**NATURAL SCIENCES TRIPOS Part II**

Monday 5 June 2006 09.00 – 12.00

ASTROPHYSICS - PAPER 1

**Before you begin read these instructions carefully.**

*The paper is divided into Section* **A** *and Section* **B**. *Candidates may attempt Parts from* **ALL** *questions in Section* **A** *and from not more than* **THREE** *questions in Section* **B**.

*Each question is divided into Part* **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of questions.*

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*A master cover sheet listing all Parts of all questions attempted must also be completed.*

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| *Tags* | |

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## SECTION A

### 1X Relativity

**(i)** In special relativity, let the electromagnetic field tensor $E_{\mu\nu}$ be derived from a potential $A$ by

$$E_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \, .$$

Suppose the 4-current density is $J^{\mu}$.

Show that Maxwell's equation,

$$E^{\mu\nu}{}_{,\mu} = J^{\mu} \, ,$$

implies that the 4-current density satisfies

$$J^{\mu}{}_{,\mu} = 0 \, .$$

Interpret this result physically.

**(ii)** Consider a static metric

$$ds^2 = e^{-2\phi/c^2} dt^2 - g_{ij} dx^i dx^j$$

where $g_{ij}$ and $\phi$ are independent of $t$.

Let $\xi^{\alpha}$ be the vector field $(1, \mathbf{0})$. Derive Killing's equation,

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \, ,$$

by evaluating the relevant connections.

For a freely falling particle with 4-velocity $U^{\alpha}$, show that $\xi_{\alpha} U^{\alpha}$ is conserved along the worldline.

Explain this result physically.

[**CONTINUED...**

**2Y Astrophysical Fluid Dynamics**

**(i)** Show that, for a spherical system in hydrostatic equilibrium under its own gravity, the pressure $p(r)$ and density $\rho(r)$ satisfy the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho. \qquad (*)$$

Estimate the temperature of the gas in a cluster of galaxies of mass $M = 2\times 10^{14} M_\odot$ and radius $R = 1\text{Mpc}$.

**(ii)** A spherical star has a polytropic equation of state

$$p = K\rho^{1+1/n},$$

where $K$ and $n$ are constants. The dimensionless variables $\theta$ and $\xi$ are defined by

$$\rho = \rho_c \theta^n, \qquad r = a\xi,$$

where $\rho_c$ is the central density of the star and $a$ is a constant. Show from (*) that in hydrostatic equilibrium, the variables $\theta$ and $\xi$ must satisfy the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \qquad (**)$$

Derive an expression for the constant $a$.

By transforming to the variable $y = \xi\theta$, or otherwise, find the solution of $(**)$ for the special case $n = 1$.

Show that for $n = 1$, the radius of the star is given by

$$R = \left(\frac{K\pi}{2G}\right)^{1/2}.$$

[**TURN OVER**

**3X Statistical Physics**

**(i)** Show that the photon flux, $F$, and number density of photons, $n$, are related by

$$F = \frac{1}{4} n c \,.$$

Hence, assuming the form of the Planck spectrum, show that the mean energy per photon for radiation in a cavity at temperature $T$ is

$$\bar{\varepsilon} = kT \left[ \frac{\int_0^\infty \frac{x^4 dx}{e^x - 1}}{\int_0^\infty \frac{x^3 dx}{e^x - 1}} \right] . \qquad (*)$$

**(ii)** Show that the pressure due to a photon gas is related to the energy density by

$$P = \frac{1}{3} u \,.$$

Assume that the internal energy of radiation in thermal equilibrium with the walls of a cavity with the volume $V$ and temperature $T$ is given by

$$U = aT^4 V \,.$$

Derive the relationship between $T$ and $V$ for reversible adiabatic changes of the system.

Show, using $(*)$ or otherwise, that the number of photons in the cavity is invariant in the case of reversible adiabatic changes.

Assume that the cavity contains a trace of gas. How does the ratio of gas pressure to radiation pressure vary during such changes?

[**End of Section A**]

[**CONTINUED...**

## SECTION B

### 4Z Structure and Evolution of Stars

**(i)** According to Heisenberg's uncertainty principle, the energy width of an absorption line, $\Delta E$, formed in the photosphere of a star satisfies $\Delta E \Delta t \sim \hbar$, where $\Delta t$ is the lifetime of the electron in a particular energy state prior to the radiative transition. Assume that the lifetime is set by the mean time between interatomic collisions with cross section $\sigma$. Derive an approximate condition for the number density, $n$, and temperature, $T$, of the photosphere for which the line width due to this mechanism exceeds the thermal width of a line of frequency $\nu_0$. Hence provide a qualitative explanation for the different line widths of giants and main sequence stars of given effective temperature.

**(ii)** A population of homogeneous, radiative stars shares the same chemical composition. The opacity $K$ is given by $K = K_0 Z(1+X)\rho T^{-3.5}$ and the energy generation rate per unit mass is given by $\varepsilon = \varepsilon_0 X^2 \rho T^4$ where $X$ is the mass fraction of hydrogen and $Z$ the mass fraction of heavy elements, $\rho$ is the density and $T$ is the temperature within the star. $\varepsilon_0$ and $K_0$ are constants. Gas pressure is dominant and the mean molecular weight $N$ is equal to $4/(3+5X)$. For stars of fixed mass, use homology arguments to show that the radius varies with $X$ and $Z$ according to:

$$R \propto X^{4/13}(1+X)^{2/13}(3+5X)^{7/13}Z^{2/13}\,.$$

Assuming without proof that the corresponding scaling expression for luminosity is

$$L \propto X^{-2/13}(1+X)^{-14/13}(3+5X)^{-101/13}Z^{-14/13} \tag{1}$$

sketch on a Hertzsprung-Russell diagram how the position of a star of fixed mass and fixed mass fraction of heavy elements, $Z$, varies with helium content.

**[TURN OVER**

**5X Physical Cosmology**

**(i)** A galaxy spectrum shows two emission lines at wavelength 1.08 $\mu$m and 5.832 $\mu$m respectively. Assume that the lines belong to the Lyman and Balmer series of hydrogen, respectively. What is the redshift of the galaxy?

[The rest frame wavelength of the Lyman and Balmer series of hydrogen can be written as

$$\lambda_{\rm Lyman} = 9.1176 \left(\frac{\rm n^2}{\rm n^2-1}\right).10^{-8}{\rm m} \quad {\rm n} = 2, 3, 4, ... \ ,$$

$$\lambda_{\rm Balmer} = 9.1176 \left(\frac{\rm 4n^2}{\rm n^2-4}\right).10^{-8}{\rm m} \quad {\rm n} = 3, 4, 5, ...]$$

**(ii)** Define what is meant by the angular diameter distance $d_\theta$.

Assume a flat ($k = 0$) Friedmann-Robertson-Walker Universe with pressureless matter of density $\rho_{\rm mat}$ and a cosmological constant $\Lambda$. Derive the dependence of $d_\theta$ on $z$, $\Omega_{\rm mat,0}$ $\Omega_{\Lambda,0}$ and $H_0$ where $\Omega_{\rm mat,0}$ $\Omega_{\Lambda,0}$, $H_0$ are the present-day values of $\Omega_{\rm mat} = \frac{8\pi G\rho_{\rm mat}}{3H^2}$, $\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$ and the Hubble parameter $H = \frac{\dot{R}}{R}$.

What is the angular size of a galaxy with proper size 10 kpc at $z = 3$ if $\Omega_{\rm mat,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$ and $H_0 = 70$ kms$^{-1}$ Mpc$^{-1}$?

[You may assume that $\int_0^3 \frac{1}{0.3(1+z')^3+0.7}dz' \approx 0.884$.]

**[CONTINUED...**

**6Y Stellar Dynamics & Structure of Galaxies**

**(i)** A binary system consists of two stars of masses $m_1$ and $m_2$ in circular orbits with a separation $R$. Show that the angular frequency $\Omega$ in an inertial frame is given by

$$\Omega^2 R^3 = G(m_1 + m_2)\,.$$

**(ii)** For the system in part (i), show that if a particle of negligible mass is to experience a net acceleration due to the gravitational field of the two stars directed towards the centre of mass of the system, then the particle must be equidistant from the two stars.

A particle is placed equidistant between the two stars along the line connecting the two stars and with a velocity that is perpendicular to the line connecting the two stars. Show that if this velocity is chosen such that the angular velocity about the centre of mass is $2\sqrt{2}\Omega$ (where $\Omega$ is the angular frequency given in part (i)) then its centripetal acceleration exactly matches its gravitational acceleration towards the centre of mass.

Discuss whether the particle would remain in a circular orbit about the centre of mass.



[TURN OVER

**7Z Topics**

**(i)** An $O$ star producing $10^{49}$ ionising photons per second is formed in the centre of a spherical cloud of radius 1 pc and hydrogen number density $10^9 \mathrm{m}^{-3}$. Estimate the minimum time required to ionise the whole cloud.

Explain why in practice cloud ionisation will take longer than this.

Show that in the case of a very low density uniform cloud, the radius of the ionised region, $R_{\rm ion}$, grows with time according to $R_{\rm ion} \propto t^{1/3}$.

**(ii)** Stars of mass $m$ travel perpendicularly, with velocity $V$, through a gas disc in the nucleus of a galaxy. Show that gas located at a perpendicular distance $x$ from the flight path of the centre of the star will be strongly perturbed by the encounter if $x \lesssim x_{crit} = \frac{Gm}{V^2}$.

Assume that the distance $x_{\rm crit}$ defines a cylindrical volume of gas which strongly interacts with the star. Estimate the mass of gas with which a white dwarf ($M = 1M_\odot,\ R = 5 \times 10^6$m) interacts each time it crosses a disc of thickness 0.1 pc, density $10^{-13}$ kg m$^{-3}$, and velocity $V = 200$ kms$^{-1}$.

Estimate the number of orbits over which the white dwarf's orbit is significantly affected by the gas disc.

Explain why the number of orbits over which the star's orbit is perturbed is different in the case of a red giant star ($M = 1M_\odot,\ R = 10^{10}$m).

**END OF PAPER**

## NATURAL SCIENCES TRIPOS Part II

Wednesday 7 June 2006 09.00 – 12.00

ASTROPHYSICS - PAPER 2

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**1X Relativity**

**(i)** Show that if a vector **S** is parallel-transported along an affine-parametrised geodesic with tangent vector **T**, then the angle between **S** and **T** is constant along the curve.

**(ii)** Let events in Newtonian spacetime be parametrised by coordinates $(t, x^i)$. Show that the equations of motion of a freely falling particle are

$$
\begin{aligned}
\frac{d^2t}{du^2} &= 0 \\
\frac{d^2x^i}{du^2} &+ \frac{\partial\phi}{\partial x^i}\left[\frac{dt}{du}\right]^2 = 0
\end{aligned}
$$

for a suitable parameter $u$, where $\phi(x^i)$ is the Newtonian potential.

By comparison with the geodesic equation, show that the connection has components

$$
\begin{aligned}
\Gamma^i_{oo} &= \frac{\partial\phi}{\partial x^i} \\
\Gamma^a_{bc} &= 0 \quad \text{otherwise}\,.
\end{aligned}
$$

Hence show that

$$R^i_{ojo} = -R^i_{ooj}\,,$$

where $R^i_{jkl}$ is the Riemann tensor.

Now suppose that connection is a metric connection. Deduce that

$$R_{iojo} \neq R_{oijo}\,,$$

and explain the significance of this result.

[**CONTINUED...**

**2Y Astrophysical Fluid Dynamics**

**(i)** Write down the equations governing the evolution of the perturbed quantities $\Delta\rho$ and $\Delta u_x$ (to first order in perturbed quantities) in the case of an initially static, uniform medium (density $\rho_0$) where the perturbations obey a barotropic equation of state ($p = p(\rho)$) and where $\frac{dp}{d\rho} \equiv c_s^2$.

Show that these equations are satisfied by perturbations of the form:

$$\begin{aligned} \Delta\rho &= f_1(x - c_s t) + f_2(x + c_s t) \\ \Delta u_x &= \frac{c_s}{\rho_0}[f_1(x - c_s t) - f_2(x + c_s t)], \end{aligned}$$

and hence derive expressions for $f_1$ and $f_2$ in terms of the form of the perturbation ($\Delta\rho(x, 0)$; $\Delta u_x(x, 0)$) at $t = 0$.

The initial perturbation of a uniform static system (density $\rho_o = 1$) consists of a 1% boost to the density in the region $-1 < x < 1$, with the fluid remaining at zero velocity. Sketch profiles of the perturbed quantities $\Delta\rho$ and $\Delta u$ as a function of $x$ (being as quantitative as you can) a) at $t = 0$ and b) at $t = 10$ for the case that $c_s = 1$.

**(ii)** Explain physically why shocks may form in supersonic fluid flows.

Starting from the Rankine-Hugoniot relations, show that the post-shock to pre-shock density ratio across an adiabatic shock is given by

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{(\gamma + 1)p_1 + (\gamma - 1)p_2}$$

where $\gamma$ is the ratio of specific heats (assumed constant across the shock) and $p_2$ and $p_1$ are the post-shock and pre-shock pressures respectively.

Show also that the ratio of pressures across the shock is given by

$$\frac{p_2}{p_1} = \frac{(\gamma - 1) + 2M^2}{(\gamma + 1)}.$$

where $M$ is the pre-shock Mach number of the flow. The relationship between pressure and density in the pre-shock and post-shock region is given respectively by $p_1 = K_1\rho_1^\gamma$ and $p_2 = K_2\rho_2^\gamma$. Explain why the ratio $K_2/K_1$ is greater than 1 and why this is compatible with the shock being adiabatic.

[TURN OVER

**3X Statistical Physics**

**(i)** Particles of mass $m$, in thermal equilibrium at temperature $T$ are subject to gravitational acceleration in the $z$ direction of the form $g_z = -w^2 z$ where $w$ is constant.

Determine how

a) the velocity of an individual particle

b) the rms velocity in the z direction of the particle distribution

and

c) the particle number density

varies with $z$.

Briefly explain how the answers to a) and b) can be reconciled with each other.

**(ii)** Particles in thermal equilibrium at temperature $T$ move in one dimension between a pair of barriers with separation $L$. Determine the specific heat capacities at constant volume, $C_V$, and constant pressure $C_p$ for such a gas.

Assume that if $L$ is slowly varied so that the system undergoes a reversible adiabatic change. Calculate how $T$ depends on $L$.

Produce a phase space diagram (position versus velocity) for the case that the barrier separation is $L_0$, and the temperature is $T_0$, by sketching the position of a number of particles so as to indicate the thermal equilibrium distribution in phase space. Provide indicative values on the axes.

The system is adiabatically compressed so that the barrier separation is $\lambda L_0$. Using the result derived above for the dependence of $T$ on $L$, construct a new phase space diagram with appropriately labelled axes.

How has the density of points in phase space changed as a result of this adiabatic compression?

[**End of Section A**]

**[CONTINUED ...**

## SECTION B

### 4Z Structure and Evolution of Stars

**(i)** Show that under suitable approximations (which you should state) that the mean molecular mass in the stellar interior is given by

$$N = \frac{4}{(3+5X)} \,,$$

where $X$ is the hydrogen mass fraction.

Derive an approximate expression for the relation between the hydrogen mass fraction and the number of free electrons per nucleon within a stellar interior.

**(ii)** In a star of mass $M$, the density as a function of radius $r$ is

$$\rho = \rho_c \left[ 1 - \left( \frac{r}{R} \right)^2 \right]$$

where $\rho_c$ is the central density and $R$ is the radius of the star.

Show that $\rho_c = \frac{15M}{8\pi R^3}$ and the central pressure, $P_c$, is given by $P_c = \frac{15GM^2}{16\pi R^4}$.

Hence if the star obeys an ideal gas equation of state determine how the central temperature depends on $M$ and $R$.

Explain why the role of selection degeneracy pressure imposes a minimum mass on hydrogen burning stars. Given that the equation of state of a fully degenerate gas is given by $P_{\rm deg} = 10^7 \left( \frac{\rho_c}{\rm kg\,m^{-3}} \right)^{5/3} N{\rm m}^{-2}$ and that the central temperature required for hydrogen burning is $4 \times 10^6 K$, estimate the minimum mass of a hydrogen burning star.

**[TURN OVER**

**5X Physical Cosmology**

**(i)** Describe what are meant by the terms "recombination" and "surface of last scattering" in the context of the cosmic microwave background. You may find it helpful to provide a sketch of the last scattering surface.

**(ii)** Assume that the epoch of last scattering of the cosmic microwave background occurred when the universe had a temperature of 3000K.

Use the Friedman equation to calculate the Hubble constant at the epoch of last scattering.

Calculate the age of the universe at the epoch of last scattering and compare it to the age of the universe at the present time. You may neglect the contribution of radiation to the energy density. Carefully justify all other assumptions you make.

**6Y Stellar Dynamics & Structure of Galaxies**

**(i)** A spherical cluster of stars has a radial luminosity density $j(r)$, where $r$ is the radial distance from the centre.

Show that the surface brightness $I(R)$ is given by

$$I(R) = 2\int_R^\infty \frac{j(r)r dr}{\sqrt{r^2 - R^2}}\,, \qquad (*)$$

where $R$ is the radial distance projected on to the plane of the sky.

A hypothetical system has $j(r) = j_0$ for $r \le r_0$

$$= 0 \text{ for } r > r_0\,.$$

Show that $I(R)$ falls to 1/2 its central value when

$$R = \frac{\sqrt{3}}{2} r_0\,.$$

**(ii)** Show that $I(R)$ and $j(r)$ as defined in part (i) satisfy

$$j(r) = -\frac{1}{2\pi r}\frac{d}{dr}\int_r^\infty \frac{I(R)RdR}{\sqrt{R^2 - r^2}}\,.$$

**[CONTINUED...**

Show how this should be modified if the cluster is tidally stripped and therefore has a maximum radius $r_0$.

[You may assume that $\int_a^b \frac{dt}{(b-t)^{1/2}(t-a)^{1/2}} = \pi$ ]



**7Z Topics**

**(i)** Two images of a gravitationally lensed object are separated by $3''$. Image A shows a sudden increase in brightness. Image B shows the same sudden increase in brightness 400 days later. Estimate the mass and approximate distance, $D$, of the lensing object. What type of object is it likely to be?

[Gravitational lensing deflects light by an angle $\theta_E \approx \left(\frac{4GM}{c^2 D}\right)^{\frac{1}{2}}$ where $M$ is the mass of the lens.]

**(ii)** A star in the Large Magellanic Cloud (LMC), distance 50kpc, is lensed by a compact object of mass $0.1M_{\odot}$ located halfway to the LMC and moving with the tangential velocity of $v = 200$ kms$^{-1}$. Estimate the duration of the lensing event and show that for a uniform mass density $\rho$ of lensing objects in the halo of the Galaxy, the "optical depth" for lensing (i.e. the chance that the line of sight to the star is within $\theta_E$ of a lens) is $\tau \sim G\rho D^2/c^2$. Why is there no dependence on the mass of the lensing objects?

**END OF PAPER**

**NATURAL SCIENCES TRIPOS Part II**

Thursday 8 June 2006 13.30 – 16.30

ASTROPHYSICS - PAPER 3

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## SECTION A

### 1X Relativity

**(i)** The energy momentum tensor of a perfect fluid with pressure $p$ and 4-velocity $u^a$ is

$$T^{ab} = \left(\frac{p}{c^2} + \rho\right) u^a u^b - p g^{ab} .$$

Let $h^{ab}$ be the tensor

$$h^{ab} = g^{ab} - u^a u^b ,$$

where $g^{ab}$ is the metric tensor.

Show that $h^{ab} u_b = 0$ and interpret this result geometrically.

Hence derive the relativistic Euler equation

$$\left(\frac{p}{c^2} + \rho\right) c^2 u^a{}_{;b}\, u^b = h^{ab} p_{,b} .$$

**(ii)** The Schwarzschild metric is

$$\begin{aligned} ds^2 = c^2 \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 \quad &- \quad r^2 d\theta^2 \\ &- \quad r^2 \sin^2\theta d\phi^2 . \end{aligned}$$

Show that a particle starting at rest at a radial distance $r = r_0$ has motion described by

$$\begin{aligned} \frac{dr}{d\tau} &= -\sqrt{\frac{2GM}{r}} \\ \frac{dr}{dt} &= -\left(1 - \frac{2GM}{rc^2}\right)\sqrt{\frac{2GM}{r}} , \end{aligned}$$

where $\tau$ is the time measured by a clock at rest with respect to the particle and $t$ is the time measured by an observer at infinity.

Hence show that

(i) the particle takes a finite time to reach the singularity at $r = 0$

(ii) an observer at infinity sees the particle take an infinite time to reach $r = 2GM/c^2$.

[**CONTINUED...**

## 2Y Astrophysical Fluid Dynamics

**(i)** Show that the magnetic field $\mathbf{B}$ in a fluid of conductivity $\sigma$ and non-relativistic flow velocity $\mathbf{v}$ satisfies the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}. \qquad (*)$$

Give a physical interpretation of the two terms on the right hand side of $(*)$.

[You may assume $\mathbf{j} = \sigma (\mathbf{E} + \mathbf{u} \wedge \mathbf{B})$ and any relevant formulae on the formulae sheet.]

**(ii)** Consider an axially symmetric fluid of infinite conductivity rotating with angular velocity $\Omega(r, z)\hat{\mathbf{z}}$, where $R, \phi, z$ define a cylindrical coordinate system. The fluid is threaded by a steady poloidal magnetic field with components $(B_R,\ 0,\ B_z)$. Show that the magnetic field can be written in terms of a scalar function $\psi(R, z)$,

$$B_R = -\frac{1}{R}\frac{\partial \psi}{\partial z}, \quad B_z = \frac{1}{R}\frac{\partial \psi}{\partial R}.$$

Use the induction equation (*) to show that the function $\psi(R, z)$ must satisfy

$$\frac{\partial \Omega}{\partial z}\frac{\partial \psi}{\partial r} = \frac{\partial \Omega}{\partial r}\frac{\partial \psi}{\partial z}.$$

Show that this relation requires that $\Omega$ is a function of $\psi$ only.

Show that $\mathbf{B}.\nabla\Omega = 0$ and hence describe how the magnetic field structure relates to surfaces of constant $\Omega$.

[**TURN OVER**

### 3X Statistical Physics

**(i)** A diatomic molecule $X_2$ consists of two atoms, mass $m$, separated by a bond of length $a$, whose vibrational motion can be described as a simple harmonic oscillator with frequency $\omega$.

Determine a condition for the rotational modes of the molecule to be excited at lower temperature than the vibrational modes.

Sketch, for this case, the variation of a) $C_V$ and b) $\gamma$ with $T$. Indicate the temperatures separating each regime and the respective values of $C_V$ and $\gamma$.

Why can the contribution of rotation around the bond to the value of $C_V$ be neglected?

[ You may assume that the quantised energy levels for molecular vibration and rotation are respectively $E_n = \left(n + \frac{1}{2}\right)\hbar w$ and $E_J = \frac{J(J+1)}{2I}\hbar^2$ where $I$ is the moment of inertia ]

**(ii)** A sealed box contains a collection of particles, all of which travel a distance of exactly $\lambda$ before colliding with other particles (irrespective of whether the previous collision was with other particles or the box walls). A small patch of area $A$ on the inside of the box is centred at the origin of a spherical coordinate system $(r, \theta, \phi)$ where $\theta$ is measured with respect to the inward normal to the box wall. Show, clearly stating any assumptions, that for particles colliding at $(\lambda, \theta, \phi)$, the fraction of collisions involving particles whose last collision was with the patch is given by:

$$f_{\text{patch}} = \frac{A\cos\theta}{2\pi\lambda^2(1+\cos\theta)}\,.$$

The box is placed in an evacuated chamber and the patch is replaced by a hole of area $A$. Describe how the properties of the gas at $(\lambda, \theta, \phi)$ are modified by the presence of the hole, and how this depends on the value of $A$.

Assume that the flux of escaping particles is given by $F = \frac{1}{4}n\overline{v}$ (where $n$ is the number density and $\overline{v}$ the mean particle speed). Determine the timescale on which the gas density in the box falls by a factor of 10, for the case that the gas temperature, $T$, is 300K, the area of the hole is $1\text{mm}^2$ and the box volume is $1\text{m}^3$.

**[CONTINUED...**

**3X Statistical Physics** - continued

Assume that the box contains 1kg of nitrogen (molecular mass $m = 28$ atomic mass units, molecular diameter 1 nm). Discuss whether the expression $F = \frac{1}{4}n\overline{v}$ is likely to be a good approximation to the flux of escaping particles.

$\left[\text{You may assume without proof that } \overline{v} = \sqrt{\frac{8kT}{\pi m}}\right]$

[**End of Section A**]

**SECTION B**

**4Z Structure and Evolution of Stars**

**(i)** Explain why the fusion reaction

$${}^{12}_{6}C +^{12}_{6} C \rightarrow^{24}_{12} Mg + \gamma$$

is a) energetically possible and b) not a major energy source for stars on the main sequence.

**(ii)** Write an essay describing the later stages of the evolution of stars that lead to the creation of white dwarfs, neutron stars and black holes. Include estimates of the masses of the remnants and of the mass of the stars on the main sequence that produce each type of remnant.

What observational evidence is there for the existence of each type of remnant?

**5X Physical Cosmology**

**(i)** Explain what is meant by the term "particle horizon". Show that in a flat Universe which started to expand at $t = 0$ the (proper) horizon is

$$d_H(t) = R(t) \int_0^t \frac{cdt'}{R(t')} .$$

Deduce that, if $R(t) \propto t^\alpha$ with $\alpha > -1$, $d_H = \frac{1}{-\alpha+1}ct$.

**(ii)** Assume that electron-positron annihilation occurs when the Universe was 5s old.

What is the temperature of the Universe at this time?

Explain briefly why, at the time of electron-positron annihilation, the Universe can be treated as flat.

Find the proper size of the particle horizon at this time, and its corresponding size in the current universe.

[CONTINUED ...

**6Y Stellar Dynamics & Structure of Galaxies**

**(i)** Outline the 'proof' of the Jeans Theorem, which states that (i) any steady state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of the motion in a static potential, and that (ii) any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation.

**(ii)** Show that if the distribution function for a spherical system has the form

$$f(\epsilon, L) = f_0(\epsilon) L^{-1},$$

where $\epsilon$ is the relative energy, $L$ is the absolute value of the angular momentum and $f_0(\epsilon)$ is an arbitrary function, then at any point the velocity dispersions in spherical coordinates are related by

$$\overline{V_\theta^2} = \overline{V_\phi^2} = \frac{1}{2}\overline{V_r^2}.$$

**7Z Topics**

**(i)** Two percent of solar-type stars have "hot jupiters", i.e Jupiter-sized planetary companions in circular orbits of 0.05 AU radius. A transit is seen when the flux from the star is dimmed by the planet when it passes in front of it. By how much, in magnitudes, is the luminosity of the star decreased during the transit?

Calculate what fraction of solar-type stars would be expected to exhibit hot Jupiter transits, stating clearly any assumptions you make.

[ diameter of the sun = $1.4 \times 10^6$km

diameter of Jupiter = $1.4 \times 10^5$ km

1 AU = $1.5 \times 10^8$ km ]

**[TURN OVER**

**(ii)** An initially non rotating white dwarf, mass $M_0$, accretes material from a disc in Keplerian rotation at the white dwarf surface. Assuming that the mass-radius relation for a white dwarf is $M \propto R^{-3}$ show that when the white dwarf mass has grown to $M$, its angular velocity is given by

$$\frac{\Omega}{\Omega_K} = \frac{3}{4k^2}\left[1 - \left(\frac{M_0}{M}\right)^{4/3}\right] ,$$

where $\Omega_K$ is the Keplerian angular velocity of material at the surface of the white dwarf. The moment of inertia of the white dwarf is given by $I = k^2 M R^2$, where $k^2$ is a constant. State any additional assumptions in your derivation.

In a classical nova system, the white dwarf is observed to rotate with an angular velocity close to $\Omega_K$. Estimate $M_0$ for this system.

If the white dwarf is accreting matter at $10^{-8} M_\odot$ yr$^{-1}$, estimate for how long it has been accreting. [You may assume $k^2 \sim 0.6$ for a white dwarf.]

**END OF PAPER**

# NATURAL SCIENCES TRIPOS Part II

Friday 9 June 2006 09.00 – 12.00

ASTROPHYSICS - PAPER 4

**Before you begin read these instructions carefully.**

*The paper is divided into Section* **A** *and Section* **B**. *Candidates may attempt Parts from* **ALL** *questions in Section* **A** *and from not more than* **THREE** *questions in Section* **B**.

*Each question is divided into Part* **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts of questions.*

*The number of marks for each question is the same, with Part* **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

*Write on* **one side** *of the paper only and* **begin each answer on a separate sheet.**

*Answers must be tied up in separate bundles, marked* **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X, 5X** *should be in one bundle and* **2Y, 6Y** *in another bundle.)*

*A master cover sheet listing all Parts of all questions attempted must also be completed.*

**It is essential that every cover sheet bear the candidate's examination number and desk number.**

| STATIONERY REQUIREMENTS | SPECIAL REQUIREMENTS |
|---|---|
| *Script Paper* | *Formulae Booklet* |
| *Blue Cover Sheets* | *Approved Calculators Allowed* |
| *Yellow Master Cover Sheets* | |
| *1 Rough Work Pad* | |
| *Tags* | |

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

## SECTION A

### 1X Relativity

**(i)** Consider a metric of the form

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \qquad (*)$$

where $a(t)$ is a scale-factor and $k = 0,\ 1$ or $-1$ is a constant.

By transforming from $(t, r, \theta, \phi)$ coordinates to $(\eta, \chi, \theta, \phi)$ coordinates, where $\eta$ and $\chi$ are to be found, show that the metric can be cast into the form

$$ds^2 = b^2(\eta)[d\eta^2 - d\chi^2 - f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]$$

for suitable choices of $b(\eta)$ and $f(\chi)$.

**(ii)** Consider a universe described by a metric of the form $(*)$. The energy density $\rho$ and pressure $p$ of the universe are related to the scale factor $a$ and the constant $k$ through

$$\begin{aligned} \frac{\dot{a}^2 + k}{a^2} &= \frac{8\pi G\rho}{3} \\ \frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} &= -8\pi G\frac{p}{c^2} \end{aligned} \qquad (**)$$

Show that if $(\rho c^2 + 3p) > 0$, then the Universe must have had a singular beginning at some time $t_{\text{sing}}$ such that

$$t_o - t_{\text{sing}} < \left(\frac{a}{\dot{a}}\right)_{t_o}$$

where $t_o$ is the present time.

Show that the equations $(**)$ can be rewritten as

$$c^2\frac{d}{dt}(\rho a^3) = -p\frac{da^3}{dt}\,.$$

How does $\rho$ vary with $a$ if the equation of state has the form $p = w\rho c^2$?

Find how $a$ varies with $t$ if $k = 0$.

[**CONTINUED...**

**2Y Astrophysical Fluid Dynamics**

**(i)** The energy per unit volume of a fluid of density $\rho$ is

$$E = \rho\left(\frac{1}{2}v^2 + \Phi + \varepsilon\right),$$

where $v$ is the velocity of the fluid, $\Phi$ is the gravitational potential, and $\varepsilon$ is the internal energy per unit mass. Show that $E$ (for an ideal fluid with pressure $p$) satisfies the equation,

$$\frac{\partial E}{\partial t} + \nabla.((E+p)v) = -\rho\dot{Q} + \rho\frac{\partial \Phi}{\partial t},$$

where $\dot{Q}$ is the coding rate minus the heating rate per unit mass.

**(ii)** Explain without detailed calculation why the condition for thermal instability is of the form

$$\left.\left(\frac{\partial \dot{Q}}{\partial T}\right)\right|_p < 0,$$

where $\dot{Q}$ is the cooling rate minus the heating rate per unit mass.

The net cooling rate for a gas depends on its temperature and pressure according to

$$\dot{Q} = Ap\left[\left(\frac{T}{T_1} - 4\right)^3 - 3\left(\frac{T}{T_1} - 4\right) + 52\right] - B,$$

where $A$, $B$ and $T_1$ are constants. Determine the pressure range over which there are three phases in thermal equilibrium.

Determine the temperatures of these three phases if

$$p = \frac{B}{52A}.$$

Which of these phases, if any, is thermally stable?

[**TURN OVER**

**3X Statistical Physics**

**(i)** Monatomic hydrogen gas of density $10^{12}$ particles m$^{-3}$ and temperature 10K collapses to form a star. By using a plausible estimate for the temperature and density within the star, show that the gas cannot undergo this change through reversible adiabatic compression. Suggest a mechanism by which the necessary change in entropy can be achieved.

**(ii)** Outline the argument why a zero temperature gas of $N$ fermions, contained in volume $V$ occupies energy states up to a maximum value

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} .$$

Show that the mean energy per fermion is $\frac{3}{5}E_F$. Explain without detailed calculation why the energy of a Fermi gas at low temperature is of the form

$$E = \frac{3}{5}NE_F + AN\left(\frac{kT}{E_F}\right)kT\,,$$

where $A$ is a constant of order unity.

The volume occupied by a zero-temperature thermally-isolated Fermi gas with energy $E_0$ is slowly increased from $V$ to $V(1+\varepsilon)$ (where $\varepsilon \ll 1$). The gas is then heated at constant volume until its energy is restored to its original value ($E_0$). Show that its temperature at this point is given by

$$kT \sim \left(\frac{2\varepsilon}{5A}\right)^{1/2} E_F\,.$$

Explain, without detailed calculation, the sign of the entropy changes of the gas during the two phases of this experiment.

[**End of Section A**]

[**CONTINUED...**

## SECTION B

### 4Z Structure and Evolution of Stars

**(i)** The extinction due to the presence of interstellar dust in the galaxy is given by $A_\lambda = 0.8\lambda^{-1}$mag pc$^{-1}$ where $\lambda$ is the wavelength of light in nm. Calculate the extinction experienced by light from a star located 5kpc away in the U, B and V bands. [The effective wavelengths of U, B and V bands may be taken as 365, 430 and 560 nm respectively]

The effect of interstellar extinction is to shift the positions of stars in a U-B versus B-V two colour diagram. Sketch the shift in the U-B, B-V two colour plane, indicating its slope and clearly indicating the direction of increasing reddening. Discuss under what circumstances the position of a star in the U-B, B-V two colour plane can be used to infer the colour that the star would have in the absence of extinction?

**(ii)** A gas cloud collapses to form a star cluster. The number of stars formed with masses in the range $(M, M + dM)$ is given by

$$dN = \psi(M)dM$$

where $\psi(M) \propto M^{-2.35}$. As stars leave the main sequence, those with mass $M > M_{SN}$ return 90% of their mass to the interstellar medium in a supernova explosion. Stars with mass $M < M_{\rm SN}$ produce white dwarfs with mass $M_{\rm WD} = 0.6M_\odot$, returning the remainder of their main sequence mass to the interstellar medium.

Derive an expression for the fraction, $f_{\rm ret}$, of the total mass of the cluster returned to the interstellar medium when the main sequence turn off occurs at a mass of $1M_\odot$.

Choose suitable values for the maximum and minimum stellar mass in the cluster, and the mass, $M_{\rm SN}$, above which stars experience a supernova explosion.

Provide a numerical estimate of $f_{\rm ret}$. Explain why this answer is relatively insensitive to the value of $M_{\rm MAX}$, the maximum stellar mass.

**[TURN OVER**

**5X Physical Cosmology**

**(i)** The density contrast of a fluctuation $\delta(t)$ in a matter-dominated universe evolves according to the equation

$$\ddot{\delta} + 2\frac{\dot{R}}{R}\dot{\delta} - 4\pi G\rho\delta = 0\,, \qquad (*)$$

where $R(t)$ is the scale factor, $\rho$ is the mean density and $G$ is the gravitational constant. Explain the assumptions involved in deriving this equation, and the physical meaning of each term.

Solve this equation for the case of a static universe.

**(ii)** Find the solution of equation $(*)$ in the Einstein-de Sitter $(\Lambda = 0, \Omega = 1)$ case.

If a fluctuation had $\delta = 10^{-4}$ and $\dot{\delta} = 0$ when the universe was 1 million years old, how large is the fluctuation (in linear theory) when the universe is $10^{10}$ years old?



[**CONTINUED...**

**6Y Stellar Dynamics & Structure of Galaxies**

**(i)** Show that the crossing time for a typical star in a cluster of $N$ stars each of mass $m$ which has radius $R$ is

$$t_{\text{cross}} \simeq \sqrt{\frac{R^3}{GNm}}\,.$$

A star crossing the cluster suffers perturbations in velocity of $\Delta v_\perp \simeq \frac{Gm}{b^2}\frac{2b}{v}$ perpendicular to its motion when it passes a star with velocity $v$ at impact parameter $b$. Show that the collisionless approximation breaks down on a timescale

$$t_{\text{relax}} \simeq \frac{N}{16 \ln N} t_{\text{cross}}.$$

**(ii)** Using the results of part (i), estimate the relaxation timescale for a globular cluster with radius 50pc consisting of $10^6$ stars each of $0.6 M_\odot$.

What is the relaxation time for the cluster core, which contains $10^4$ stars within a 1.5pc radius?

How do these timescales compare with the ages of globular clusters in our galaxy?

Describe the evolution of a globular cluster through this and other processes.

**[TURN OVER**

**7Z Topics**

**(i)** Suppose the space density of quasars with luminosity $> L$ evolves with redshift $z$ according to $n \propto L^{-2}\,(1+z)^6$. Assuming that space is Euclidean, calculate how the number density of quasars per unit redshift selected to be brighter than a fixed apparent magnitude depends on redshift.

**(ii)** A galaxy with initial gas mass $M_g$ forms a supermassive black hole of mass $M_{bh}$ in its nucleus by accretion of gas. The typical binding energy per unit mass of gas in the galaxy is $\frac{1}{2}V^2$. Show that the energy liberated by accretion on to the black hole is enough to unbind the residual gas in the galaxy when the hole has grown to a mass $M_{bh} = M_{\rm unb}$ where

$$M_{\rm unb} = \frac{M_g}{\left(1 + \frac{2\epsilon c^2}{V^2}\right)}$$

[$\epsilon$ is the efficiency with which rest mass energy is converted into radiation in the case of accretion onto a black hole].

Estimate $M_{\rm unb}$ for a typical large galaxy and compare it with typical estimates for supermassive black hole masses in galactic nuclei. Suggest two reasons why in practice supermassive black holes can grow to masses greater than $M_{\rm unb}$.

**END OF PAPER**