UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 3 June 2019 13.30pm – 16.30pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

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A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) In an inertial reference frame S, there is a uniform electric field $\vec{E}$ and a uniform magnetic field $\vec{B}$. Another inertial frame S' is related to S by a Lorentz boost along the x-axis with speed v. Starting from the Maxwell field-strength tensor, or otherwise, express the Cartesian components of the electric and magnetic fields in S' in terms of those in S.

[*The components of the field-strength tensor are related to those of the electric and magnetic fields by*

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix} .]$$

If $\vec{E}$ and $\vec{B}$ are orthogonal in S, find a condition on their magnitudes if the magnetic field vanishes in some other inertial frame.

(ii) A uniform electric field of magnitude E lies along the y-axis of an inertial reference frame in Minkowski spacetime, and a uniform magnetic field of magnitude E/c lies along the z-axis. A particle of mass m and charge q moves under the influence of these fields along the worldline $x^\mu(\tau)$, where τ is proper time. The equation of motion for the particle's 4-velocity $u^\mu = dx^\mu/d\tau$ is

$$m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu \, .$$

If the particle starts from rest at the origin of the spacetime at $\tau = 0$, show that

$$c\frac{dt}{d\tau} - \frac{dx}{d\tau} = c \, ,$$

where $x^\mu = (ct, x, y, z)$.

Hence, or otherwise, find $x^\mu(\tau)$.

Show that

$$[x(\tau)]^2 = \frac{2}{9}\frac{qE}{mc^2}[y(\tau)]^3 \, .$$

Describe the motion for $qE(c\tau) \gg mc^2$.

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a spherical rocky planet that possesses constant density ρ throughout. Derive an expression for the maximum possible mass of this planet, assuming that this mass corresponds to the point where the rock at the planetary centre exceeds a critical pressure $p_{\rm cr}$ (and hence gets crushed).

Suppose now that this object is the rocky core of a giant gas planet. Would the maximum mass of the rocky core be changed relative to the bare rocky core case considered above and, if so, in what sense?

(ii) Consider a thin isothermal atmosphere on the surface of a neutron star, with surface gravity g. Let ρ_0 be the density at the base of the atmosphere and c_s^2 be the isothermal sound speed.

Calculate the density structure of the atmosphere in the following cases:

(a) no magnetic fields are present;

(b) a uniform vertical magnetic field is present;

(c) a uniform horizontal magnetic field is present; and

(d) the gas possesses a horizontal magnetic field such that $B = \beta_0^{-1}\sqrt{\mu_0 p_{\rm gas}}$, where β_0 is a constant and $p_{\rm gas}$ is the gas pressure.

Now consider a situation where a horizontal sheet of horizontal magnetic flux is situated close to the base of an otherwise unmagnetized atmosphere. The magnetized sheet is initially in pressure equilibrium with the gas above and below it. Qualitatively describe the subsequent behaviour of the atmosphere as a function of β_0.

TURN OVER...

Question 3Z - Introduction to Cosmology

(i) Consider an expanding universe with a zero cosmological constant and $\rho + 3p/c^2 > 0$, where ρ is the density and p is the pressure. Show that the age of the universe satisfies $t_0 < 1/H_0$, where H_0 is the present-day value of the Hubble parameter.

Show that in such a universe, and ignoring radiation,

$$\frac{1}{\Omega_{\rm m}(z)} - 1 = \left(\frac{1}{\Omega_{\rm m,0}} - 1\right)(1+z)^{-1}\,, \qquad (*)$$

where $\Omega_{\rm m}(z)$ is the matter density as a fraction of the critical density at redshift z and the subscript 0 indicates the present time. Comment on the implication of $(*)$ for the evolution of the universe at early times.

(ii) At redshift $z = 1100$, just before recombination, estimate:

(a) the number density of free electrons;

(b) the mean free path of photons; and

(c) the average time a photon travels between scatterings with free electrons.

[*You may ignore elements other than hydrogen in your calculations. You should take the baryon density parameter* $\Omega_{\rm b,0} = 0.05$, *and the present-day critical density of the universe to be* $8.4 \times 10^{-27}\,\mathrm{kg\,m^{-3}}$.]

What do your results imply about the opacity of the universe at early times? Give reasons for your answer.

After recombination, the gas in the universe remains neutral until an early generation of stars reionises it at $z = z_{\rm re}$, after which the gas remains fully ionised to the present day. Show that in a matter-dominated Friedmann–Robertson–Walker universe the optical depth for scattering by the free electrons is

$$\tau = \frac{2}{3}\frac{n_{\rm e,0}\sigma_{\rm T} c}{H_0}\left[(1+z_{\rm re})^{3/2} - 1\right]\,,$$

where $n_{\rm e,0}$ is the present-day electron density and $\sigma_{\rm T}$ is the cross-section for Thomson scattering.

Calculate $z_{\rm re}$ if $\tau = 0.1$ and $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Question 4Y - Structure and Evolution of Stars

(i) Briefly discuss the different timescales that govern stellar structure and evolution. For each case, provide a brief explanation and give an expression that can be used to estimate the timescale.

(ii) Describe the life cycle of a solar-mass star from the main sequence to its final state. At each stage of the evolution describe the key nuclear processes, time scales, typical radius, and luminosity.

Sketch the stellar evolution discussed above on a Hertzsprung-Russell diagram in the Temperature–Luminosity plane. Label the axes and units appropriately. Identify the locations of different types of stars on the Hertzsprung-Russell diagram.

TURN OVER...

Question 5Z - Statistical Physics

(**i**) A thermal system (of constant volume) undergoes a phase transition at temperature T_{c}. The specific heat of the system is measured to be

$$C_V = \begin{cases} \alpha T & T < T_{\mathrm{c}} \\ \beta & T > T_{\mathrm{c}}\,, \end{cases}$$

where α and β are constants. Show that the entropy for $T > T_{\mathrm{c}}$ is given by

$$S = \beta \ln(T/T_c) + \gamma$$

for some constant γ.

How can the value of γ be verified using macroscopically measurable quantities?

(**ii**) Explain, from a macroscopic and microscopic point of view, what is meant by an *adiabatic change*.

Define the *Carnot cycle* for a system operating between two heat baths at temperatures T_1 and T_2, with $T_2 > T_1$.

The efficiency η is the ratio of work done to heat extracted from the heat bath at T_2 per cycle. Determine the efficiency, η_{Carnot}, of the Carnot cycle in terms of T_1 and T_2 in the case of an ideal gas by explicitly calculating the heat transferred in terms of the associated volume changes of the gas.

Show that no other cyclic process operating between T_1 and T_2 can be more efficient than the Carnot cycle, and that all reversible cyclic processes have the same efficiency η_{Carnot}.

Question 6Z - Principles of Quantum Mechanics

(i) A three-dimensional isotropic harmonic oscillator of mass μ and frequency ω has lowering operators

$$\mathbf{A} = \frac{1}{\sqrt{2\mu\hbar\omega}} \left(\mu\omega\mathbf{X} + \mathrm{i}\mathbf{P}\right) ,$$

where $\mathbf{X}$ and $\mathbf{P}$ are the position and momentum operators. Assuming the standard commutation relations for $\mathbf{X}$ and $\mathbf{P}$, evaluate the commutators $[A_i^\dagger, A_j^\dagger]$, $[A_i, A_j]$ and $[A_i, A_j^\dagger]$ among the components of the raising ($\mathbf{A}^\dagger$) and lowering operators.

How is the ground state $|\mathbf{0}\rangle$ of the oscillator defined?

How are normalised higher-excited states obtained from $|\mathbf{0}\rangle$?

(ii) By expressing the orbital angular momentum operator $\mathbf{L}$ in terms of the raising and lowering operators given in Part (i), show that each first-excited state of the isotropic oscillator there has total orbital angular momentum quantum number $\ell = 1$.

Find a linear combination $|\psi\rangle$ of these first-excited states obeying $L_z|\psi\rangle = +\hbar|\psi\rangle$ and $\langle\psi|\psi\rangle = 1$.

TURN OVER...

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) A test particle moves on a bound orbit in a spherical gravitational potential $\Phi(r)$. Show that the mean square velocity on any bound orbit is

$$\langle v^2 \rangle = \left\langle r \frac{\mathrm{d}\Phi}{\mathrm{d}r} \right\rangle ,$$

where $\langle \cdot \rangle$ denotes a time average.

(ii) A test particle moves in the gravitational potential of a point mass M. Show that for suitable choice of constant h and function $u(r)$, where r is the distance between test particle and point mass, the orbit can be described (using a plane-polar coordinates r, ϕ) by an equation of the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2},$$

where G is the gravitational constant.

Show that

$$\dot{r} = \frac{GMe}{h} \sin(\phi - \phi_0),$$

where e and ϕ_0 are constants of integration, and the dot denotes differentiation with respect to time.

Show that for an unbound Kepler orbit

$$e^2 = 1 + \left(\frac{h\, v_\infty}{GM} \right)^2 ,$$

where v_∞ is the velocity of the test particle at infinity.

Which properties of the orbit do e and ϕ_0 describe?

Question 8Y - Physics of Astrophysics

(i) The spectra of stars derived from a magnitude-limited survey of a small patch of the sky are examined and all stars that have spectral features of solar-type stars (but no evidence of spectral features of more massive stars) are put into a solar-type database. This database contains a mixture of single solar-type stars and binary stars where the primary star is solar-type. You may assume that single solar-type stars and binaries with solar-type primaries are equally abundant in the Galaxy and that the fraction of binaries with secondary masses, M_2, in the range M_2 to $M_2 + dM_2$ scales with $M_2 dM_2$. Show that if stellar luminosity scales with stellar mass as $L \propto M^3$ then the number of solar-type primaries in the database per unit interval of distance from the Earth can be written in the form

$$n(d) = Ad^2\left(1 - \left[2(d/d_{\max})^2 - 1\right]^{2/3}\right),$$

for $0.5 < (d/d_{\max})^2 < 1$, where $d_{\max}$ is the maximum distance to which solar-type binaries could be detected in the original survey and A is a normalisation constant.

Write down also the form of $n(d)$ for solar-type binaries and single stars for $(d/d_{\max})^2 < 0.5$.

(ii) A rotating cloud of gas of mass $1\,M_\odot$ and radius 10^4 au collapses to form a rotationally-supported disc. If the initial kinetic energy of the cloud is 1% of the magnitude of its gravitational potential energy, estimate the radius of the disc.

When imaged with a radio telescope, the disc appears elliptical in shape with angular diameters along the two principal axes of 1 arcsec and 0.5 arcsec. Use your value of the disc radius to estimate the distance of the source from the Earth.

A non-relativistic jet, aligned with the angular momentum vector of the disc, emerges from the central regions of the disc and is seen in projection against the disc behind it. In two successive observations separated by five years, the distance between a bright emission knot in the jet and the centre of the disc is observed to change from 5.0 arcsec to 5.1 arcsec. Estimate the jet speed relative to the disc centre on the assumption that the knot co-moves with the jet.

If the knot emits a spectral line of carbon monoxide with rest wavelength 1.3 mm, and ignoring any motion of the disc centre with respect to the Earth, to what frequency should the radio receiver be tuned in order to detect this spectral line from the knot?

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 4 June 2019 13.30pm-16.30pm

ASTROPHYSICS - PAPER 2

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Question 1X - Relativity

(i) A curve $x^a(\lambda)$ is parameterised by λ. Explain what it means for λ to be an *affine* parameter.

By starting from the geodesic equation in an affine parameterisation λ, or otherwise, show that a geodesic curve with respect to a general, not necessarily affine, parameter u satisfies

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = \frac{dx^a}{du}\frac{d^2\lambda/du^2}{d\lambda/du}\,. \tag{$*$}$$

(ii) Two metric tensor fields, $g_{ab}(x)$ and $\tilde{g}_{ab}(x)$, are said to be conformally related if $g_{ab}(x) = \Omega^2(x)\tilde{g}_{ab}(x)$ for some scalar field $\Omega(x)$. Show that the metric connections constructed from the two metrics are related by

$$\Gamma^a_{bc} = \tilde{\Gamma}^a_{bc} + \left(\delta^a_c\frac{\partial}{\partial x^b}\ln\Omega + \delta^a_b\frac{\partial}{\partial x^c}\ln\Omega - \tilde{g}_{bc}\tilde{g}^{ad}\frac{\partial}{\partial x^d}\ln\Omega\right)\,.$$

Suppose that $x^a(\tilde{\lambda})$ is an affinely-parameterised *null* geodesic of the metric $\tilde{g}_{ab}$. Show, using $(*)$ or otherwise, that $x^a(\tilde{\lambda})$ is also a null geodesic of the metric g_{ab} but that the parameter $\tilde{\lambda}$ is generally not affine and is instead related to an affine parameter λ by $\Omega^2 \propto d\lambda/d\tilde{\lambda}$.

The line element for a homogeneous and isotropic cosmological model with flat spatial sections takes the form

$$ds^2 = a^2\eta_{\mu\nu}dx^\mu dx^\nu\,, \tag{$\dagger$}$$

where $a(t)$ is the scale factor as a function of cosmic time t and $\eta_{\mu\nu}$ is the Minkowski metric. The conformal time x^0 satisfies $a dx^0 = d(ct)$ and the x^i $(i = 1, 2, 3)$ are comoving Cartesian coordinates. Write down the general form of null geodesics in Minkowski spacetime in terms of an affine parameter $\tilde{\lambda}$ and hence show that the components of the 4-momentum of a photon in the cosmological spacetime described by $(\dagger)$ are

$$p^\mu \propto \frac{1}{a^2}(1, e^i)\,,$$

where e^i are the Euclidean components of a constant unit 3-vector.

Show further that the energy E of the photon, as measured by (comoving) observers with constant x^i, evolves as $E \propto 1/a$.

Question 2Y - Astrophysical Fluid Dynamics

(i) Starting with the fluid equations in conservative form, derive the Rankine-Hugoniot jump conditions for a fluid passing normally through a shock. Assume that the shock is adiabatic, and the gas has adiabatic index γ.

(ii) It can be shown (but you do not need to do so) that the Rankine-Hugoniot jump conditions imply

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}^2}{(\gamma+1)+(\gamma-1)(\mathcal{M}^2-1)},$$
$$\frac{p_2}{p_1} = \frac{(\gamma+1)+2\gamma(\mathcal{M}^2-1)}{(\gamma+1)},$$

where ρ_1 is the pre-shock density, ρ_2 is the post-shock density, p_1 is the pre-shock pressure, p_2 is the post-shock pressure, and $\mathcal{M}$ is the Mach number of the pre-shocked flow. Comment on the behaviour of ρ_2/ρ_1 and p_2/p_1 in the limit of a very strong shock.

Now consider a very weak shock passing through a fluid with $\gamma = 5/3$, such that $\mathcal{M}^2 = 1 + \epsilon$ with $\epsilon \ll 1$. Show that the jump in the adiabatic constant K, where $p = K\rho^\gamma$, of the gas across the shock

$$\frac{\Delta K}{K} \approx \alpha\epsilon^3,$$

where α is the value of a constant you do not need to determine.

At a certain location in a galaxy cluster, the intracluster medium has pressure $p = 6 \times 10^{-12}\,\mathrm{Pa}$, temperature $T = 5 \times 10^7\,\mathrm{K}$, and is cooling via X-ray emission at a volumetric rate of $2 \times 10^{-28}\,\mathrm{J\,s^{-1}\,m^{-3}}$. A jetted active galactic nucleus at the centre of the cluster generates a series of weak shocks with $\mathcal{M} = 1.15$ that pass through this region. Taking $\alpha = 5/48$, estimate the time interval between the shocks needed for shock heating to balance, on average, the radiative cooling. Give your answer in terms of years.

[You may recall that the energy equation in entropy form for a fluid with adiabatic index γ is

$$\frac{1}{K}\frac{DK}{Dt} = -\frac{(\gamma-1)}{p}\rho\dot{Q}_{\mathrm{cool}}.]$$

TURN OVER...

Question 3Z - Introduction to Cosmology

(i) In a universe with zero cosmological constant ($\Omega_{\Lambda,0} = 0$), and ignoring radiation, the angular diameter distance at redshift z is given by

$$d_{\mathrm{A}}(z) = 2\frac{c}{H_0}\frac{1}{\Omega_{\mathrm{m},0}^2\,(1+z)^2}\left[\Omega_{\mathrm{m},0}z + (\Omega_{\mathrm{m},0} - 2)\left(\sqrt{1+\Omega_{\mathrm{m},0}z} - 1\right)\right],$$

where $\Omega_{\mathrm{m},0}$ is the present-day matter density parameter and H_0 is the Hubble constant. Using this relation show that, if $\Omega_{\mathrm{m},0} \ll 1$, the angular diameter θ of a galaxy of fixed proper size:

(a) decreases as z^{-1} for $z \ll 1$;

(b) increases as z for $z \gg \Omega_{\mathrm{m},0}^{-1}$; and

(c) remains roughly constant in the interval $1 < z < \Omega_{\mathrm{m},0}^{-1}$.

(ii) In the early universe, the cross-section for neutrino interactions as a function of temperature T is given by

$$\sigma(T) = 3.5 \times 10^{-67}\,(T/\mathrm{K})^2\,\mathrm{m}^2\,,$$

and the number density of a single neutrino type is given by

$$n(T) = 7 \times 10^6\,(T/\mathrm{K})^3\,\mathrm{m}^{-3}\,.$$

Estimate the mean time between interactions at temperature T.

Show that when the universe is radiation dominated, the Hubble parameter $H \propto T^2$ and estimate the proportionality constant. You may ignore all relativistic particles other than photons in your calculation.

As the universe cools, at what temperature do neutrinos decouple?

When electrons and positrons annihilate the temperature of the photons increases by a factor of $(11/4)^{1/3}$. Explain why the temperature of the neutrinos is not similarly affected and, assuming that neutrinos are massless, determine the temperature of the relic neutrinos present in the universe today?

It is now known that at least some types of neutrinos do have a small mass. Give a brief qualitative explanation of why observations of the large-scale structure of galaxies can be used to place limits on the sum of the neutrino masses.

Question 4Y - Structure and Evolution of Stars

(i) The specific intensity of a blackbody as a function of temperature T and wavelength λ is given by Planck's law

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1},$$

where k_B is Boltzmann's constant, h is Planck's constant, and c is the speed of light. Using the approximation for the blackbody spectrum valid at long wavelengths, explain why limb darkening is more pronounced in blue light than red light.

(ii) Explain what is meant by radiative opacity, and provide a definition of it in terms of the photon mean free path.

Briefly discuss the main sources of radiative opacity in main sequence stars, differentiating between line and continuum opacity.

Briefly discuss two differences between spectral features of a solar-like G star and a late M dwarf.

In the centre of cool white dwarfs composed of pure carbon, detailed calculations show that the *conduction* opacity can be described as

$$\kappa_{\rm cond} \simeq \Lambda \left(\frac{T}{\rho}\right)^2 \, {\rm cm}^2 \, {\rm g}^{-1},$$

where T is the temperature, ρ is the density, and $\Lambda = 5 \times 10^{-7} \, {\rm g \, K^{-2} \, cm^{-4}}$. By assuming that radiative opacity is dominated by Thomson scattering, show that in the centre of such stars, where $T \simeq 10^7 \, {\rm K}$ and $\rho \simeq 10^6 \, {\rm g \, cm^{-3}}$, conduction completely dominates energy transport.

TURN OVER...

Question 5Z - Statistical Physics

(**i**) Using the classical statistical mechanics of a monatomic gas of N identical point particles with negligible interactions, show that the free energy is

$$F = -Nk_{\rm B}T \ln \left[\frac{eV}{N} \left(\frac{mk_{\rm B}T}{2\pi\hbar^2} \right)^{3/2} \right] , \qquad (*)$$

where V is the volume of the system, T is the temperature, and m is the mass of each particle.

[*You may wish to use*

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \left(\frac{\pi}{\alpha} \right)^{\frac{1}{2}} .]$$

(**ii**) For the gas in Part (i), derive the ideal gas law relating pressure P, volume V, and temperature T starting from the free energy $(*)$.

Explain briefly whether this law is also applicable to a gas of molecules with internal structure (but still ignoring interactions).

For a monatomic gas at low density with negligible interactions, calculate the entropy S as a function of T, V, and the number of particles, N. Deduce the equation relating the pressure and volume of the gas on a curve in the P–V plane along which S and N are constant.

Question 6Z - Principles of Quantum Mechanics

(i) Let $|i\rangle$ and $|j\rangle$ be two eigenstates of a time-independent Hamiltonian H_0, separated in energy by $\hbar\omega_{ij}$. At time $t = 0$ the system is perturbed by a small, time-independent operator V. The perturbation is turned off at time $t = T$. Show that if the system is initially in state $|i\rangle$, the probability of a transition to state $|j\rangle$ is approximately

$$P_{ij} = 4|\langle i|V|j\rangle|^2 \frac{\sin^2(\omega_{ij}T/2)}{(\hbar\omega_{ij})^2}\,. \qquad (*)$$

(ii) An uncharged spin-half particle with magnetic moment μ travels at speed v through a region of uniform magnetic field of strength B. Over a length L of its path, an additional perpendicular magnetic field b is applied. The spin-dependent part of the Hamiltonian is

$$H(t) = \begin{cases} -\mu(B\sigma_z + b\sigma_x) & \text{while } 0 < t < L/v\,, \\ -\mu B\sigma_z & \text{otherwise,} \end{cases}$$

where σ_z and σ_x are Pauli matrices. The particle initially has its spin aligned along the direction of B. Find the probability that it makes a transition to the state with opposite spin by assuming $b \ll B$ and using $(*)$ from Part (i).

Determine also this probability by finding the exact evolution of the state and show that your exact result reduces to the perturbative one at leading order in b^2/B^2.

[*You may find the identity* $e^{i\mathbf{a}\cdot\boldsymbol{\sigma}} = (\cos a)I + (i\sin a)\,\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}$ *useful. The Pauli matrices are*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\Bigg]$$

TURN OVER...

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) Show that the gravitational potential of an infinitesimally-thin disc, lying in the $z = 0$ plane, and with axially-symmetric surface density $\Sigma(R)$ can be written in the form

$$\Phi(R, z) = \int_0^\infty f(k) e^{-k|z|} J_0(kR) \mathrm{d}k,$$

for some function $f(k)$, where $J_0(x)$ is the solution of the equation

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y = 0$$

that is finite at $x = 0$.

Derive an integral expression relating $\Sigma(R)$ and $f(k)$.

(ii) A particle is in a circular orbit of radius R_0, in the $z = 0$ plane of an axisymmetric potential of the form $\Phi(R, |z|)$. Show that the circular velocity is given by

$$V_\mathrm{c}^2 = R_0 \left(\frac{\partial\Phi}{\partial R}\right)_{R=R_0, z=0}.$$

The orbit is now slightly perturbed in the z-direction. Show that the particle undergoes vertical oscillations with some frequency ν that you should determine in terms of Φ.

If the potential is entirely due to a thin disc of matter in the $z = 0$ plane, show that

$$\nu^2 = -\frac{1}{R}\frac{\mathrm{d}V_\mathrm{c}^2}{\mathrm{d}R}.$$

What can one deduce about the distribution of gravitating matter in a region of a spiral galaxy where the rotation speed is increasing with radius?

[*You may assume*

$$\nabla^2 F = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial F}{\partial R}\right) + \frac{\partial^2 F}{\partial z^2}.$$]

Question 8Y - Physics of Astrophysics

(i) Fabuloso, a yet-to-be-discovered moon of Jupiter, describes an eccentric orbit ($e = 0.7$) with a period of 3 days. Estimate the maximal level of tidal distortion of Fabuloso and suggest a possible consequence of this.

Comment briefly on whether you expect the orbit of Fabuloso to be significantly affected by the tidal effect of the Sun.

[*Mass of Jupiter* $M_{\rm J} = 10^{-3}\,{\rm M}_\odot$*; Mass of Fabuloso* $M_{\rm F} = 10^{-7}\,{\rm M}_\odot$*; radius of Fabuloso* $= 2000\,{\rm km}$*; orbital radius of Jupiter around the Sun* $= 5\,{\rm au}$]

(ii) A massive star drives a bipolar jet whose ejecta are confined within a pair of cones of half-opening angle θ (with $\theta \ll 1$). The mass flow rate in the jet is $10^{-6}\,{\rm M}_\odot\,{\rm yr}^{-1}$ and the jet velocity is $200\,{\rm km\,s}^{-1}$. The star is situated in the core of a gas cloud of mass $10^5\,{\rm M}_\odot$, radius 10 pc. Estimate the value of θ such that the momentum input from the jet is sufficient to drive all the cloud material in its path from out of the cloud's gravitational potential. Hence estimate how many jet sources are required in order to disrupt the entire cloud.

The stars are formed in a burst of star formation in which the number of stars per logarithmic mass bin scales as $m^{-1.35}$ for $m > 0.1{\rm M}_\odot$. If powerful jets are driven by stars more massive than $10\,{\rm M}_\odot$ and all such jets have the parameters described above, show that cloud disruption would require the formation of a total mass in stars that exceeds the total cloud mass.

Show that in the case that only 20% of the cloud mass is turned into stars, the kinetic energy that is expelled in the jets over the (typical) 10 Myr lifetime of the jet sources exceeds the gravitational binding energy of the cloud. Explain why the cloud is nevertheless not disrupted in this case.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 6 June 2019 09:00am – 12:00pm

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Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 7X** *should be in one bundle and* **2Y, 4Y and 8Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper (lined on one side)	*Astrophysics Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X - Relativity

(i) A test particle of mass m is at rest at radius $r > 2\mu$ in the Schwarzschild spacetime outside a spherical mass distribution of mass M, with $\mu = GM/c^2$. The line element is

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2\,. \qquad (*)$$

If the 4-velocity of the particle is u^μ, determine its 4-acceleration $Du^\mu/D\tau$, where τ is the particle's proper time.

Determine the magnitude of the particle's acceleration in its rest frame and hence the force required to hold it at rest.

(ii) Consider free-falling test particles moving radially in the spacetime with line element $(*)$. Show that the paths of massive and massless particles obey

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k \quad \text{(both massive and massless)}$$

and

$$\left(1 - \frac{2\mu}{r}\right)c^2\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 = \begin{cases} c^2 & \text{massive}\,, \\ 0 & \text{massless}\,, \end{cases}$$

where k is a constant and dots denote differentiation with respect to an affine parameter, which is proper time in the case of a massive particle.

A massive particle falls freely from rest at $r = \infty$. When the particle is at radius r, with $r > 2\mu$, determine its speed relative to a stationary observer there (i.e., an observer at rest at radius r).

The infalling particle emits light at frequency ν in its instantaneous rest frame. Light emitted at radius r is observed by a stationary observer at a larger radius $r_{\mathrm{rec}} > 2\mu$ to have frequency ν_{rec}. Show that

$$\frac{\nu_{\mathrm{rec}}}{\nu} = \left(\frac{1 - 2\mu/r}{1 - 2\mu/r_{\mathrm{rec}}}\right)^{1/2}\frac{1 - \sqrt{2\mu/r}}{\sqrt{1 - 2\mu/r}}\,.$$

Interpret this result physically.

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider an axisymmetric, rotating, viscous flow described in a cylindrical polar coordinate system (R, ϕ, z) with density ρ, velocity field $\mathbf{v} = (v_R, v_\phi, v_z)$, and coefficient of kinematic viscosity ν. Suppose that the flow is restricted to have $v_z = 0$ and $v_\phi = v_\phi(R, t)$. Show that

$$\frac{\partial}{\partial t}(\rho R v_\phi) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\left[R\rho v_\phi v_R - \rho\nu R^2 \frac{\partial}{\partial R}\left(\frac{v_\phi}{R}\right)\right]\right) = 0.$$

Interpret this equation physically.

[*The azimuthal component of the momentum equation for this system is*

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_R v_\phi}{R}\right) = \frac{\partial}{\partial R}\left[\rho\nu\left(\frac{\partial v_\phi}{\partial R} - \frac{v_\phi}{R}\right)\right] + \frac{2\rho\nu}{R}\left(\frac{\partial v_\phi}{\partial R} - \frac{v_\phi}{R}\right),$$

and the divergence operator in cylindrical geometry is

$$\nabla \cdot \mathbf{A} = \frac{1}{R}\frac{\partial(RA_R)}{\partial R} + \frac{1}{R}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}.$$]

(ii) Consider a local patch of a geometrically-thin accretion disk orbiting with angular velocity Ω in the Newtonian gravitational potential of a central object. Assume that the patch is in vertical hydrostatic equilibrium with a profile such that the isothermal sound speed $c_s^2 \equiv p/\rho$, where p is pressure, is constant. Show that the vertical profile of the disk atmosphere is

$$\rho = \rho_c e^{-z^2/(2h^2)},$$

with $h = c_s/\Omega$.

In black hole accretion disks, radiation pressure can become important for the vertical structure of the disk. The relevant equation of state is

$$p = \frac{\mathcal{R}^*\rho T}{\mu} + \frac{4\sigma}{3c}T^4,$$

where $\mathcal{R}^*$ is the modified gas constant, T is the temperature of the gas and radiation (assumed to be in thermodynamic equilibrium with each other), μ is the mean molecular weight of the gas, σ is the Stefan-Boltzmann constant, and c is the speed of light. Consider a local patch of such a black hole disk sufficiently far from the hole that the gravitational field can be taken to be Newtonian. Assume that the vertical structure of the patch is such that $c_s^2 \equiv p/\rho$ is constant with height. Show that, for a given surface density Σ, the sound speed c_s is related to the mid-plane temperature T_c of the disk by

QUESTION CONTINUES OVER...

$$c_s = \alpha T_c^4 + \sqrt{\alpha^2 T_c^8 + \beta T_c},$$

where α and β should be determined in terms of Σ and physical constants.

For a fixed surface density, it can be shown that the vertically-integrated heating rate $\dot{Q}_+ \propto c_s^2$ whereas the vertically-integrated cooling $\dot{Q}_- \propto T_c^4$. Show that there exists a critical temperature T_1 such that the disk is thermally unstable (at constant Σ) when $T_c > T_1$ but is otherwise thermally stable. What is the physical relevance of the temperature T_1?

Question 3Z - Introduction to Cosmology

(i) A brief period of exponential cosmic expansion, termed *inflation*, is thought to have occurred between times $t_{\rm start} = 10^{-36}\,$s and $t_{\rm end} = 10^{-34}\,$s. Inflation would explain why our universe is spatially flat, and almost homogeneous over distances larger than the horizon scale at early times.

(a) If the present-day total density parameter $|\Omega_{\rm tot}(t_0) - 1| = 0.1$, calculate $|\Omega_{\rm tot}(t_{\rm end}) - 1|$ at the end of inflation for the case of a radiation-dominated universe.

(b) Repeat step (a) for a matter-dominated universe.

(c) Find the time dependence of the Hubble parameter, $H(t)$, during the exponential expansion.

(d) If $|\Omega_{\rm tot}(t_{\rm start}) - 1| = 1$ at the start of inflation, calculate the amount of inflation, $a(t_{\rm end})/a(t_{\rm start})$ where $a(t)$ is the scale factor, required to reach the value of $|\Omega_{\rm tot}(t_{\rm end}) - 1|$ obtained in step (a).

(e) If the expansion time (i.e., the time taken for the universe to expand by a factor e) during inflation is $H^{-1} \simeq 10^{-36}\,$s, calculate the factor by which the universe expanded in the inflationary era. Compare your answer to the expansion factor requirement derived in step (d).

(ii) Sketch the $\Omega_{\rm m,0}$–$\Omega_{\Lambda,0}$ plane for $0 \le \Omega_{\rm m,0} \le 3$ and $-2 \le \Omega_{\Lambda,0} \le 3$, where $\Omega_{\rm m,0}$ and $\Omega_{\Lambda,0}$ are, respectively, the cosmic density parameters of matter and of the cosmological constant today. Determine boundaries that divide the plane into open and closed universes, and into accelerating and decelerating universes, and include these in your sketch, clearly labelling the different regions. Indicate on your sketch the approximate parameters of our universe. (You may assume that the present-day density of radiation is negligible, i.e., $\Omega_{\rm rad,0} = 0$).

Determine whether a universe with $\Omega_{\rm m,0} = 0.5$, $\Omega_{\Lambda,0} = 3.5$ is open or closed. Show that the values $\Omega_{\rm m,0} = 0.5$, $\Omega_{\Lambda,0} = 3.5$ can correspond to a universe with no big bang. What is the maximum observed redshift in such a universe?

[*You may require the factorisation* $x^3 - 6x^2y + 7y^3 = (x + y)\,(x^2 - 7xy + 7y^2)$.]

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) Describe briefly the definitions of *Visual Binaries*, *Spectroscopic Binaries*, and *Eclipsing Binaries*. By which observational technique is the binary nature of each of these classes of binary stars recognised?

An optical spectrum of a solar-like star showed that the spectral lines are Doppler shifted relative to their rest-frame-line-positions. Follow-up observations revealed that the magnitude of the Doppler shift varied periodically in time. High-resolution optical images did not reveal a binary companion to the star. What are the possible explanations for these observations? What additional information or observations would you need to confirm the cause?

(ii) A $1\,\mathrm{M}_\odot$ white dwarf accretes matter of solar composition from a companion star for 10^5 yr at a rate of $1 \times 10^{-10}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$. At the end of this accretion period, a mass fraction f of the accreted hydrogen ignites and burns at a constant luminosity for 100 days. Solve for the critical mass fraction of burnt hydrogen, f_{crit}, above which mass is ejected from the system. You can assume that the main source of opacity is electron scattering.

Comment on the relevance of the above process for our understanding of thermonuclear supernovae.

Suggest observational tests that may help distinguish between different proposals for the origin of thermonuclear supernovae.

Question 5Z - Statistical Physics

(**i**) What is meant by the *chemical potential* μ of a thermodynamic system?

Derive the Gibbs distribution giving the probability of a system at temperature T and chemical potential μ (and fixed volume) with variable particle number N being in a particular microstate.

(**ii**) Consider a non-interacting, two-dimensional gas of fermionic particles in a region of fixed area, at temperature T and chemical potential μ. Using the Gibbs distribution, find the mean occupation number $n(\varepsilon)$ of a one-particle quantum state of energy ε.

Show that the density of one-particle states is independent of ε.

For a gas of N such particles at temperature T, deduce that the mean number of particles dN with energies between ε and $\varepsilon + d\varepsilon$ is very well approximated for $k_{\mathrm{B}}T \ll \varepsilon_{\mathrm{F}}$ by

$$dN = \frac{N}{\varepsilon_{\mathrm{F}}} \frac{d\varepsilon}{e^{(\varepsilon-\varepsilon_{\mathrm{F}})/(k_{\mathrm{B}}T)} + 1},$$

where ε_{F} is the Fermi energy.

Show that, for $k_{\mathrm{B}}T \ll \varepsilon_{\mathrm{F}}$, the specific heat of the gas has a power-law dependence on T, and find the power-law index.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Consider the Hamiltonian $H = H_0 + V$, where V is a small perturbation. If $H_0|n\rangle = E_n|n\rangle$, show that the energy eigenvalues of H correct to second order in the perturbation are

$$E_n^{\text{full}} = E_n + \langle n|V|n\rangle + \sum_{m \neq n} \frac{|\langle m|V|n\rangle|^2}{E_n - E_m} \,,$$

assuming the energy levels of H_0 are non-degenerate.

How is this result changed if there are energy levels of H_0 that are degenerate with $|n\rangle$?

(ii) In a certain three-state system, the unperturbed Hamiltonian, H_0, and perturbation, V, take the form

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \qquad \text{and} \qquad V = V_0 \begin{pmatrix} 0 & \epsilon & \epsilon^2 \\ \epsilon & 0 & 0 \\ \epsilon^2 & 0 & 0 \end{pmatrix} \,,$$

with V_0 and ϵ real, positive constants and $\epsilon \ll 1$.

(a) Consider first the case $E_1 = E_2 \neq E_3$ and $|\epsilon V_0/(E_3 - E_2)| \ll 1$. Use the results of degenerate perturbation theory to obtain the energy eigenvalues correct to order ϵ.

(b) Now consider the different case $E_3 = E_2 \neq E_1$ and $|\epsilon V_0/(E_2 - E_1)| \ll 1$. Obtain the energy eigenvalues correct to order ϵ^2.

(c) Obtain the exact energy eigenvalues in case (b), and compare these to your perturbative results by expanding to second order in ϵ.

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) The number density of stars in a spherically-symmetric galaxy drops as $n_*(r) \propto r^{-2}$ with radius r. Show that if one counts the number of stars within a projected radius R on the sky, it exceeds the number of stars within a sphere of radius R by a factor $\pi/2$.

[*You may assume* $\int_{-\infty}^{\infty} \ln(1 + 1/x^2)\,\mathrm{d}x = 2\pi$.]

(ii) Consider a spherically-symmetric stellar-dynamical system with distribution function

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\,\sigma^2)^{3/2}} \exp\left(\mathcal{E}/\sigma^2\right),$$

where $\mathcal{E} = \Psi - \frac{1}{2}v^2$ is the reduced energy, Ψ is the reduced potential, v is the velocity, σ is the (constant) velocity dispersion and ρ_1 is constant. Find the density ρ of the system, and write down Poisson's equation for the system.

The equation of hydrostatic support for an isothermal gas of density $\rho(r)$ at temperature T in a spherically symmetric system is

$$\frac{k_{\mathrm{B}}T}{m_0}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho\,\frac{Gm(r)}{r^2},$$

where k_{B} is Boltzmann's constant, m_0 is the mass per particle, and $m(r)$ the total mass interior to the radius r. Show that the stellar-dynamical system and the gaseous system have the same density structure when

$$\sigma^2 = \frac{k_{\mathrm{B}}T}{m_0}.$$

For this (Maxwellian) velocity distribution function, show that the mean particle speed is $\bar{v} = (8/\pi)^{1/2}\sigma$.

[*You may assume without proof that*

$$\int_0^{\infty} \exp\left(-\alpha x^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, \qquad \int_0^{\infty} x^3 \exp\left(-\alpha x^2\right) = \frac{1}{2\alpha^2},$$

and that in spherical coordinates

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}.\ ]$$

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) A disc around a white dwarf is very optically thick to ultraviolet photons (optical depth $\tau = 100$). Estimate the time taken for a photon to diffuse from the mid-plane to the surface of the disc if its surface density is 300 kg m^{-2} and its mean (vertically-averaged) mass density is $3 \times 10^{-4}\,\mathrm{kg\,m^{-3}}$.

A foreground molecular cloud passes between the white dwarf and the observer. Explain why it is possible for the emission lines from the disc to become bluer as a result while the continuum is reddened.

(ii) A region of a galaxy behaves as a 'closed box', i.e., there are no inflows or outflows of gas or stars to/from the region. Gas is converted to stars by star formation. For unit mass of stars formed, the *nucleosynthetic yield* (y) is defined as the mass of heavy elements synthesised and returned to the interstellar medium. If heavy elements are synthesised and returned to the gas phase (almost) immediately after star formation and if the return of hydrogen to the gas phase has a negligible effect on the composition of the gas, the system evolves according to

$$\frac{\mathrm{d(AM_g)}}{\mathrm{dt}} = A\frac{\mathrm{dM_g}}{\mathrm{dt}} - y\frac{\mathrm{dM_g}}{\mathrm{dt}},$$

where M_g is the mass in the gas phase and A is the abundance (the mass fraction of heavy elements in the gas phase). Explain the physical significance of each term in the above equation.

Show that if the region contains a mass M_{tot} which is initially all in the gas phase with $A = 0$, the mass of stars that subsequently form in this region with abundance less than A is given by

$$M_*(< A) = M_{\mathrm{tot}}\big(1 - \mathrm{e}^{-A/y}\big).$$

Derive also, in the case that the star formation rate scales as M_{g}^n, where M_{g} is the instantaneous gas mass, an expression for the rate of change of A. Hence show that A varies linearly with time if $n = 1$ but logarithmically with time for $n > 1$.

In the solar neighbourhood, where the present-day abundance of iron is 1.75 times the yield, y, it is found that 2% of stars have an abundance of iron that is less than $0.5y$. Comment on whether this result is consistent with the solar neighbourhood behaving as a 'closed box'; if not, suggest in what way this assumption should be relaxed in order to explain the data.

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 7 June 2019 09.00am-12.00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

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Blue Cover Sheets	*Approved Calculators Allowed*
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1 Rough Work Pad	
Tags	

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Question 1X - Relativity

(i) Starting from the Bianchi identity satisfied by the Riemann curvature tensor,

$$\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0\,,$$

derive the contracted Bianchi identity

$$\nabla^a \left(R_{ab} - \frac{1}{2} R g_{ab} \right) = 0\,,$$

where $R_{ab} \equiv R_{cab}{}^c$ is the Ricci tensor and $R \equiv g^{ab} R_{ab}$ is the Ricci scalar.

Briefly explain why it is appropriate to take the field equations of general relativity to be of the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}\,, \qquad (*)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter and Λ is a constant.

(ii) Show that the Einstein field equations given in $(*)$ in Part (i) are equivalent to

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \qquad (**)$$

in vacuum.

Consider the following static, spherically-symmetric line element in spherical coordinates, $x^\mu = (ct, r, \theta, \phi)$ with $\mu = 0, 1, 2, 3$:

$$ds^2 = \mathcal{A}(r) d(ct)^2 - \frac{1}{\mathcal{A}(r)} dr^2 - r^2 d\Omega^2\,,$$

where $\mathcal{A}(r)$ is a function of r only. Solve the $\theta\theta$ component of the vacuum Einstein field equations $(**)$ to determine the general solution for $\mathcal{A}(r)$ and verify that, with this, the other components of the field equations are automatically satisfied.

By comparing with the Schwarzschild solution, determine the form of $\mathcal{A}(r)$ that describes the spacetime with $\Lambda > 0$ outside a spherically-symmetric mass M centred at the origin.

Given that a free-falling massive particle has $\mathcal{A}(r) c\dot{t} =$ constant, where dots denote differentiation with respect to proper time, show that for radial motion

$$\ddot{r} = -\frac{GM}{r^2} + \frac{\Lambda c^2}{3} r\,.$$

Give a physical interpretation of the Λ term in this radial equation of motion.

[*The non-zero components of the Ricci tensor are*

$$R_{00} = -\mathcal{A}^2 R_{rr}\,, \quad R_{rr} = \frac{\mathcal{A}''}{2\mathcal{A}} + \frac{\mathcal{A}'}{r\mathcal{A}}\,, \quad R_{\theta\theta} = r\mathcal{A}' + \mathcal{A} - 1\,, \quad R_{\phi\phi} = \sin^2\theta R_{\theta\theta}\,,$$

where primes denote differentiation with respect to r.]

TURN OVER...

Question 2Y - Astrophysical Fluid Dynamics

(i) The energy equation in entropy form for a fluid with adiabatic index γ is

$$\frac{1}{K}\frac{DK}{Dt} = -\frac{(\gamma-1)}{p}\rho\dot{Q}_{\rm cool},$$

where $K = p\rho^{-\gamma}$, and p is pressure, ρ is density, and K is the adiabatic constant. We take the cooling law to have the form $\dot{Q}_{\rm cool} = \rho\Lambda(T) - \mathcal{H}_0$, where $\mathcal{H}_0$ is a heating term that is spatially uniform and constant in time. Briefly describe the physical processes that would give a cooling law of this form.

Consider a fluid that is spatially homogeneous, static, and in thermal equilibrium. Starting from the above energy equation, show that density and pressure perturbations are related by

$$\frac{\partial}{\partial t}(\delta p) - c_s^2\frac{\partial}{\partial t}(\delta\rho) = -(\gamma-1)\rho_0\delta(\dot{Q}_{\rm cool}),$$

where c_s is the adiabatic sound speed in the equilibrium state.

(ii) Consider the gas from Part (i) which, in the equilibrium state, is at rest with uniform density ρ_0, and uniform pressure p_0. The gas is subject to cooling and heating processes described by the cooling law from Part (i) with $\Lambda = \Lambda_0 T^\alpha$, where α is a constant. In the initial state, these cooling and heating processes are in balance. Small plane-wave perturbations (e.g., where $\rho \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$) are now introduced into the system. Show that the corresponding dispersion relation is

$$\omega(\omega^2 - c_s^2k^2) = -i\omega_c\left[\alpha\omega^2 - \frac{(\alpha-1)c_s^2}{\gamma}k^2\right],$$

where c_s is the adiabatic sound speed of the gas, and ω_c a real constant that should be given in terms of parameters already specified.

State the solution of the dispersion relation in the limits of $\omega_c = 0$ and give a physical interpretation of this solution.

Consider the case when ω_c is small but non-zero. By expanding the dispersion relation away from the $\omega_c = 0$ solution, or otherwise, show that the perturbations can grow or be damped.

Give a criterion involving α and γ that determines whether the perturbations grow or damp.

Question 3Z - Introduction to Cosmology

(i) Sketch, with clearly-labelled axes, the angular power spectrum of the cosmic microwave background (CMB) temperature fluctuations.

Briefly discuss the physical processes that give rise to the main features of the CMB power spectrum.

(ii) Show that in the radiation-dominated era (in which the curvature term can be neglected) the age of the universe, t, can be related to the photon temperature, $T_{\rm r}$, via

$$t = \left(\frac{3c^2}{32\pi G a} \right)^{1/2} \frac{1}{T_{\rm r}^2} \, ,$$

where a is the radiation density constant. (You may neglect relativistic species other than photons.)

How would additional relativistic species (e.g., neutrinos) affect this age?

Big Bang nucleosynthesis took place when $k_{\rm B} T_{\rm r} \simeq 0.1\,{\rm MeV}$. Ignoring relativistic species other than photons, how old was the universe then?

Outline the main steps by which light elements are built up in Big Bang nucleosynthesis. Your answer should include the evolution of the neutron-to-proton ratio before nucleosynthesis, the role of deuterium, and an explanation of why nucleosynthesis ends.

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) Briefly describe the different sources of pressure possible within stars. For each case, state the functional form of the equation of state and give an example of where it is the dominant source of pressure.

(ii) Assume that the density $\rho(r)$ of a spherically symmetric star varies linearly from the centre to the surface (i.e., from radius $r = 0$ to $r = R$):

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right).$$

Derive an expression for the total mass of the star, $M(R)$.

Derive an expression for the pressure as a function of radius, $P(r)$. You may assume that $P(R) = 0$.

Use the above result to deduce the dependence of *central* pressure on stellar radius and mass, $P_c = f(R, M)$.

Derive an expression for the temperature as a function of radius, and show that $T(R) = 0$.

Assuming the star is composed only of totally-ionized hydrogen, derive an expression for the central temperature T_c as a function of stellar radius.

Compare the central temperature computed in this way to that of the Sun, where $T_c = 1.6 \times 10^7\,\mathrm{K}$. What conclusions do you draw from this comparison?

Question 5Z - Statistical Physics

(**i**) Give an outline of the Landau theory of phase transitions for a system with one real order parameter ϕ.

(**ii**) Describe the phase transitions that can be modelled by the Landau potentials

(a) $$\Phi = \frac{1}{4}\phi^4 + \frac{1}{2}\varepsilon\phi^2 \ ,$$

(b) $$\Phi = \frac{1}{6}\phi^6 + \frac{1}{4}\ g\phi^4 + \frac{1}{2}\varepsilon\phi^2 \,,$$

where ε and g are control parameters that depend on the temperature.

In case (b), find the curve of first-order phase transitions in the g–ε plane, and find the regions where superheating or supercooling is possible.

TURN OVER...

Question 6Z - Principles of Quantum Mechanics

(i) Define the *spin-raising* and *spin-lowering* operators, S_+ and S_-.

Show that

$$S_\pm|s,\sigma\rangle = \hbar\sqrt{s(s+1)-\sigma(\sigma\pm 1)}\,|s,\sigma\pm 1\rangle\,,$$

where $S_z|s,\sigma\rangle = \hbar\sigma|s,\sigma\rangle$ and $S^2|s,\sigma\rangle = s(s+1)\hbar^2|s,\sigma\rangle$.

(ii) Two spin-half particles, with spin operators $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$, have a Hamiltonian

$$H = \alpha\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} + \mathbf{B}\cdot(\mathbf{S}^{(1)} - \mathbf{S}^{(2)})\,,$$

where α and $\mathbf{B}$ are constants. Express H in terms of the z-components of the two particles' spin operators and their spin-raising and lowering operators. Hence find the eigenvalues of H and comment on their behaviour as $|\mathbf{B}|\to 0$.

Question 7X - Stellar Dynamics and the Structure of Galaxies

(i) Assume that stars in the solar neighbourhood have a velocity distribution that is isotropic and Maxwellian, with velocity distribution $\sigma = 50\,\mathrm{km\,s^{-1}}$, that the stars all have mass $1M_\odot$ and that the stellar mass density is $0.04M_\odot\mathrm{pc}^{-3}$. Neglecting any gravitational-focusing, what is the closest approach that a star is likely to have made to the Sun during its lifetime of 4.5 Gyr, assuming that the Sun's environment has always been similar to the present solar neighbourhood.

Is the neglect of gravitational-focusing justified for this calculation?

(ii) State the Jeans theorem and discuss its relevance for the modelling of stellar systems, carefully explaining each of the terms: phase-space distribution function, two-body relaxation, relaxation time, collisionless stellar system, collisionless Boltzmann equation, and integrals of motions.

Discuss what can be inferred for the properties of the stellar density distribution and the velocity distribution of a collisonless stellar system if its (phase space) distribution function is a function of the energy of stellar orbits only.

How do these properties change if the distribution function is instead a function of energy and (total) angular momentum of stellar orbits.

$\Big[$ *You may assume that the mean speed of a Maxwellian distribution is* $(8/\pi)^{1/2}$ *times its velocity dispersion.* $\Big]$

TURN OVER...

Question 8Y - Physics of Astrophysics

(i) A point mass, M, which moves supersonically through gas at speed v sweeps up all the gas that lies within a radius GM/v^2 of its path. Show that a black hole moving in this way through a medium of density ρ should attain infinite mass in a finite time $t_{\rm inf}$.

Evaluate $t_{\rm inf}$ in the case that the mass is a black hole of initially $M = 1000\,\mathrm{M}_\odot$, moving at $v = 200\,\mathrm{km\,s^{-1}}$ through a medium of density $\rho = 10^{-16}\,\mathrm{kg\,m^{-3}}$.

Suggest two reasons why the black hole would never in practice be able to grow arbitrarily large.

(ii) It has been suggested that Galactic dark matter is provided by a large number of black holes of mass $10^6\,\mathrm{M}_\odot$. Estimate how many black holes would be required, interior to the orbit of the Sun, if their collective gravity dominated the gravitational dynamics at the location of the Sun (8.5 kpc from the Galactic Centre) and if the speed of the Sun's circular orbit around the centre of the Galaxy is 230 km $\mathrm{s^{-1}}$.

The black holes orbit through gas of mean density $10^{-21}\,\mathrm{kg\,m^{-3}}$. Using the information given in Part (i) about the accretion rate of gas onto a moving point mass, estimate the X-ray luminosity of each accreting black hole if 10% of the rest mass energy of the accreted material is converted into X-ray radiation. Assume that the black hole velocities are typical of the Galactic potential.

Given that there are a total of 30 X-ray sources in the Galaxy with luminosity in excess of $10^{30}\,\mathrm{W}$, discuss whether the above model for Galactic dark matter is consistent with observational constraints.

It has alternatively been suggested that the dark matter is in the form of $10^6\,\mathrm{M}_\odot$ clusters of solar-mass black holes. If the accretion rate on to each black hole is unaffected by it being in a cluster environment, compare the total accretion luminosity per cluster with that of a single $10^6\,\mathrm{M}_\odot$ black hole and discuss whether the observational limits given above on the number of bright X-ray sources in the Galaxy would be violated in this scenario.

If X-ray observatories can typically detect sources with luminosities down to $10^{20}\,\mathrm{W}$ at a distance of 100 pc, discuss what fraction of the black hole clusters in the Galaxy would be detectable.

Suggest, without detailed calculation, what processes would limit the lifetime of the black hole clusters in the limits of: (a) large; and (b) small cluster radius.

END OF PAPER