NST2AS NATURAL SCIENCES TRIPOS Part II

ASTROPHYSICS **Examinations: 2, 4, 5 & 8 June 2020**

Astrophysics Constants and Formulae

Available for examinations in the Natural Sciences Tripos Part II – Astrophysics

Physical Constants

velocity of light	$c = 2.998 \times 10^{8}\mathrm{ms}^{-1}$
Newtonian gravitational constant	$G = 6.673 \times 10^{-11}\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Planck's constant	$h = 6.626 \times 10^{-34}\mathrm{Js}$
	$\hbar = 1.055 \times 10^{-34}\mathrm{Js} = h/2\pi$
Stefan–Boltzmann constant	$\sigma = 5.671 \times 10^{-8}\mathrm{Wm}^{-2}\mathrm{K}^{-4} = ac/4$
gas constant	$\mathcal{R} = 8.314\ \mathrm{Jmol}^{-1}\mathrm{K}^{-1}$ $\qquad \mathcal{R}^* = 10^3\mathcal{R}$
Avogadro's number	$N_\mathrm{A} = 6.022 \times 10^{23}\mathrm{mol}^{-1}$
Boltzmann's constant	$k_\mathrm{B} = 1.381 \times 10^{-23}\mathrm{JK}^{-1} = \mathcal{R}/N_\mathrm{A}$
atomic mass unit	$m_\mathrm{u} = 1.661 \times 10^{-27}\mathrm{kg}$
proton rest mass	$m_\mathrm{p} = 1.673 \times 10^{-27}\mathrm{kg}$ $= 938\ \mathrm{MeV}$
neutron rest mass	$m_\mathrm{n} = 1.675 \times 10^{-27}\mathrm{kg}$ $\quad (m_\mathrm{n} - m_\mathrm{p} = 1.29\,\mathrm{MeV})$ $= 939\,\mathrm{MeV}$
electron rest mass	$m_\mathrm{e} = 9.109 \times 10^{-31}\mathrm{kg}$ $\quad (m_\mathrm{p}/m_\mathrm{e} = 1836)$ $= 0.511\ \mathrm{MeV}$
classical electron radius	$r_\mathrm{e} = 2.818 \times 10^{-15}\mathrm{m}$
Compton wavelength	$\lambda_\mathrm{C} = 2.426 \times 10^{-12}\mathrm{m} = h/(m_e c)$
Thomson cross section	$\sigma_\mathrm{T} = 6.652 \times 10^{-29}\mathrm{m}^2 = (8\pi/3)r_e^2$
inverse fine-structure constant	$\alpha^{-1} = 137.0$
radiation density constant	$a = 7.566 \times 10^{-16}\mathrm{Jm}^{-3}\mathrm{K}^{-4} = 8\pi^5 k^4/(15c^3h^3)$
electronic charge	$e = 1.602 \times 10^{-19}\mathrm{C}$
Bohr radius	$a_0 = 5.292 \times 10^{-11}\mathrm{m} = h^2/(\pi\mu_0 c^2 m_\mathrm{e} e^2)$
temperature of triple point of water	$T_\mathrm{tr} = 273.16\mathrm{K}$
vacuum magnetic permeability	$\mu_0 = 4\pi \times 10^{-7}\mathrm{Hm}^{-1}$ (exact)
vacuum electrical permittivity	$\epsilon_0 = 8.854 \times 10^{-12}\mathrm{Fm}^{-1} = c^{-2}\mu_0^{-1}$
ionization potential of hydrogen	$\chi_\mathrm{H} = 13.598\,\mathrm{eV}$

Conversion Factors

electron volt	$1\text{eV} = 1.602 \times 10^{-19}\text{J}$
erg	$1\text{ erg} = 10^{-7}\text{J}$
year	$1\,\text{yr} = 3.156 \times 10^{7}\text{s}$

Astronomical Measures

astronomical unit	$1\,\text{au} = 1.496 \times 10^{11}\text{m}$
parsec	$1\,\text{pc} = 3.086 \times 10^{16}\text{m}$
light year	$1\,\text{ly} = 9.461 \times 10^{15}\text{m}$
solar mass	$1\,\text{M}_\odot = 1.989 \times 10^{30}\text{kg}$
solar radius	$1\,\text{R}_\odot = 6.960 \times 10^{8}\text{m}$
solar luminosity	$1\,\text{L}_\odot = 3.90 \times 10^{26}\text{W}$
solar effective temperature	5780K
absolute magnitude of sun	$M_\text{V} = 4.83$
distance modulus	$m - M = 5\log_{10} D - 5$
Earth mass	$1\text{M}_\oplus = 5.976 \times 10^{24}\text{kg}$
Earth radius	$1\text{R}_\oplus = 6.371 \times 10^{6}\text{m}$
Jansky	$1\text{Jy} = 10^{-26}\text{Wm}^{-2}\text{Hz}^{-1}$

Vector Relations

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \mathbf{B}\,(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla \cdot (\phi\mathbf{A}) = \phi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\phi$$

$$\nabla \wedge (\phi\mathbf{A}) = \phi\nabla \wedge \mathbf{A} - \mathbf{A} \wedge \nabla\phi$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \wedge (\nabla \wedge \mathbf{B}) + \mathbf{B} \wedge (\nabla \wedge \mathbf{A})$$

$$\nabla \cdot (\mathbf{A} \wedge \mathbf{B}) = \mathbf{B} \cdot (\nabla \wedge \mathbf{A}) - \mathbf{A} \cdot (\nabla \wedge \mathbf{B})$$

$$\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

$$\nabla \wedge (\nabla\phi) = 0$$

$$\nabla \cdot (\nabla \wedge \mathbf{A}) = 0$$

$$\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

$$\mathbf{A} \wedge (\nabla \wedge \mathbf{A}) = \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{A}$$

Maxwell's Equations (*in vacuo*)

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mu_0^{-1} \nabla \wedge \mathbf{B} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$$

Lorentz force density:

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \wedge \mathbf{B}$$

Electromagnetic energy density:

$$W = \frac{1}{2}\left(\epsilon_0 E^2 + \mu_0^{-1} B^2\right)$$

Poynting's Vector:

$$\mathbf{S} = \mathbf{E} \wedge \mathbf{B}/\mu_0$$

Poynting's Theorem:

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}$$

Candidates may be asked to derive any of the subsequent formulae in this booklet.

Relativity

Type (2,0) tensor:

$$T'^{mn} = T^{rs} \frac{\partial x'^m}{\partial x^r} \frac{\partial x'^n}{\partial x^s}$$

Type (0,2) tensor:

$$T'_{mn} = T_{rs} \frac{\partial x^r}{\partial x'^m} \frac{\partial x^s}{\partial x'^n}$$

Type (1,1) tensor:

$$T'^m{}_n = T^r{}_s \frac{\partial x'^m}{\partial x^r} \frac{\partial x^s}{\partial x'^n}$$

Kronecker delta:

$$\delta^r_s = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}$$

Interval:

$$ds^2 = g_{mn} dx^m dx^n$$

Metric tensor:

$$g_{mn} = g_{nm}, \qquad g_{mr} g^{ms} = \delta_r^s$$

Magnitude of a vector:

$$|X|^2 = g_{mn} X^m X^n$$

Orthogonality condition:

$$g_{mn} X^m Y^n = 0$$

Raising and lowering indices:

$$X^m = g^{mn} X_n, \qquad X_m = g_{mn} X^n$$

Metric connection:

$$[mn, r] = \frac{1}{2} \left(\frac{\partial g_{mr}}{\partial x^n} + \frac{\partial g_{nr}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r} \right)$$

$$\Gamma^r_{mn} = g^{rs} [mn, s]$$

Metric connection for diagonal metric g_{mn}:
($m \neq n \neq r$, no sums over repeated indices)

$$\Gamma^m_{nr} = 0, \quad \Gamma^n_{mm} = -\frac{1}{2g_{nn}} \frac{\partial g_{mm}}{\partial x^n}, \quad \Gamma^m_{nm} = \Gamma^m_{mn} = \frac{1}{2g_{mm}} \frac{\partial g_{mm}}{\partial x^n},$$

$$\Gamma^m_{mm} = \frac{1}{2g_{mm}} \frac{\partial g_{mm}}{\partial x^m}.$$

Geodesic:

$$\frac{d^2 x^r}{ds^2} + \Gamma^r_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0$$

or

$$\frac{du_i}{ds} = \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} u^j u^k$$

where $u^i = \frac{dx^i}{ds}$.

Absolute derivative:

$$\frac{DT^i}{Ds} = \frac{dT^i}{ds} + \Gamma^i_{jk} T^k \frac{dx^j}{ds}$$

Parallel propagation:

$$\frac{DT^i}{Ds} = 0$$

Covariant derivative:

$$\nabla_s T^r = \frac{\partial T^r}{\partial x^s} + \Gamma^r_{sm} T^m$$

$$\nabla_s T_r = \frac{\partial T_r}{\partial x^s} - \Gamma^m_{sr} T_m$$

$$\nabla_t g_{rs} = 0$$

Riemann curvature tensor:

$$R_{mnr}{}^s = -\frac{\partial}{\partial x^m} \Gamma^s_{nr} + \frac{\partial}{\partial x^n} \Gamma^s_{mr} + \Gamma^p_{mr} \Gamma^s_{np} - \Gamma^p_{nr} \Gamma^s_{mp}$$

Non-commutative covariant differentiation:

$$\nabla_n \nabla_m T_r - \nabla_m \nabla_n T_r = R_{nmr}{}^{s} T_s$$

Relation of curvature tensor to metric:

$$\begin{aligned} R_{rsmn} = & \frac{1}{2}\left(\frac{\partial^2 g_{rn}}{\partial x^s \partial x^m} + \frac{\partial^2 g_{sm}}{\partial x^r \partial x^n} - \frac{\partial^2 g_{rm}}{\partial x^s \partial x^n} - \frac{\partial^2 g_{sn}}{\partial x^r \partial x^m}\right) \\ & + g^{pq}([rn,p][sm,q] - [rm,p][sn,q]) \end{aligned}$$

Identities:

$$R_{rsmn} = -R_{srmn} = -R_{rsnm} = R_{mnrs}$$

$$R_{rsmn} + R_{mrsn} + R_{smrn} = 0$$

Conditions for flatness:

$$R_{rsmn} = 0$$

Bianchi identity:

$$\nabla_t R_{rsmn} + \nabla_r R_{stmn} + \nabla_s R_{trmn} = 0$$

Ricci tensor:

$$R_{rm} = R_{mr} = R_{nrm}{}^{n}$$

Ricci scalar:

$$R = R^n{}_n$$

Einstein tensor:

$$G^n{}_t = R^n{}_t - \frac{1}{2}\delta^n_t R$$

$$\nabla_n G^n{}_t = 0$$

Einstein field equation:

$$G_{nt} = -\frac{8\pi G}{c^4} T_{nt}$$

Geodesic deviation:

$$\frac{D^2 \xi^i}{Ds^2} - R_{klj}{}^{i} \xi^k \frac{dx^j}{ds}\frac{dx^l}{ds} = 0$$

Schwarzschild metric:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

Astrophysical Fluid Dynamics

Mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum:

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = & -\frac{1}{\rho}\nabla p - \nabla \Phi \\ & + \frac{1}{\rho}\mathbf{J} \wedge \mathbf{B}\end{aligned}$$

Energy (for a stationary gravitational potential):

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] = -\rho \dot{Q}$$

where

$$E = \rho \left(\frac{1}{2}u^2 + \Phi + e \right), \qquad e = \frac{1}{\gamma - 1}\frac{p}{\rho}.$$

Poisson's Equation:

$$\nabla^2 \Phi = 4\pi G \rho$$

Induction (in flux-freezing approximation):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B})$$

Rankine–Hugoniot Conditions:

$$\begin{aligned}\rho_1 u_1 &= \rho_2 u_2 \\ p_1 + \rho_1 u_1^2 &= p_2 + \rho_2 u_2^2 \\ \frac{1}{2}u_1^2 + e_1 + \frac{p_1}{\rho_1} &= \frac{1}{2}u_2^2 + e_2 + \frac{p_2}{\rho_2}\end{aligned}$$

Bernoulli's constant:

$$H = \frac{1}{2}u^2 + \int \frac{dp}{\rho} + \Phi$$

Schwarzschild criterion:

$$\text{Stability requires } \frac{d}{dr}\left(\frac{p}{\rho^\gamma}\right) > 0$$

Navier-Stokes equation in cylindrical polar coordinates with constant ν:

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} =$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right] - \nu\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z} + \frac{u_\phi u_r}{r} =$$

$$-\frac{1}{r\rho}\frac{\partial p}{\partial \phi} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} - \frac{u_\phi}{r^2}\right] + \nu\frac{2}{r^2}\frac{\partial u_r}{\partial \phi}$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} =$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right]$$

Statistical Physics

First law of thermodynamics:

$$dE = TdS - pdV$$

Enthalpy:

$$H = E + pV$$

Free energy:

$$F = E - TS = -k_{\mathrm{B}}T \ln Z$$

Gibbs free energy:

$$G = E + pV - TS$$

Grand canonical potential:

$$\Phi = F - \mu N$$

Entropy:

$$S = k_{\mathrm{B}} \ln \Omega = -k_{\mathrm{B}} \sum_n p(n) \ln p(n)$$

Bose–Einstein mean occupancy of single-particle state r:

$$\langle n_r \rangle = \frac{1}{e^{\beta(E_r - \mu)} - 1}$$

where $\beta \equiv 1/(k_{\mathrm{B}}T)$

Fermi–Dirac mean occupancy of single-particle state r:

$$\langle n_r \rangle = \frac{1}{e^{\beta(E_r-\mu)}+1}$$

Maxwell–Boltzmann distribution:

$$F(v)\,dv \propto v^2 e^{-mv^2/(2k_{\mathrm{B}}T)}\,dv$$

Planck radiation law:

$$E(\omega)\,d\omega = \frac{V\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega}-1}\,d\omega$$

Radiation pressure:

$$p = \frac{4\sigma}{3c}T^4$$

Perfect gas:

$$p = \frac{Nk_{\mathrm{B}}T}{V}$$

Stirling's formula:

$$\ln n! \sim n(\ln n - 1) \text{ as } n \to \infty$$

Cosmology

Robertson–Walker metric:

$$ds^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

where $R(t)$ is the scale factor.
Friedmann equations:

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{-8\pi G}{c^2}p + \Lambda c^2$$

Conservation laws:

Dust: $\rho R^3 = \text{constant}$ Radiation: $T^4 R^4 = \text{constant}$

Hubble constant:

$$H_0 = \left.\frac{\dot{R}}{R}\right|_0$$

Deceleration parameter:

$$q_0 = -\left.\frac{\ddot{R}R}{\dot{R}^2}\right|_0$$

Proper distance:

$$d_{\rm prop}(t) = R(t) \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

Luminosity distance for $\Lambda = 0$:

$$d_L = R^2(t_0)\frac{r_1}{R(t_1)} = \frac{c}{H_0 q_0^2}\left\{q_0 z + (q_0 - 1)\left[(1+2q_0 z)^{1/2} - 1\right]\right\}.$$

Angular diameter distance:

$$d_\theta = R(t_1) r_1 = d_L (1+z)^{-2}$$

Equations of Stellar Structure

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho$$

$$\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2 \rho\epsilon$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho L}{16\pi a c r^2 T^3} \qquad \text{(radiative)}$$

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \qquad \text{(convective)}$$

Energy released in conversion of hydrogen to helium: 0.007 mc^2.

Stellar Dynamics

The Collisionless Boltzmann Equation:

$$\frac{Df}{Dt} = 0 = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial v_i}\frac{dv_i}{dt}$$

Cartesian coordinates: (x, y, z)

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

$$\ddot{\mathbf{r}} = \dot{v}_x\mathbf{i} + \dot{v}_y\mathbf{j} + \dot{v}_z\mathbf{k}$$

$$\begin{aligned}\frac{Df}{Dt} = 0 &= \frac{\partial f}{\partial t} + v_x\frac{\partial f}{\partial x} + v_y\frac{\partial f}{\partial y} + v_z\frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v_x} - \frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial v_y} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} \\ &= \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}}\end{aligned}$$

Jeans equations:

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \overline{v_i})}{\partial x_i} = 0 \quad \text{where} \quad \nu = \int f d^3\mathbf{v}$$

$$\overline{v_i} = \frac{1}{\nu} \int f v_i d^3\mathbf{v}$$

$$\frac{\partial}{\partial t}\left(\nu \overline{v_j}\right) + \frac{\partial\left(\nu \overline{v_i v_j}\right)}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0 \quad \text{where} \quad \overline{v_i v_j} = \frac{1}{\nu} \int v_i v_j f d^3\mathbf{v}$$

$$\nu \frac{\partial \overline{v_j}}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial\left(\nu \sigma_{ij}^2\right)}{\partial x_i} \quad \text{where} \quad \sigma_{ij}^2 = \frac{1}{\nu} \int \left(v_i - \overline{v_i}\right)\left(v_j - \overline{v_j}\right) f d^3\mathbf{v}$$

Bound orbit in point mass potential:

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\phi}$$

Oort's constants:

$$A \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right)\Bigg|_{R_0}$$

$$B \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right)\Bigg|_{R_0}$$

$$\kappa^2 = -4B(A - B)$$

$$\frac{\kappa}{\Omega} = 2\sqrt{\frac{-B}{A - B}}$$

Useful mathematical relations

$$\int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}\text{erf}(x), \qquad \text{erf}(\pm\infty) = \pm 1$$

Last revision: 8 April 2018