PAPER 1.

Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
RELATIVITY		1
Examiner	• In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen. • Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness. • Write on this side only and between the margins. • Not more than one solution per sheet please.	**Paper** 1 / 2 / 3 / 4
Lecturer CHALLINOR		**Section** I / II

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(ii) $h\nu \rightsquigarrow$ $\qquad\qquad$ $\overset{e^-}{\longleftarrow}$

$$p^\mu = \frac{h\nu}{c}(1,1,0,0) \qquad q^\mu = \left(\frac{E}{c}, -p, 0, 0\right)$$

After scattering, have 

$\bar p^\mu = \frac{h\bar\nu}{c}(1,\cos\theta,\sin\theta,0)$

Conservation of 4-momentum $\Rightarrow p^\mu + q^\mu = \bar p^\mu + \bar q^\mu$.

final electron

Follows that $\bar q^\mu = (p^\mu + q^\mu - \bar p^\mu)$ and so, using $\eta_{\mu\nu}\bar q^\mu \bar q^\nu = (m_e c)^2$, we have

$$(m_e c)^2 = \underbrace{\eta_{\mu\nu}p^\mu p^\nu}_{0} + \underbrace{\eta_{\mu\nu}\bar p^\mu \bar p^\nu}_{0} + (m_e c)^2 + 2\eta_{\mu\nu}p^\mu q^\nu$$
$$-2\eta_{\mu\nu}p^\mu \bar p^\nu - 2\eta_{\mu\nu}q^\mu \bar p^\nu$$

[8] $$\Rightarrow \frac{h\nu}{c}\left(\frac{E}{c}+p\right) - \left(\frac{h\nu}{c}\right)\left(\frac{h\bar\nu}{c}\right)(1-\cos\theta) - \frac{h\bar\nu}{c}\left(\frac{E}{c}+p\cos\theta\right) = 0$$

$$\Rightarrow \nu(E+pc) = \bar\nu\left(E + pc\cos\theta + h\nu(1-\cos\theta)\right)$$

$$\Rightarrow \bar\nu = \frac{\nu(E+pc)}{E + pc\cos\theta + h\nu(1-\cos\theta)}$$

Comments

SIMILAR TO COMPTON SCATTERING EXAMPLE IN NOTES

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$\bar{\nu}$ maximised when $E + pc\cos\theta + h\nu(1-\cos\theta)$ minimised.

For $pc > h\nu$, this is at $\underline{\theta = \pi}$.

[2] We then have $\bar{\nu}_{max} = \dfrac{\nu(E+pc)}{[(E-pc)+2h\nu]}$.

Taking $h\nu \sim k_B T$, have $\nu \sim 2.08 \times 10^{15}$ Hz (i.e., UV) and $h\nu \approx 8.62$ eV. $E = 0.71$ MeV and so $pc = 0.49$ MeV ($\gg h\nu$). Follows that $\bar{\nu}_{max} \sim \nu\left(\dfrac{E+pc}{E-pc}\right) \sim 5.56\,\nu$

[4] $\bar{\nu}_{max} \sim \underline{10^{16}\text{ Hz}}$.

With $E = \gamma m_e c^2$, have $\gamma \gg 1$ if $E \gg m_e c^2$ and $\beta \approx 1$

We have $E + pc = m_e c^2(\gamma + \gamma\beta) \approx 2\gamma m_e c^2$, and

$$E - pc = m_e c^2 \gamma (1-\beta).$$

Since $\gamma^{-2} = 1-\beta^2 = (1+\beta)(1-\beta) \approx 2(1-\beta)$, $E - pc \approx \dfrac{m_e c^2 \gamma}{2\gamma^2}$.

[6] Follows that $\dfrac{\bar{\nu}_{max}}{\nu} = \dfrac{2\gamma m_e c^2}{\frac{m_e c^2}{2\gamma} + 2h\nu} = \dfrac{4\gamma^2}{1 + \frac{4h\nu\gamma}{m_e c^2}}$.

In the limit $h\nu\gamma \ll m_e c^2$ (i.e., photon energy in e^- rest frame $\ll m_e c^2$), have $\bar{\nu}_{max} \approx 4\gamma^2 \nu$ [In this limit, no energy change of photon in rest frame of e^-.]

Comments

UNSEEN CALCULATION

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Paper I

2i) ρ_0 p_0 ν $\quad e^{i(kx-\omega t)}$: form of perturbations (AFD)

Show $\quad \omega^2 + \frac{4}{3}i\omega\nu k^2 - C_s^2 k^2 = 0$

Navier Stokes is

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \nu\left[\nabla^2\underline{u} + \frac{1}{3}\nabla(\nabla\cdot\underline{u})\right]$$

Continuity is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$$

Introduce small perturbations about a static uniform eq.

$\rho = \rho_0 + \delta\rho$

$p = p_0 + \delta p_0$

$\underline{u} = \delta\underline{u}$

Perturbations are isothermal so

$\delta p = C_s^2\,\delta\rho$, where C_s^2 is the isothermal sound speed

$$C_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_T = \left.\frac{\partial}{\partial \rho}\right|_T\left(\frac{\mathcal{R}_*\rho T}{\mu}\right)$$

$$\Rightarrow C_s^2 = \frac{\mathcal{R}_* T}{\mu}$$

2//

Linearised equations are:

$$\frac{\partial}{\partial t}\delta \underline{u} = -\frac{c_s^2}{\rho_0}\nabla \delta\rho + \nu\left[\nabla^2\delta\underline{u} + \frac{1}{3}\nabla(\nabla\cdot\delta\underline{u})\right] \qquad (1)$$

$$\frac{\partial}{\partial t}\delta\rho + \rho_0\nabla\cdot\delta\underline{u} = 0 \qquad (2)$$

$$\frac{\partial}{\partial t}(1) \Rightarrow \frac{\partial^2}{\partial t^2}\delta\underline{u} = \frac{-c_s^2}{\rho_0}\nabla\left(\frac{\partial}{\partial t}\delta\rho\right) + \nu\left[\nabla^2\frac{\partial}{\partial t}\delta\underline{u} + \frac{1}{3}\nabla\left(\nabla\cdot\frac{\partial}{\partial t}\delta\underline{u}\right)\right]$$

Sub for $\delta\rho$ from (2)

$$\frac{\partial^2\delta\underline{u}}{\partial t^2} - c_s^2\nabla^2\delta\underline{u} = \nu\left[\nabla^2\frac{\partial}{\partial t}\delta\underline{u} + \frac{1}{3}\nabla\left(\nabla\cdot\frac{\partial}{\partial t}\delta\underline{u}\right)\right] \qquad (3)$$

Assume plane wave solution:

$$\delta\underline{u} = \delta\underline{u}_1 e^{i(\underline{k}\cdot\underline{x} - \omega t)}$$

$$\therefore\ (-\omega^2 + c_s^2k^2)\delta\underline{u}_1 = \nu\left[i\omega k^2\delta\underline{u}_1 + \frac{i\omega}{3}\underline{k}(\underline{k}\cdot\delta\underline{u}_1)\right]$$

Dot through by $\underline{k}$:

$$(-\omega^2 + c_s^2k^2)(\underline{k}\cdot\delta\underline{u}_1) = i\omega\nu\left[k^2(\underline{k}\cdot\delta\underline{u}_1) + \frac{1}{3}k^2(\underline{k}\cdot\delta\underline{u}_1)\right]$$

Provided $\underline{k}$ is not $\perp$ to $\delta\underline{u}_1$, divide by $\underline{k}\cdot\delta\underline{u}_1$ to get

$$-\omega^2 + c_s^2 k^2 = i\omega\nu\frac{4}{3}k^2$$

$$\omega^2 + \frac{4}{3}i\omega\nu k^2 - c_s^2 k^2 = 0$$

5

Compute λ_{crit} in terms of ν and c_s

Solve quadratic:

$$\omega = \frac{1}{2}\left(-\frac{4}{3}i\nu k^2 \pm \left(-\frac{16}{9}\nu^2 k^4 + 4c_s^2 k^2\right)^{1/2}\right)$$

1

This describes propagating, but damped, modes provided it has a real part

$$\Rightarrow 4c_s^2 k^2 - \frac{16}{9}\nu^2 k^4 > 0$$

$$\Rightarrow \quad k^2 < \frac{9}{4}\frac{c_s^2}{\nu^2}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda} < \frac{3}{2}\frac{c_s}{\nu}$$

$$\therefore \lambda_{crit} = \frac{4}{3}\pi\frac{\nu}{c_s}$$

2

ii) Introduce perturbations into N-S equation in rotating frame and linearise

$$\frac{\partial}{\partial t}\delta\underline{u} + \underbrace{\cancel{(\delta\underline{u}\cdot\nabla)\delta\underline{u}}}_{2^{nd}\text{ order}} = -\frac{1}{\rho_0}\nabla\delta p + 2\delta\underline{u}\times\underline{\Omega} - \frac{1}{2}\nabla\left[(\underline{\Omega}\times\delta\underline{r})^2\right]$$

$$+ \nu\left[\nabla^2\delta\underline{u} + \frac{1}{3}\nabla(\nabla\cdot\delta\underline{u})\right]$$

Perturbations incompressible, so $\frac{1}{\rho_0}\nabla\delta p = \nabla\left(\delta p/\rho_0\right)$

Take curl of above and remove all gradient terms $(\nabla\times(\nabla f)=0)$

$$\frac{\partial}{\partial t}\nabla\times\delta\underline{u} = 2\nabla\times(\delta\underline{u}\times\underline{\Omega}) + \nu\nabla\times(\nabla^2\delta\underline{u})$$

$$\Rightarrow \frac{\partial\delta\underline{\omega}}{\partial t} = 2\left[\underbrace{\cancel{\delta\underline{u}(\nabla\cdot\underline{\Omega})}}_{\underline{\Omega}=\text{const.}} - \underbrace{\cancel{\underline{\Omega}(\nabla\cdot\delta\underline{u})}}_{\text{incompressible}} + (\underline{\Omega}\cdot\nabla)\delta\underline{u} - \underbrace{\cancel{(\delta\underline{u}\cdot\nabla)\underline{\Omega}}}_{\underline{\Omega}=\text{const}}\right]$$

$$+ \nu\nabla^2(\nabla\times\delta\underline{u})$$

$$\Rightarrow \frac{\partial\delta\underline{\omega}}{\partial t} = 2(\underline{\Omega}\cdot\nabla)\delta\underline{u} + \nu\nabla^2\delta\underline{\omega} \qquad (*)$$

5

Let $\nu=0$ and $\underline{\Omega}=\Omega\hat{\underline{z}}$. Then $(*)$ reads

$$\frac{\partial}{\partial t}\nabla\times\delta\underline{u} = 2\Omega\frac{\partial}{\partial z}\delta\underline{u}$$

Introduce plane wave perturbation: $\delta \underline{u} = \delta \underline{u}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

$$\Rightarrow \omega \underline{k} \times \delta \underline{u}_1 = 2\Omega k_z i \delta \underline{u}_1$$ ⊕

$$\Rightarrow \omega \underline{k} \times (\underline{k} \times \delta \underline{u}_1) = 2i\Omega k_z (\underline{k} \times \delta \underline{u}_1)$$

$$\Rightarrow \omega^2 [\underline{k}(\underline{k}\cdot\delta \underline{u}) - \delta \underline{u}_1 \cdot k^2] = 2i\Omega k_z (\omega \underline{k} \times \delta \underline{u}_1)$$

$$\Rightarrow \omega^2 [(\underline{k}\cdot\delta \underline{u}_1)\underline{k} - k^2 \delta \underline{u}_1] = -4\Omega^2 k_z^2 \delta \underline{u}_1$$

Dot with $\underline{k}$

$\Rightarrow$ LHS $= 0$

RHS $= -4\Omega^2 k_z^2 \underline{k}\cdot\delta \underline{u}_1 = 0$

2

$\Rightarrow$ waves are transverse

Cross with $\underline{k}$

$$\Rightarrow \omega^2 k^2 (\underline{k} \times \delta \underline{u}_1) = +4\Omega^2 k_z^2 (\underline{k} \times \delta \underline{u}_1)$$

$$\Rightarrow \omega^2 = +\frac{4\Omega^2 k_z^2}{k^2}$$

$$\omega = \pm 2\Omega \frac{|k_z|}{k}$$

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Now include non-zero ν

$$\frac{\partial}{\partial t} \nabla \times \delta \underline{u} = 2\Omega \frac{\partial}{\partial z} \delta \underline{u} + \nu \nabla^2 \delta \underline{\omega}$$

$$= 2\Omega \frac{\partial}{\partial z} \delta \underline{u} + \nu \nabla^2 (\nabla \times \delta \underline{u})$$

Plane waves:

$$\omega \underline{k} \times \delta \underline{u}_1 = 2i\Omega k_z \delta \underline{u}_1 + \nu \left(-k^2 i \underline{k} \times \delta \underline{u}_1\right)$$

$$= 2i\Omega k_z \delta \underline{u}_1 - i\nu k^2 \underline{k} \times \delta \underline{u}_1$$

$$\Rightarrow \left(\omega + i\nu k^2\right)\left(\underline{k} \times \delta \underline{u}_1\right) = 2i\Omega k_z \delta \underline{u}_1$$

This is exactly the same as $\oplus$, with $\omega \to \omega + i\nu k^2$

So dispersion relation is

$$\omega = -i\nu k^2 \pm 2\Omega \frac{|k_z|}{k}$$

4//

Interpretation:

— propagating transverse waves

— decay rate νk^2

1//

Paper I

Cosmology

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COSMOLOGY		3X
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Lecturer EFSTATHIOU		**Section** I / II

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(i) Consider radial null geodesics in $\phi = 0$ separated by $\Delta\theta$. Proper separation at comoving distance $\chi(z)$ is $R(t)\chi\,\Delta\theta$. Angular diameter distance is thus

[2]
$$D_A(z) = R(t)\,\chi(z)$$

We have $(1+z)^{-1} = R(t)/R_0$ and, since $c^2dt^2 = R^2 d\chi^2$,

$$\chi(z) = \int_{t(z)}^{t_0} \frac{c\,dt'}{R(t')} = \int_0^z c\,\frac{dt'}{dz'}\,\frac{1}{R(t')}\,dz'.$$

As $1+z = R_0/R$, $\frac{dz}{dt} = -\frac{R_0}{R^2}\frac{dR}{dt} = -(1+z)H(z)$, so

$$\chi(z) = \int_0^z \frac{c}{(1+z')H(z')}\,\frac{(1+z')}{R_0}\,dz' = \frac{1}{R_0}\int_0^z \frac{c\,dz'}{\cancel{(1+z')}H(z')}$$

Follows that

[4]
$$D_A(z) = \frac{c}{1+z}\int_0^z \frac{dz'}{H(z')}.$$

Let a source have luminosity L. It is observed to have flux S where

$$S = \frac{L}{4\pi D_L^2},$$

which defines $D_L(z)$.

If the source is at frequency ν_s, it emits $\frac{L}{h\nu_s}$ photons per unit time, so in time interval dt_s

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CHALLINOR		(1) / 2 / 3 / 4
Lecturer		Section
EFSTATHIOU		(I) / II

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Comments

it emits $\frac{L\, dt_s}{h\nu_s}$ photons. These are observed in a time interval dt_0 at frequency ν_0 where

$$\nu_0/\nu_s = dt_s/dt_0 = 1/(1+z)$$

The photons are spread over a sphere of proper radius $R_0 \chi(z)$ so flux S is also equal to

$$S = \underbrace{\frac{L\, dt_s}{h\nu_s}}_{\text{number}} \times \underbrace{h\nu_0}_{\text{observed energy}} \times \underbrace{\frac{1}{dt_0}}_{\text{per observer time interval}} \times \underbrace{\frac{1}{4\pi R_0^2 \chi^2(z)}}_{\text{area}},$$

i.e.,
$$\frac{L}{4\pi D_L^2(z)} = L\left(\frac{\nu_0}{\nu_s}\right)^2 \times \frac{1}{4\pi R_0^2 \chi^2(z)}$$

$$\Rightarrow D_L(z) = \left(\frac{\nu_s}{\nu_0}\right) R_0 \chi(z) = R_0(1+z)\chi(z)$$

$$= c(1+z)\int_0^z \frac{dz'}{H(z')}$$

[4]

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(ii) $q = -R\ddot{R}/\dot{R}^2$.

We have $\frac{d}{dt} = \frac{dz}{dt}\frac{d}{dz} = -(1+z)H(z)\frac{d}{dz}$,

and $\dot{R} = R\,H(z) = R_0\,H(z)/(1+z)$

Follows that $q = -\frac{\ddot{R}}{R}\frac{1}{H^2} = +\frac{R_0(1+z)H(z)}{RH^2}\frac{d}{dz}\left(\frac{H}{1+z}\right)$

$$= +\frac{(1+z)^2}{H}\frac{d}{dz}\left(\frac{H}{1+z}\right)$$

$$= +\frac{(1+z)^2}{H}\left(\frac{H'}{1+z} - \frac{H}{(1+z)^2}\right)$$

$$= +(1+z)\frac{H'}{H} - 1$$

$$= -1 + \frac{(1+z)}{2}\frac{(H^2)'}{H^2}$$

[5]

$$\ddot{R} = -(1+z)H(z)\frac{d}{dz}\left(R_0\frac{H}{1+z}\right) = -R_0\left(\frac{1}{2}(H^2)' - \frac{H^2}{(1+z)}\right),$$

$$\text{so } \dddot{R} = R_0(1+z)H(z)\frac{d}{dz}\left(\frac{1}{2}(H^2)' - \frac{H^2}{1+z}\right)$$

$$= R_0(1+z)H(z)\left(\frac{1}{2}(H^2)'' - \frac{(H^2)'}{1+z} + \frac{H^2}{(1+z)^2}\right),$$

and $\dot{q} = \frac{1}{R}\frac{1}{H^3}R_0(1+z)H\left(\frac{1}{2}(H^2)'' - \frac{(H^2)'}{1+z} + \frac{H^2}{(1+z)^2}\right)$

$$= \frac{(1+z)^2}{H^2}\times\left(\frac{1}{2}(H^2)'' - \frac{(H^2)'}{1+z} + \frac{H^2}{(1+z)^2}\right)$$

[5]

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Lecturer: Efstathiou		Section: I / (II)

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So $j = 1 - (1+z)\frac{(H^2)'}{H^2} + \frac{1}{2}(1+z)^2\frac{(H^2)''}{H^2}$

Spatially flat, with non-relativistic matter and Λ →

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) = H_0^2\Omega_m(1+z)^3 + H_0^2\Omega_\Lambda ,$$

since $\rho_m \propto R^{-3}$; $\rho_\Lambda = \text{const}$ — [3]

with $\Omega_\Lambda + \Omega_m = 1$ it follows that

$$H(z) = H_0\left[\Omega_m(1+z)^3 + \Omega_\Lambda\right]^{\frac{1}{2}}$$

$H^2 = H_0^2\Omega_m(1+z)^3 \Rightarrow (H^2)' = 3H_0^2\Omega_m(1+z)^2$ and $(H^2)'' = 6H_0^2\Omega_m(1+z)$. Follows that

$$q(z) = -1 + \frac{1}{2}(1+z)\frac{3H_0^2\Omega_m(1+z)^2}{H_0^2[\Omega_m(1+z)^3 + 1 - \Omega_m]}$$

[1] $$= -\Omega_m(1+z)^3 - 1 + \Omega_m + \frac{3}{2}\Omega_m(1+z)^3$$

$\Rightarrow q_0 = -1 + \frac{3}{2}\Omega_m$

[1] Also $j_0 = 1 - 3\Omega_m + \frac{1}{2}\times 6\Omega_m = 1$

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Lecturer		Section
EFSTATHIOU		I / II

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If we expand $H(z) = H_0 + z H'(0) + \frac{1}{2} z^2 H''(0) + \cdots$,

can calculate $1/H(z)$ and hence $D_L(z)$.

[Not required, but result is

$$D_L(z) = \frac{cz}{H_0}\left[1 + z - \tfrac{1}{2} z(1+z)\frac{H'}{H}\Big|_0 - \tfrac{1}{6} z^2 \frac{H''}{H}\Big|_0 + \tfrac{1}{3} z^2 \left(\frac{H'}{H}\right)^2 + \cdots\right]$$

We can now solve for $H'|_0$ from q_0 and $H''|_0$ from q_0 and j_0 etc. For example

(3) $$q_0 = -1 + \frac{H'}{H}\Big|_0$$

[Not required, but $\frac{H''}{H}\Big|_0 = j_0 - q_0^2$]

Case (a): $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7 \Rightarrow D_L(z=0.5) = \frac{cz}{H_0} \times 1.319$

(b) $\Omega_m = 1$, $\Omega_\Lambda = 0 \Rightarrow D_L(z = 0.5) = \frac{cz}{H_0} \times 1.094$

(2)

The supernova will be fainter in (a)

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(i) For bosons/fermions,

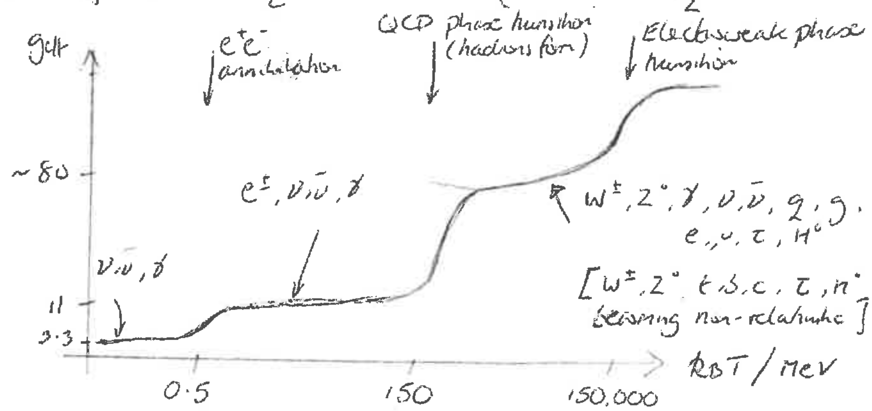
$$\rho_i c^2 = g_i \frac{4\pi c}{h^3}\int_0^\infty \frac{p^3\,dp}{e^{pc/k_BT}\mp 1}$$

$$= g_i \frac{4\pi c}{h^3}\left(\frac{k_BT}{c}\right)^4 \underbrace{\int_0^\infty \frac{x^3\,dx}{e^x \mp 1}}_{\frac{\pi^4}{15}\ (\text{bosons})\ \text{or}\ \frac{7}{8}\frac{\pi^4}{15}\ (\text{fermions})}$$

[using given integrals]

Follows that $\rho_i c^2 = \left(\frac{g_i}{2}\right)\underbrace{\frac{8\pi^5 k_B^4}{(hc)^3 15}}_{\equiv a} \times \begin{cases} 1 & \text{bosons} \\ 7/8 & \text{fermions}\end{cases}$ [5]

Note $g_{\rm eff}$ is defined so that the contribution of relativistic species in equilibrium to $\rho_i c^2$ is $\frac{g_{\rm eff}}{2} a T_\gamma^4$.



[5]

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Page 1

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Stars Q1 Paper I

1

4i) μ_n = mean molecular weight of fully neutral gas

$$\frac{1}{\mu_n} = \sum_j A_j^{-1} F_j$$

A_j = atomic mass number of element j

F_j = mass fraction of element j

for $X = 0.7$, $Y = 0.3$ $(Z = 0)$:

$$\frac{1}{\mu_n} = \frac{X}{1} + \frac{Y}{4} = 0.7 + \frac{0.3}{4} \simeq 1.29$$ //

3

μ_{ion} for a fully ionized gas given by

$$\frac{1}{\mu_{ion}} = \sum_j \frac{1+Z_j}{A_j} F_j$$

where Z_j is the atomic number of element j, i.e. number of electrons liberated in complete ionisation of the atom.

$$\therefore \frac{1}{\mu_{ion}} = 2X + \frac{3}{4}Y$$

$\mu_{ion} \simeq 0.62$ //

3

Stars Q1

- Dynamical Stability of transition zone?

First consider where such a transition might take place.

For the Sun, fully ionized in interior, partially ionized near the surface (See neutral H, He absorption lines)

$\therefore$ transition zone occurs near surface

For ideal gas $P = \frac{1}{\mu m_H} \rho k T$

So, in limit of transition, where P & T = constant across ionisation front

then $\rho \propto \mu$

$\therefore \rho_n / \rho_{ion} \simeq 1.29/0.62 \sim 2$, factor 2 increase in density

$\Rightarrow$ a layer of dense neutral gas overlying a layer of lower density ionised gas

= unstable

4

4i) Angular Size of the Star

0.1 arcsec $\Rightarrow$ distance $10pc = 3.1\times10^{17}m$

$$\Theta_{2R} = \frac{2R_\odot}{d} = \frac{2\times 6.96\times10^{8}}{3.1\times10^{17}} = 4.5\times10^{-9}$$

$$= 4.5\times10^{-9}\times\frac{180}{\pi}\times 3600 \text{ arcsec}$$

$$= 9.3\times10^{-4} \text{ arcsec}$$

Angular Size of Semi major axis

$$\theta_a = 250\,\theta_{2R} = 0.23 \text{ arcsec}$$ ①

Percentage accuracy

$$\frac{\sigma_{\theta_{2R}}}{\theta_{2R}} = \frac{0.01}{9.3\times10^{-4}}\times100 = 1075\%$$

$$\frac{\sigma_{\theta_a}}{\theta_a} = \frac{0.01}{0.23}\times100 = 4.3\%$$ ②

Parallax error, $\sigma_\pi = 0.01$ arcsec, accuracy on mass?

from Kepler 3: $GM\left(\frac{P^2}{4\pi^2}\right) = a^3$ $\quad P$ = period, $M = m_1 + m_2$

$$\therefore\ M = \frac{4\pi^2}{GP^2}a^3$$

Stars Q1

error in M from $\frac{dM}{da}$

$$\frac{dM}{da} = 3 . \frac{4\pi^2}{G p^2} a^2 = \frac{3M}{a}$$

So, $\frac{\sigma_M}{M} = \frac{3\sigma_a}{a}$

from previously $\frac{\sigma_a}{a} = 4.3\%$, ignoring error on parallax

∴ now, adding in quadrature

$$\frac{\sigma_a}{a} = \left(\left(\frac{\sigma_{\theta_a}}{a}\right)^2 + \left(\frac{\sigma_\pi}{\pi}\right)^2 \right)^{\frac{1}{2}}$$

$$= \left(\left(4.3 \times 10^{-2}\right)^2 + \left(\frac{0.01}{0.1}\right)^2 \right)^{\frac{1}{2}}$$

$$= 11\%$$

∴ $\frac{\sigma_m}{M} = 33\%$ //

7

Assume blackbody emission, $T_{eff} = 5800$ K

$$x \equiv \frac{F_\nu(10^{14}\,Hz)}{F_\nu(10^{15}\,Hz)} = \frac{2h\nu_{14}^3/c^2 \left(e^{h\nu_{15}/kT} - 1\right)}{2h\nu_{15}^3/c^2 \left(e^{h\nu_{14}/kT} - 1\right)}$$

$$= \left(\frac{\nu_{14}}{\nu_{15}}\right)^3 \frac{e^{h\nu_{15}/kT} - 1}{e^{h\nu_{14}/kT} - 1}$$

Stars Q1

to find uncertainty in T from x, need dx/dT

$$\frac{dx}{dT} = \left(\frac{\nu_{14}}{\nu_{15}}\right)^3 \left[\frac{\left(e^{h\nu_{15}/kT}-1\right)}{\left(e^{h\nu_{14}/kT}-1\right)^2} \cdot \frac{h\nu_{14}}{kT^2} e^{h\nu_{14}/kT} - \frac{h\nu_{15}}{kT^2} \frac{e^{h\nu_{15}/kT}}{e^{h\nu_{14}/kT}-1} \right]$$

$$= \left(\frac{\nu_{14}}{\nu_{15}}\right)^3 \frac{e^{h\nu_{15}/kT}-1}{e^{h\nu_{14}/kT}-1} \left[\frac{h\nu_{14}}{kT^2} \frac{e^{h\nu_{14}/kT}}{e^{h\nu_{14}/kT}-1} - \frac{h\nu_{15}}{kT^2} \frac{e^{h\nu_{15}/kT}}{e^{h\nu_{15}/kT}-1} \right]$$

$$= x \left[\frac{h\nu_{14}}{kT^2} \frac{e^{h\nu_{14}/kT}}{e^{h\nu_{14}/kT}-1} - \frac{h\nu_{15}}{kT^2} \frac{e^{h\nu_{15}/kT}}{e^{h\nu_{15}/kT}-1} \right]$$

Since $\nu_{14} = 0.1\nu_{15}$, $\sigma_x = \frac{dx}{dT}\sigma_T$

$$\frac{\sigma_T}{T} = \frac{1}{T}\left(\frac{dx}{dT}\right)^{-1}\sigma_x$$

$$= \frac{\sigma_x}{x} \cdot \left[\frac{h\nu_{14}}{kT} \frac{e^{h\nu_{14}/kT}}{e^{h\nu_{14}/kT}-1} - \frac{h\nu_{15}}{kT} \frac{e^{h\nu_{15}/kT}}{e^{h\nu_{15}/kT}-1} \right]^{-1}$$

$$= 10\% \cdot \left(\frac{h\nu_{15}}{kT}\right)^{-1} \left[0.1 \frac{e^{h\nu_{14}/kT}}{e^{h\nu_{14}/kT}-1} - \frac{e^{h\nu_{15}/kT}}{e^{h\nu_{15}/kT}-1} \right]^{-1}$$

$$= 10\% \cdot 1.5\times10^{-1}$$

$$\approx 1.5\%$$

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Stat Phys - Paper 1, Question 5, Part (i)

- Ideal gas: equation of state $pV = Nk_BT$ ③
  internal energy $E = C_V T$

- First law $dE = đQ - p\,dV$

$$dV = (\partial V/\partial p)_T\, dp + (\partial V/\partial T)_p\, dT$$
$$= -(Nk_BT/p^2)\, dp + (Nk_B/p)\, dT$$
$$\therefore C_p = (\partial E/\partial T)_p + p(\partial V/\partial T)_p$$
$$= dE/dT + Nk_B$$
$$= \underline{C_V + Nk_B}$$
④

- $$\gamma = C_p/C_V$$
$$= (C_V + Nk_B)/C_V$$
$$= \underline{1 + Nk_B/C_V}$$
②

- Adiabatic: no heat in or out of system ①
  (or entropy is constant)

Stat Phys – Paper 1, Question 5, Part (ii)

- Adiabatic $\rightarrow$ $đQ = 0$

$\therefore$ First law $\Rightarrow$ $dE = -p\,dV$

$\therefore C_V\,dT = -(Nk_BT/V)\,dV$

$\therefore dT/T = -(Nk_B/C_V)\,dV/V$

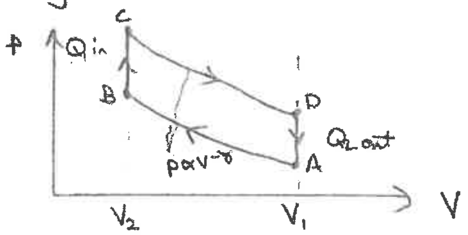
$\therefore \ln T = -(Nk_B/C_V)\ln V + \text{const}$ (4)

$\therefore T V^{Nk_B/C_V} = \text{const}$

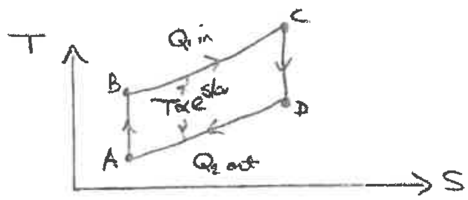
$\therefore p V^{1+Nk_B/C_V} = \text{const}$

$\therefore \underline{pV^{\gamma} = \text{const}}$

- Otto cycle



(3)



Constant volume $\Rightarrow$ $dE = T\,dS$

$\therefore C_V\,dT = T\,dS$

$\therefore \ln T = S/C_V + \text{const}$ (3)

$T \propto e^{S/C_V}$

- Efficiency $\eta = W/Q_1 = (Q_1 - Q_2)/Q_1 = 1 - Q_2/Q_1$

Constant volume $\Rightarrow$ $Q_1 = C_V\,(T_C - T_B)$ where $C_V = (\partial E/\partial T)_V = \frac{3}{2}Nk_B$

$Q_2 = C_V\,(T_D - T_A)$

$\therefore Q_2/Q_1 = (T_D - T_A)/(T_C - T_B)$

Adiabatic $\rightarrow pV^{\gamma} = \text{const}$ or $TV^{\gamma-1} = \text{const}$

$\therefore T_A V_1^{\gamma-1} = T_B V_2^{\gamma-1}$

$T_D V_1^{\gamma-1} = T_C V_2^{\gamma-1}$

$\therefore T_A/T_D = T_B/T_C$

$\therefore Q_2/Q_1 = T_D(1 - T_B/T_C)/[T_C(1 - T_B/T_C)] = T_D/T_C$

$\therefore \eta = 1 - T_D/T_C$

$= 1 - (V_2/V_1)^{\gamma-1}$ (8)

$= \underline{1 - r^{1-\gamma}}$

- Higher $r = V_1/V_2$ $\rightarrow$ more efficient

Larger $\gamma = 1 + Nk_B/C_V$ $\rightarrow$ more efficient (2)

Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
PQM		6X
Examiner		Paper
CHALLINOR		1 / 2 / 3 / 4
Lecturer		Section
SKINNER		I / II

- In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.
- Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.
- Write on this side only and between the margins.
- Not more than one solution per sheet please.

Draft Mark Scheme

Comments

(i) Born rule: if system is in state $|\psi\rangle$,

Probability of obtaining the value a when an observable $\hat{A}$ is measured (where a is an eigenvalue of $\hat{A}$) is

$$P = |\langle x|\psi\rangle|^2,$$

where $\hat{A}|x\rangle = a|x\rangle$.

If transform the state and the eigenstates similarly, i.e., transform the entire system, P is unchanged. If the transformed states are $|\psi'\rangle = U(g)|\psi\rangle$ and $|x'\rangle = U(g)|x\rangle$, must have $U^\dagger U = 1$ for $\langle\psi'|x'\rangle = \langle\psi|x\rangle$ $\forall$ $|\psi\rangle$ & $|x\rangle$

If U represents the group composition law projectively (i.e., up to a phase), we must have

$$U(g_1)U(g_2)|\psi\rangle = e^{i\phi(g_1,g_2)}U(g_1 g_2)|\psi\rangle$$

phase (doesn't affect any observable)

composition

[4] The projective homomorphism can only be a phase since U is unitary.

Page 1

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PQM		6X

Examiner		Paper
CHALLINOR	• In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen.	(1) / 2 / 3 / 4
Lecturer	• Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness.	Section
SKINNER	• Write on this side only and between the margins. • Not more than one solution per sheet please.	(I) / II

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If $U(s)$ is a symmetry of $\hat{H}$, it will commute with $\hat{H}$,

$$[U(s), \hat{H}] = 0.$$

[3] This also means that $U(s)$ commutes with the time-evolution operator $e^{-i\hat{H}t/\hbar}$.

Suppose $|\psi\rangle$ is an eigenstate of the Hermitian generator of the transformation, and hence of $U(s)$. Then the time-evolved state is also, since

$$U(s)\, e^{-i\hat{H}t/\hbar}|\psi\rangle = e^{-i\hat{H}t/\hbar} \underbrace{U(s)|\psi\rangle}_{\lambda|\psi\rangle}$$

$$= \lambda\left(e^{-i\hat{H}t/\hbar}|\psi\rangle\right).$$

[3] It follows that the eigenvalue is conserved.

[Other arguments, such as conservation of expected value of $U(s)$, or its generator, in any state also fine.]

/10

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Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
PQM		6X
Examiner CHALLINOR	• In the right margin please indicate which parts of the answer are based on lecture notes, example sheets, recent previous exam questions, and/or are unseen. • Please remember that external examiners, etc. appreciate the merits of accuracy, legibility, and neatness. • Write on this side only and between the margins. • Not more than one solution per sheet please.	Paper 1 / 2 / 3 / 4
Lecturer SKINNER		Section I / II

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(ii) Since $U(t) = e^{-i\hat{H}t/\hbar}$, if $U(t)T = TU(-t)$,

for infinitesimal t we have

$$\left(1 - i\frac{\hat{H}t}{\hbar}\right)T = T\left(1 + i\frac{\hat{H}t}{\hbar}\right)$$

If T is linear, this implies that $-\hat{H}T = T\hat{H}$.

Let $|\phi\rangle$ be an eigenstate of $\hat{H}$ with energy E. Then

$$\hat{H}T|\phi\rangle = -T\hat{H}|\phi\rangle = -ET|\phi\rangle,$$

so $T|\phi\rangle$ is also an eigenstate with energy $-E$!

(8) The ionized states of hydrogen have arbitrarily large energy, so the time-reversed states would have arbitrarily negative energy $\rightarrow$ no stable ground state.

We still have $-i\hat{H}T = Ti\hat{H}$, but now since anti-linear, this implies $[\hat{H}, T] = 0$ since

$$-i\hat{H}T = Ti\hat{H} = -iT\hat{H}.$$

Repeating the argument above, $T|\phi\rangle$ is now an eigenstate with the same E since

$$\hat{H}T|\phi\rangle = T\hat{H}|\phi\rangle = TE|\phi\rangle = ET|\phi\rangle$$ since E is real.

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Course	NST Astrophysics Part II: 2020-21 Model Solution	Question Number
PQM		6X
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Lecturer SKINNER	• Write on this side only and between the margins. • Not more than one solution per sheet please.	Section I / II

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The original ground state is still the ground state and is stable.

[6]

The position-space wavefunction is $\psi(\underline{x})$ where

$$|\psi\rangle = \int d^3\underline{x}\ |\underline{x}\rangle \underbrace{\langle \underline{x}|\psi\rangle}_{\psi(\underline{x})}$$

If we transform $|\psi\rangle \to T|\psi\rangle$,

$$T|\psi\rangle = \int d^3x\ T(\psi(\underline{x})|\underline{x}\rangle)$$

$$= \int d^3\underline{x}\ \psi^*(\underline{x})\, T|\underline{x}\rangle \quad \text{(anti-linear)}$$

$$= \int d^3\underline{x}\ \psi^*(\underline{x})|\underline{x}\rangle .$$

[3] Follows that $\underline{\psi(\underline{x}) \to \psi^*(\underline{x})}$.

Momentum basis states $\langle \underline{x}|\underline{p}\rangle = e^{i\underline{p}\cdot\underline{x}/\hbar}$ so

$$|\underline{p}\rangle = \int d^3\underline{x}\ |\underline{x}\rangle\langle\underline{x}|\underline{p}\rangle = \int d^3\underline{x}\ e^{i\underline{p}\cdot\underline{x}/\hbar}|\underline{x}\rangle$$

[3] so $T|\underline{p}\rangle = \int d^3\underline{x}\ e^{-i\underline{p}\cdot\underline{x}/\hbar}\, T|\underline{x}\rangle = \int d^3\underline{x}\ e^{-i\underline{p}\cdot\underline{x}/\hbar}|\underline{x}\rangle$

$$= \int d^3\underline{x}\ \langle\underline{x}|-\underline{p}\rangle|\underline{x}\rangle$$

$$= \underline{|-\underline{p}\rangle} .$$

/20

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Dynamics – Paper 1, Question 7, Part (i)

- Potential $\Phi = \Phi(r)$

$$\therefore \underline{F} = -\nabla\Phi = -(d\Phi/dr)\,\hat{\underline{r}} = \ddot{\underline{r}}$$

Specific angular momentum $\underline{h} = \underline{r} \wedge \dot{\underline{r}}$

$$\therefore \dot{\underline{h}} = \underline{r} \wedge \ddot{\underline{r}}$$

$$= -(d\phi/dr)\, r\, \hat{\underline{r}} \wedge \hat{\underline{r}}$$

$$= 0$$

$$\therefore \underline{h} = \text{constant}$$

(2)

- Define polar coords in orbital plane defined by $\underline{h}$: r, ϕ

Orbit : $r(\phi)$

$$\underline{h} = \underline{r} \wedge \dot{\underline{r}} = r^2 \dot{\phi}$$

Let $u = r^{-1}$

$$\therefore \dot{r} = (\partial r/\partial u)\dot{u} = -u^{-2}(du/d\phi)\dot{\phi} = -h\, du/d\phi$$

$$\therefore \ddot{r} = -h(d^2u/d\phi^2)\,\dot{\phi} = -h^2 u^2\, d^2u/d\phi^2$$

Radial component of $\ddot{\underline{r}}$: $\ddot{r} - r\dot{\phi}^2 = -d\Phi/dr = -GM/r^2$

$$\therefore -h^2u^2\, d^2u/d\phi^2 - u^{-1}h^2u^4 = -GMu^2$$

$$\therefore d^2u/d\phi^2 + u = GM/h^2$$

Let $u = A(1 + e\cos(\phi - \varpi))$ where A, e, ϖ are constants

$$\therefore d^2u/d\phi^2 = -Ae\cos(\phi - \varpi)$$

$$\therefore A = GM/h^2$$

$$\therefore r^{-1} = \left(\frac{GM}{h^2}\right)(1 + e\cos(\phi - \varpi))$$

(5)

- For a closed orbit $e < 1$

Minimum (r) occurs for Maximum $(1 + e\cos(\phi - \varpi)) = 2$

$$\therefore r > h^2/(2GM)$$

(3)

Dynamics – Paper 1, Question 7, Part (ii) [1]

- $\Phi = -GM(r-r_s)^{-1}$

  $\therefore d\Phi/dr = GM(r-r_s)^{-2}$

  Radial component of $\ddot{\underline{r}}$: $\ddot{r} - r\dot{\phi}^2 = -d\Phi/dr$

  Radial potential, so specific angular momentum $h = r^2\dot{\phi}$ = ~~constant~~

  $$\therefore \ddot{r} - h^2/r^3 = -GM(r-r_s)^{-2}$$

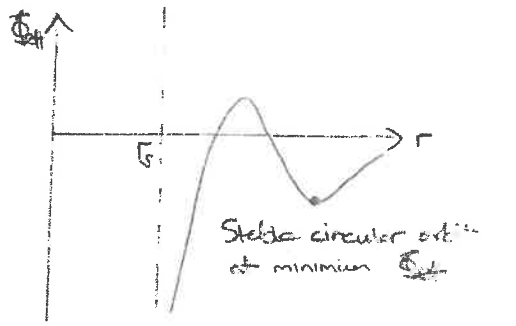
  Specific energy
  $$E = \tfrac{1}{2}\underline{v}.\underline{v} + \Phi(r)$$
  $$= \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}(r\dot{\phi})^2 + \Phi(r)$$
  $$= \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}h^2/r^2 - GM(r-r_s)^{-1} = \text{constant}$$

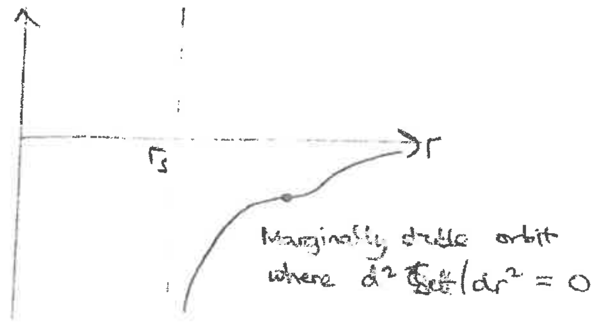
  (2)

- Define effective potential $\Phi_{eff} = \tfrac{1}{2}h^2/r^2 - GM(r-r_s)^{-1}$ s.t. $E = \tfrac{1}{2}\dot{r}^2 + \Phi_{eff}$

  As $\dot{r}^2 > 0$, $E > \Phi_{eff}$

  Plotting Φ_{eff} vs r shows two types of solution stable and unstable, with transition at marginally stable





Consider $d\Phi_{eff}/dr = -h^2r^{-3} + GM(r-r_s)^{-2}$

$\therefore d^2\Phi_{eff}/dr^2 = 3h^2r^{-4} - 2GM(r-r_s)^{-3}$

For circular orbit, $V_c^2 = r\,d\Phi/dr = rGM(r-r_s)^{-2}$, with $V_c = h/r \Rightarrow h^2 = r^3GM(r-r_s)^{-2}$

Or get from $d\Phi_{eff}/dr = 0$, as $h^2r^{-3} = GM(r-r_s)^{-2}$

So a circular orbit has
$$d^2\Phi_{eff}/dr^2 = 3r^{-3}GM(r-r_s)^{-2}r^{-4} - 2GM(r-r_s)^{-3}$$
$$= GM(r-r_s)^{-3}r^{-1}[3(r-r_s) - 2r]$$
$$= GM(r-r_s)^{-3}r^{-1}[r - 3r_s]$$
> 0 for $r > 3r_s \rightarrow$ stable

< 0 for $r_s < r < 3r_s$

Minimum stable orbit is $r = 3r_s$

(10)

Dynamics – Paper 1, Question 7, Part (ii) [2]

- $E = \frac{1}{2}\dot{r}^2 + \Phi_{eff}(r)$

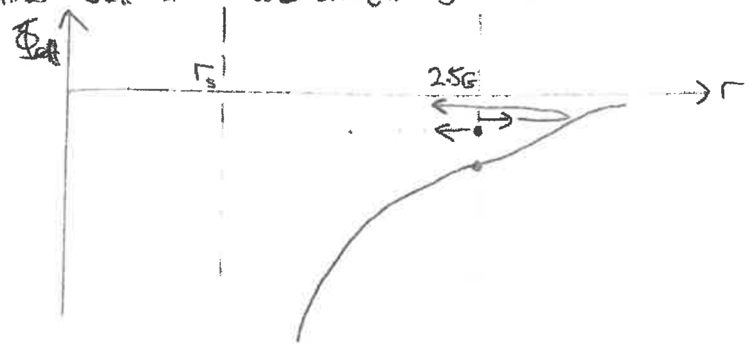
Was on circular orbit with $\dot{r} = 0 \rightarrow E = \Phi_{eff}(2.5r_s)$

$$\Phi_{eff} = \tfrac{1}{2}h^2/r^2 - GM(r-r_s)^{-1}$$
$$= \tfrac{1}{2}r^3 GM(r-r_s)^{-2} r^{-2} - GM(r-r_s)^{-1}$$
$$= \tfrac{1}{2}GM(r-r_s)^{-2}[r - 2(r-r_s)]$$
$$= \tfrac{1}{2}GM(r-r_s)^{-2}[2r_s - r]$$

$\therefore E < 0$ ...

After kick $E > \Phi_{eff}(r)$ in both cases but still $E<0$ if impulse is small enough

Neither Φ_{eff} or h are changed by impulse



For impulse $\dot{r} < 0$ $\rightarrow$ spacecraft falls towards $r = r_s$

$\rightarrow$ force is infinite at $r = r_s$

$\rightarrow$ potential not a good approximation there, but tidal destruction

$\dot{r} > 0$ $\rightarrow$ spacecraft moves outwards to a maximum radius where it turns around

$\rightarrow$ thereafter falls to $r = r_s$ as for $\dot{r} < 0$ (8)

$\rightarrow$ if impulse large could escape

Topics - Paper 1, Question 8, Part (i)

- Direction of angular momentum transfer in a tidally interacting system is from high to low Ω
- Thus the angular momentum from a rapidly spinning planet can be progressively transferred to an orbiting moon, which moves out
- If in this process the moon's orbital radius exceeds the planet's Hill sphere, then the moon is no longer bound to the planet (but is to the star) (3)

- First need angular momentum of satellite when it reaches Hill radius r_H

$$r = r_H = a_p\left(\frac{M_p}{3M_*}\right)^{1/3}$$

Here its angular speed ω is given by

$$GM_p = r^3\omega^2$$

$$\therefore \omega^2 = GM_p \Big/ a_p^3\left(\frac{M_p}{3M_*}\right) = 3GM_*/a_p^3 = 3\,\omega_p^2 \quad \text{(or recall } \omega \sim \omega_p \text{ at Hill sphere)}$$

$\Rightarrow$ Ang mom is $mr^2\omega = ma_p^2\left(\frac{M_p}{M_*}\right)^{2/3}\omega_p\, 3^{-1/6}$

- Next need angular momentum of planet

$$I = \tfrac{2}{5}M_pR_p^2 = \text{moment of inertia}$$

$\therefore$ Ang mom is $I\Omega_p = \frac{2}{5}M_pR_p^2\Omega_p$

- To be unbound, $\frac{2}{5}M_pR_p^2\Omega_p > ma_p^2\left(\frac{M_p}{M_*}\right)^{2/3}\omega_p 3^{-1/6}$

$$\therefore m/M_p < \left(\tfrac{2}{5}\cdot 3^{1/6}\right)\left(\frac{R_p}{a_p}\right)^2\left(\frac{\Omega_p}{\omega_p}\right)\left(\frac{M_*}{M_p}\right)^{2/3}$$

For Earth & Sun $\therefore m/M_\oplus < 0.48\,(6.4\times10^6/1.5\times10^{11})^2\left(\frac{2\pi/1\,\text{day}}{2\pi/365\,\text{day}}\right)(2\times10^{30}/6\times10^{24})^{2/3}$

$$< 1.5\times10^{-3}$$ (6)

- The Moon is 8 times more massive than this and so can't be evaporated (1)

Topics – Paper 1, Question 8, Part (ii)

• Radiation pressure: dust intercepts $(\pi a^2)/(4\pi R^2)$ of momentum from stellar radiation

momentum flux from star is L/c

so outward force is $\frac{1}{4}\frac{a^2}{R^2}\frac{L}{c}$

acceleration is $\left(\frac{1}{4}\frac{a^2}{R^2}\frac{L}{c}\right)/\left(\frac{4}{3}\pi\rho a^3\right) = \left(\frac{3}{16\pi}\frac{L}{c\rho}\right)a^{-1}R^{-2}$

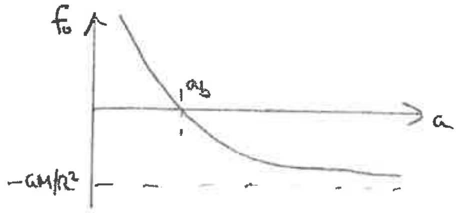
Gravity acceleration is $-GM/R^2$

$\therefore f_0 = \left[\left(\frac{3}{16\pi}\frac{L}{c\rho}\right)a^{-1} - GM\right]R^{-2}$ (6)

• For $f_0 = 0 \rightarrow a_b = 3L/(16\pi\rho c GM)$

$= 3 \times 3.9\times10^{26}/(16\pi \times 2000 \times 3\times10^{8} \times 6.7\times10^{-11} \times 2\times10^{30})$

$= 0.3\,\mu m$ (2)



(2)

• Let $f_0 = AR^{-2} = \ddot{R}$

$\therefore \dot{R}\ddot{R} = A\dot{R}R^{-2}$

$\therefore \frac{d}{dt}\left[\frac{1}{2}\dot{R}^2\right] = \frac{d}{dt}\left[-AR^{-1}\right]$

$\therefore \frac{1}{2}\dot{R}^2 = C - AR^{-1} = AR_0^{-1} - AR^{-1}$ if started at rest at R_0

Terminal velocity at $R\to\infty$ $\therefore \dot{R} = \sqrt{2A/R_0}$

If $a = \frac{1}{2}a_b \Rightarrow A = GM$ $\therefore \dot{R} = \sqrt{2\times6.7\times10^{-11}\times2\times10^{30}/1.5\times10^{11}} \simeq 40\,km/s$ (3)

• Long time after event $\rightarrow$ particles have reached terminal velocity

$\therefore R \simeq \sqrt{2A/R_0}\; t$

If $a \ll a_b \rightarrow A \simeq \left(\frac{3}{16\pi}\frac{L}{\rho c}\right)a^{-1}$ (3)

$\therefore R \propto a^{-1/2}t$

• Number of particles $R \to R+dR$ is $n(R)dR = n(a)da$

But $n(a) \propto a^{-3.5}$ and $a \propto R^{-2}$ $\therefore da \propto R^{-3}dR$

$\therefore n(R)dR \propto (R^{-2})^{-3.5}R^{-3}dR$

$\propto R^{4}dR$

Mass of grains is $n(R)dR.\frac{4}{3}\pi\rho a^3 \propto R^4 dR.(R^{-2})^3$ (4)

$\propto R^{-2}dR$