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