NATURAL SCIENCES TRIPOS Part II

Monday 4 June 2012 09:00am – 12:00pm

ASTROPHYSICS - PAPER 1

Before you begin read these instructions carefully.

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Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

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Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

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Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

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Question 1X – Relativity

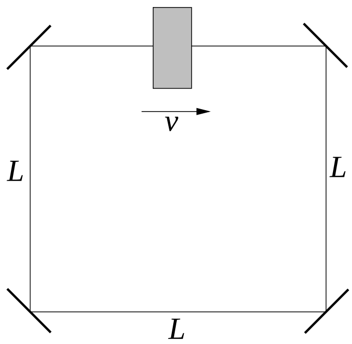
(i) A particle of mass m with total energy E collides with a stationary particle, also of mass m, to form a particle of mass $4m$. Determine the required energy E.

Compare this to the energy of each incident particle for the case where they approach each other with equal but oppositely-directed momenta.

(ii) An inertial frame S' is moving relative to another inertial frame S along the x-direction with speed v. The axes of both frames are aligned. For a particle with Cartesian velocity components (u_x, u_y, u_z) in S, use the Lorentz transformations to derive the velocity components in S'.

A medium has refractive index n, i.e. light propagates at speed c/n in the rest-frame of the medium. If the medium is at rest in S', show that the propagation speed in the frame S for light propagating along the positive x'-direction is $(1 + nv)/(n + v)$ in units with $c = 1$.

Light propagates around a square path of side L by reflecting off four mirrors. One of the sides contains a slab of material with refractive index n and proper length l moving parallel to the side with speed v relative to the mirrors (see figure).



Show that the travel times around the circuit for clockwise and anti-clockwise propagation differ by

$$\Delta t = 2l\gamma v(1 - n)\,,$$

where $\gamma \equiv (1 - v^2)^{-1/2}$ (with $c = 1$). You should ignore reflections at the faces of the slab.

For $l = 1\,\mathrm{m}$ and $n = 1.5$, how large must v be to correspond to a phase shift of π for light of free-space wavelength $\lambda = 500\,\mathrm{nm}$?

Question 2Y – Astrophysical Fluid Dynamics

(i) The Navier–Stokes equation for the velocity of a viscous fluid is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi + \nu \left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u}) \right] ,$$

where p is the pressure, ρ is the density, Φ is the gravitational potential and ν is the constant kinematic viscosity. Show that the vorticity, $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$, for a barotropic fluid obeys

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} .$$

State Kelvin's circulation theorem and describe how this is affected by viscosity.

(ii) The steady-state structure of a geometrically thin protoplanetary disc of mass $m_{\rm d}$ around a star of mass $m_\star \gg m_{\rm d}$ is described in cylindrical polar coordinates (R, ϕ, z) with the star at the origin and the z-axis perpendicular to the disc. The gas in the disc is vertically isothermal and axisymmetric. Show that the density profile is given by

$$\rho(R, z) = \rho_{\rm m}(R) e^{-z^2/(2h^2)} ,$$

where $\rho_{\rm m}(R)$ is the mid-plane density and h should be found in terms of the local sound speed $c_{\rm s}$ and the local Keplerian velocity $v_{\rm K} = \sqrt{G m_\star / R}$.

The gas pressure in the mid-plane near some fiducial radius R_0 can be written in the form $p(R, 0) = p_{\rm m0}(R/R_0)^{-n}$, where $p_{\rm m0}$ and n are constants. Show that the orbital velocity v_ϕ of the gas in the mid-plane is

$$v_\phi = v_{\rm K} \sqrt{1 - n(c_{\rm s}/v_{\rm K})^2} .$$

Hence deduce that $h/R \approx 1/M$, where M is the Mach number of the flow.

For a thin infinite sheet of uniform surface density Σ, the vertical acceleration outside the sheet is $2\pi G \Sigma$. Deduce that the disc's self-gravity can be ignored when

$$\frac{m_{\rm d}}{m_\star} \ll \frac{h}{R} .$$

An infinitesimally thin disc extends to the stellar surface at $R_\star$ and intercepts $1/4$ of the stellar luminosity $L_\star$. Estimate the accretion rate above which accretion heating is comparable to irradiation heating and show that for a Solar-like star this rate is about $10^{-8}\,{\rm M}_\odot\,{\rm yr}^{-1}$.

TURN OVER...

Question 3X – Physical Cosmology

(i) Sketch, with clearly labelled axes, the angular power spectrum of the cosmic microwave background (CMB) temperature fluctuations.

Briefly discuss the physical processes that give rise to the main features of the CMB power spectrum.

How do CMB observations support the idea that most of the matter in the Universe is non-baryonic?

(ii) Outline the main steps by which light elements are built up in big-bang nucleosynthesis. Your answer should include the evolution of the neutron-to-proton ratio before nucleosynthesis, the role of deuterium, and an explanation of why nucleosynthesis ends.

While neutrons and protons are still in chemical equilibrium (via weak interactions such as $p + e^- \leftrightarrow n + \nu_e$) before nucleosynthesis, but are non-relativistic, show that neutron-to-proton ratio is

$$\frac{n_{\rm n}}{n_{\rm p}} = \left(\frac{m_{\rm n}}{m_{\rm p}}\right)^{3/2} \exp\left[-\frac{(m_{\rm n} - m_{\rm p})c^2}{k_{\rm B}T}\right] ,$$

where T is the temperature, $k_{\rm B}$ is Boltzmann's constant, $m_{\rm n}$ is the mass of the neutron and $m_{\rm p}$ is the mass of the proton. You may assume that the leptons have zero chemical potential.

What is the neutron-to-proton ratio when $k_{\rm B}T \gg (m_{\rm n} - m_{\rm p})c^2$?

Assuming inter-conversion of neutrons and protons freezes out when $k_{\rm B}T = 0.8\,{\rm MeV}$, and nucleosynthesis starts when the age of the Universe is approximately 400 s, estimate the neutron-to-proton ratio at the start of nucleosynthesis. You may assume that the half-life of the neutron is 615 s and that $(m_{\rm n} - m_{\rm p})c^2 = 1.3\,{\rm MeV}$.

Show that at the end of nucleosynthesis the abundance of helium by mass is

$$Y = 2\left(1 + \frac{n_{\rm p}}{n_{\rm n}}\right)^{-1} ,$$

where the neutron-to-proton ratio is evaluated just before nucleosynthesis. Determine the value of Y.

Question 4Z – Structure and Evolution of Stars

(i) Explain what is meant by the Strömgren sphere around hot stars and derive an expression for its radius, $r_{\rm Strom}$, in terms of the number of ionizing photons emitted by the star per unit time, Q_*, the number density of hydrogen atoms, n, in the ambient interstellar medium (ISM) and the recombination coefficient of hydrogen, α.

Stars A and B have the same effective temperature, but the radius of star A is twice that of star B. Given that the ambient density of the ISM around star A is twice that of the ISM around star B, how would you expect the Strömgren sphere of star A to compare to that of star B? (You should justify your reasoning.)

Use simple physical arguments to explain why in true-colour photographs of external galaxies regions of on-going star formation generally appear to have a red colour.

(ii) An eclipsing-binary system with a very accurately known period has a parallax of 0.1 arcsec (with negligible error) and consists of two solar-type stars separated by $500\,{\rm R}_\odot$. What is the angular size of each of the stars and their angular separation?

If you can measure angles on the sky with a 1σ accuracy of 0.01 arcsec, what is the percentage accuracy of the measurement of the radius of each star and of their separation?

If we now include an error in the measurement of the parallax of $\sigma_\Pi =$ 0.01 arcsec, what is the percentage accuracy in the inferred mass of the system?

Assume that the stars emit as blackbodies with an effective temperature $T_{\rm eff} = 5800\,{\rm K}$. If you measure the flux ratio between $\log_{10}(\nu/{\rm Hz}) = 14.0$ and 15.0 with an accuracy of 10%, to what precision can you determine the value of $T_{\rm eff}$?

TURN OVER...

Question 5Y – Statistical Physics

(i) Given that the free energy F can be written in terms of the partition function Z as $F = -k_{\rm B} T \ln Z$ at temperature T, show that the entropy S and internal energy E are given by

$$S = k_{\rm B} \frac{\partial (T \ln Z)}{\partial T} \quad \text{and} \quad E = k_{\rm B} T^2 \frac{\partial \ln Z}{\partial T}\,,$$

where the derivatives are at constant volume V.

(ii) A meson of mass m consists of two quarks, attracted by a linear potential energy

$$V = \alpha x\,,$$

where x is the separation between the quarks and α is a constant. Treating the quarks as non-relativistic, compute the vibrational contribution to the classical partition function that arises from the separation of the two quarks.

What is the average separation of the quarks at temperature T?

Consider an ideal gas of N such mesons. A strong magnetic field fixes the orientation of the quarks so the mesons do not rotate. Compute the partition function of the gas.

What is the heat capacity at constant volume C_V of the meson gas?

[You may assume the following integrals:

$$\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad \text{and} \quad \int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

for $a > 0$.]

Question 6Y – Principles of Quantum Mechanics

(i) Write the Hamiltonian for the simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 ,$$

for a particle of mass m and classical frequency ω, in terms of the annihilation and creation operators

$$a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{m\omega}\right) \quad \text{and} \quad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{m\omega}\right) .$$

Obtain an expression for the commutator $[a, a^\dagger]$ and deduce the quantized energy levels of this system.

(ii) Consider a new operator $b = ca + sa^\dagger$, where $c \equiv \cosh\theta$ and $s \equiv \sinh\theta$ for a real constant θ and a and $a^\dagger$ are the annihilation and creation operators defined in part **(i)**. Show that

$$[b, b^\dagger] = 1 .$$

Consider the Hamiltonian

$$H = \epsilon a^\dagger a + \frac{1}{2}\lambda({a^\dagger}^2 + a^2) ,$$

where ϵ and λ are real and such that $\epsilon > \lambda > 0$. Show that the Hamiltonian can be written in the form

$$H = \frac{1}{2}\sqrt{\epsilon^2 - \lambda^2}\left(bb^\dagger + b^\dagger b\right) - \frac{\epsilon}{2} ,$$

for some suitable choice of θ which should be determined.

Hence show that the energy eigenvalues of the system are

$$E_n = -\frac{\epsilon}{2} + \left(n + \frac{1}{2}\right)\sqrt{\epsilon^2 - \lambda^2} ,$$

where n is a non-negative integer.

TURN OVER...

Question 7Z – Stellar Dynamics and the Structure of Galaxies

(i) A thin ring, mass Δm, of radius R_0 is located in the $z = 0$ plane, centred on the origin of a cylindrical coordinate system (R, z, ϕ). Show that the gravitational potential generated by the ring is

$$\Phi(R, z) = -\frac{2G\Delta m}{\pi} \frac{1}{\left[(R + R_0)^2 + z^2\right]^{1/2}} \int_0^1 \frac{dx}{\left[(1 - x^2)(1 - k^2x^2)\right]^{1/2}} , \qquad (*)$$

where

$$k^2 = \frac{4RR_0}{(R + R_0)^2 + z^2} .$$

(ii) A test particle describes a circular orbit in the plane $z = 0$ under the influence of an axisymmetric potential $\Phi(R, z)$. Derive expressions for the angular frequencies of orbital motion Ω and of epicyclic excursions ω_z of the particle in the z direction. Show also that $\Delta\phi$, the angle by which the particle advances between successive upward passages through the plane $z = 0$, is given by $\Delta\phi = 2\pi[x - \mathrm{int}(x)]$ where $x = \Omega/\omega_z$ and $\mathrm{int}(x)$ is the nearest integer to x.

In the system described in part **(i)** above, Ω and ω_z can be written in the limit $R \gg R_0$ in the form

$$\Omega = \Omega_0 \left[1 + a_1\left(\frac{R_0}{R}\right) + a_2\left(\frac{R_0}{R}\right)^2 + \cdots\right],$$

and

$$\omega_z = \omega_{z0} \left[1 + b_1\left(\frac{R_0}{R}\right) + b_2\left(\frac{R_0}{R}\right)^2 + \cdots\right],$$

where Ω_0 and ω_{z0} are, respectively, the values of Ω and ω_z in the case of a point mass ($R_0 = 0$). Expand the expression $(*)$ to order k^2 and hence show that $a_1 = b_1 = 0$. Argue further that a_2 and b_2 are of order unity (you need not calculate their values).

A satellite orbits Saturn at radius 2×10^6 km in an orbit that is slightly inclined to the plane of the rings. Approximating the rings as a single annulus of radius 1.37×10^5 km and given that the mass of Saturn's rings is 5×10^{-5} times that of Saturn itself, provide an order of magnitude estimate of $\Delta\phi$. If the minimum detectable value of $\Delta\phi$ is 0.01 rad, comment on whether this is a practicable method of measuring the mass of Saturn's rings.

[You may assume without proof that $I_0 = \pi/2$ and $I_1 = \pi/4$ where

$$I_n = \int_0^1 \frac{x^{2n}}{(1 - x^2)^{1/2}} \, dx .$$
]

Question 8Z – Topics in Astrophysics

(i) Derive the Eddington luminosity limit L of a point source of mass M, whereby outward radiation pressure balances inward gravitational attraction.

If you are now radiating at 100 W, how does that compare with your Eddington limit? Is the Eddington limit relevant to humans?

A quasar has a luminosity of 10^{40} W. Assuming that it is radiating at the Eddington limit, calculate the mass and mass-doubling timescale of the black hole. How long would it take to grow from a $20\,\mathrm{M}_\odot$ black hole, if it is always at the Eddington limit? Assume a radiative efficiency of 10%.

Explain briefly how the radiative efficiency, and thus the growth timescale, changes with the spin of the black hole?

(ii) An object is seen to pulse 100 times a second. Estimate a lower limit on its density by assuming that it is bound by its own gravity and thus cannot spin faster than a satellite can orbit it.

A rotating magnetised neutron star of radius R emits dipole radiation at a power

$$\mathcal{P} = \frac{8\pi}{3\mu_0} \frac{B^2 R^6\,\Omega^4}{c^3}\,,$$

where B is the surface magnetic field and Ω is the angular rotation rate of the star. Justify the dependence on B, R, Ω and c by considering the magnetic energy density at the speed of light cylinder, or otherwise.

Jupiter has a similar magnetic moment ($\propto BR^3$) to a pulsar, yet has a rotation period of 10 hr compared to the pulsar period of $P = 0.01$ s. How does Jupiter's luminosity in magnetic dipole radiation compare with that of the pulsar?

If the pulsar spins down at $\dot{P} = 10^{-12}\,\mathrm{s\,s^{-1}}$, what is the surface magnetic field? Assume a neutron star of mass $1\,\mathrm{M}_\odot$ and radius 10 km.

What is the braking index n, defined by $\dot{\Omega} \propto -\Omega^n$?

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Wednesday 6 June 2012 09:00am – 12:00pm

ASTROPHYSICS - PAPER 2

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Question 1X – Relativity

(i) Briefly explain how the Einstein equation in general relativity, and the Bianchi identity, ensure conservation of energy and momentum.

Use the Ricci identity, $\nabla_\mu \nabla_\nu v_\rho - \nabla_\nu \nabla_\mu v_\rho = R_{\mu\nu\rho}{}^\tau v_\tau$, to show that

$$\nabla_\mu \nabla^2 f - \nabla^2 \nabla_\mu f = R_\mu{}^\nu \nabla_\nu f$$

for any scalar function f, where $\nabla^2 \equiv \nabla_\mu \nabla^\mu$.

In a modified theory of gravity ("$f(R)$ gravity"), the Einstein equation is replaced by (with $c = 1$)

$$R_{\mu\nu} \frac{df}{dR} - \frac{1}{2} g_{\mu\nu} f(R) + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) \frac{df}{dR} = 8\pi G T_{\mu\nu} \, , \qquad (*)$$

where $f(R)$ is some function of the Ricci scalar. Show that the stress-energy tensor $T_{\mu\nu}$ is also conserved in this modified theory.

(ii) Consider a spatially-flat, homogeneous and isotropic cosmological model with line element (in Cartesian coordinates with $c = 1$)

$$ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j \, , \qquad (**)$$

where $a(t)$ is the scale factor and i and j run from 1 to 3. For a spatially-homogeneous scalar field f, show that

$$\nabla^2 f = \ddot{f} + 3H\dot{f} \, ,$$

where the Hubble parameter $H \equiv \dot{a}/a$ and overdots denote differentiation with respect to cosmic time t.

By considering the 00 component of the field equations $(*)$ in part **(i)**, and their trace (i.e. contraction with $g^{\mu\nu}$), show that for $f(R) \propto R^n$ the *vacuum* field equations for non-zero R are

$$\begin{aligned} 3n(H^2 + \dot{H})R - \frac{1}{2}R^2 - 3n(n-1)H\dot{R} &= 0 \, , \\ (n-2)R^3 - 3n(n-1)(n-2)\dot{R}^2 - 3n(n-1)R(\ddot{R} + 3H\dot{R}) &= 0 \, , \end{aligned}$$

and

$$R = 6(2H^2 + \dot{H}) \, .$$

Verify that for $n = 2$, vacuum solutions can be found with non-zero constant H and comment briefly on this result.

[You may assume that the non-zero connection coefficients for the metric in (∗∗) are

$$\Gamma^0_{ij} = a^2 H \delta_{ij} \quad \text{and} \quad \Gamma^i_{0j} = H \delta^i{}_j \,,$$

and that the non-zero components of the Ricci tensor are

$$R_{00} = 3(H^2 + \dot{H}) \qquad \text{and} \qquad R_{ij} = -a^2(3H^2 + \dot{H})\delta_{ij} \,. \quad]$$

TURN OVER...

Question 2Y – Astrophysical Fluid Dynamics

(i) An ionized fluid, composed of protons, of mass m_{p}, and electrons, of mass m_{e}, is initially neutral with a density ρ_0. By considering the effect of a small perturbation $n'(\boldsymbol{r}, t)$ to the number density of electrons, or otherwise, show that any charge imbalance obeys

$$\frac{\partial^2 n'}{\partial t^2} + \left(\frac{\rho_0 e^2}{\epsilon_0 m_{\mathrm{e}} m_{\mathrm{p}}}\right) n' = 0\,.$$

Estimate the frequency of oscillations in the Solar corona, where the density and temperature are $10^{-12}\,\mathrm{kg\,m^{-3}}$ and $10^6\,\mathrm{K}$, as well as the spatial scale over which charge neutrality is violated by spontaneous fluctuations in the plasma.

Also estimate the mean distance travelled by electrons before interacting with protons in the corona.

(ii) Consider a sphere of constant density ρ_{c} within which the pressure p at radius r is given by

$$p = p_{\mathrm{c}} - \frac{2\pi G}{3}\rho_{\mathrm{c}}^2 r^2\,,$$

where p_{c} is the central pressure. A narrow radial isothermal jet with temperature T arises at the centre of the sphere. When the effects of the jet on the radial structure in the gas sphere and of gravity on the material within the jet are negligible, show that the Mach number in the jet can be written as

$$M^2 = M_0^2 - 2\ln\left[1 - (r/r_{\mathrm{s}})^2\right],$$

where M_0 is the Mach number at the centre, and r_{s} should be found and its physical significance discussed.

Find the cross-sectional area of the jet as a function of radius and show that, for $M_0 < 1$, it has a minimum at a radius of $r_{\mathrm{s}}\sqrt{1 - e^{-(1-M_0^2)/2}}$ with an area of

$$A = A_0 M_0 e^{(1-M_0^2)/2}\,,$$

where A_0 is its area at $r = 0$.

Question 3X – Physical Cosmology

(i) Rich clusters of galaxies are observed embedded in hot gas which emits X-rays. What physical process gives rise to this radiation?

Explain why the observed X-ray surface brightness, B_{X}, is related to the electron density in the gas, n_{e}, by

$$B_{\mathrm{X}} \propto \int n_{\mathrm{e}}^2 \, dl \, ,$$

where l is distance measured along the line of sight through the cluster.

Why, at radio frequencies, is the cosmic microwave background (CMB) as seen through the cluster gas observed to have a lower temperature than it would otherwise have?

Given that the fractional temperature decrement in the CMB is given by

$$\frac{\Delta T}{T_{\mathrm{CMB}}} \propto \int n_{\mathrm{e}} \, dl \, ,$$

explain how the observables $\Delta T / T_{\mathrm{CMB}}$ and B_{X} can be used with the apparent angular diameter of the X-ray emission to estimate the angular-diameter distance to the cluster.

(ii) At redshift $z = 1100$, just before recombination, estimate: (a) the number density of free electrons; (b) the mean free path of photons; and (c) the average time a photon propagates between scatterings with free electrons. You may assume that $\Omega_{\mathrm{b}} h^2 = 0.023$ today, where Ω_{b} is the baryon density parameter and the Hubble constant is $H_0 = 100h \, \mathrm{km \, s^{-1} \, Mpc^{-1}}$, and ignore elements other than hydrogen.

What do your results imply about the opacity of the Universe at early times?

Assume that after recombination the gas remains neutral until an early generation of stars reionizes it when $z = z_{\mathrm{re}}$, after which the gas remains fully ionized to the present time. Show that in a matter-dominated Friedmann–Robertson–Walker universe, the optical depth for scattering by the free electrons is

$$\tau = \frac{2 n_{\mathrm{e},0} \sigma_{\mathrm{T}} c}{3 H_0} \left[(1 + z_{\mathrm{re}})^{3/2} - 1 \right] \, ,$$

where $n_{\mathrm{e},0}$ is the present-day electron density, H_0 is the Hubble constant, and σ_{T} is the Thomson cross-section.

Assuming $H_0 = 70 \, \mathrm{km \, s^{-1} \, Mpc^{-1}}$ and $\tau = 0.1$, calculate z_{re}.

TURN OVER...

Question 4Z – Structure and Evolution of Stars

(i) Explain what is meant by the stellar initial mass function (IMF) and give the Salpeter approximation to it.

A star cluster is born from a giant molecular cloud of mass $M_{\rm cl} = 10^6\,{\rm M}_\odot$. Assuming that 10% of the mass is converted into stars with a Salpeter IMF between $M_{\rm min} = 0.1\,{\rm M}_\odot$ and $M_{\rm max} = 100\,{\rm M}_\odot$, what is the mass of gas that goes into forming stars with masses $M \geq 10\,{\rm M}_\odot$?

Observations verify the expected number of stars with masses between 50 and $100\,{\rm M}_\odot$, but no star is observed with a mass $M > 100\,{\rm M}_\odot$. To what confidence level can astronomers conclude that in this cluster the IMF indeed has an upper mass cut-off $M_{\rm max} = 100\,{\rm M}_\odot$?

If the luminosity L of stars in the cluster obeys $L \propto M^3$, deduce the mass-to-light ratio of the cluster in Solar units and comment on its value.

(ii) If energy transport within a star is by radiative diffusion, the luminosity $L(r)$ at some radius r within the star can be written as

$$L(r) = -4\pi r^2 \, \frac{16\sigma}{3} \, \frac{T(r)^3}{\rho(r)\kappa(r)} \, \frac{{\rm d}T(r)}{{\rm d}r}\,,$$

where ρ is the density, T is the temperature, σ is the Stefan-Boltzmann constant, and the opacity κ is

$$\kappa(r) \propto \rho(r) T(r)^{-3.5}\,.$$

Show, using homology arguments, that as a star contracts along the pre-main sequence its luminosity changes with effective temperature $T_{\rm eff}$ according to

$$L \propto T_{\rm eff}^{4/5}\,.$$

The path taken by a contracting star as it approaches the main sequence in the Hertzsprung–Russell diagram is called the Henyey track. Numerical simulations show that $L \propto T_{\rm eff}^{4/5}$ is a satisfactory approximation of the Henyey tracks of massive stars, but becomes a progressively poorer fit to the tracks of stars with masses $M \lesssim 2\,{\rm M}_\odot$. What conclusions can you draw from this statement?

Question 5Y – Statistical Physics

(i) Write down the first law of thermodynamics in differential form for an infinitesimal reversible change in terms of the increments dE, dS and dV, in energy, entropy and volume. Give a brief interpretation of each term and deduce the following relations for the pressure and temperature,

$$p = -\left(\frac{\partial E}{\partial V}\right)_S \quad \text{and} \quad T = \left(\frac{\partial E}{\partial S}\right)_V .$$

(ii) Explain what is meant by an isothermal expansion and an adiabatic expansion of a gas.

By first establishing a suitable Maxwell relation, show that

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p\,,$$

and

$$\left(\frac{\partial C_V}{\partial V}\right)_T = T\left(\frac{\partial^2 p}{\partial T^2}\right)_V\,,$$

where C_V is the heat capacity at constant volume.

The energy in a gas of blackbody radiation is given by $E = aVT^4$ where a is a constant. Derive an expression for the pressure $p(V, T)$.

Show that if the radiation expands adiabatically, VT^3 is constant.

TURN OVER...

Question 6Y – Principles of Quantum Mechanics

(i) Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture.

Explain how the two pictures provide equivalent descriptions of physical results.

(ii) Derive the equation of motion for an operator in the Heisenberg picture.

For a particle of mass m moving in a one-dimensional potential $V(\hat{x})$ the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \,,$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, and the state vector is $|\Psi\rangle$. The eigenstates of $\hat{x}$ and $\hat{p}$ satisfy

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \,, \qquad \langle x|x'\rangle = \delta(x - x') \quad \text{and} \quad \langle p|p'\rangle = \delta(p - p') \,.$$

Use standard methods in the Dirac formalism to show that

$$\langle x|\hat{p}|x'\rangle = -i\hbar \frac{\partial}{\partial x} \delta(x - x')$$

and

$$\langle p|\hat{x}|p'\rangle = i\hbar \frac{\partial}{\partial p} \delta(p - p') \,.$$

Calculate $\langle x|\hat{H}|x'\rangle$ and express $\langle x|\hat{p}|\Psi\rangle$ and $\langle x|\hat{H}|\Psi\rangle$ in terms of the position-space wave function $\Psi(x)$.

Write down the momentum-space Hamiltonian for the potential

$$V(\hat{x}) = m\omega^2 \hat{x}^4/2 \,.$$

[You may use $\int_{-\infty}^{\infty} e^{ikx}\, dk = 2\pi\delta(x)$.]

Question 7Z – Stellar Dynamics and the Structure of Galaxies

(i) An object follows an elliptical orbit with semi-major axis a and eccentricity e about a point mass M. By considering the motion at periastron and apastron show that the energy per unit mass E and orbital angular momentum per unit mass L are given by

$$E = -GM/(2a) \quad , \quad L = \left[GMa(1-e^2)\right]^{1/2} .$$

Hence prove that T_{12}, the time taken for the object to travel between radius r_1 and r_2 in its orbit, satisfies

$$T_{12} = \frac{1}{\left(GMa\right)^{1/2}} \int_{r_1}^{r_2} \frac{r dr}{\left[e^2 - (r/a - 1)^2\right]^{1/2}} .$$

(ii) A space capsule is supposed to orbit a powerful X-ray emitting star of mass M in a circular orbit at radius r_0. The capsule starts its orbit at radius r_0 with the correct speed for a circular orbit there. However, the captain makes an error in the initial direction of the capsule so that its initial velocity is at an angle θ to the tangential direction. Sketch the orbit and determine the distances of greatest and least approach to the star.

Show that the total X-ray energy intercepted by the capsule on a half-orbit between periastron and apastron can be written as

$$D = f_0 \left(\frac{r_0^3}{GM}\right)^{1/2} A \int_{-1}^{+1} \frac{dy}{(1-y^2)^{1/2}(1+ey)} ,$$

where f_0 is the X-ray flux at radius r_0, A is the area of the capsule and $y = (r - r_0)/(er_0)$ with e the eccentricity of the orbit.

The crew complain that they have been put at risk by the error in initial direction because their new orbit takes them closer to the star where the X-ray flux is higher. The captain argues that the new orbit is actually safer because the capsule spends longer at larger distances from the star where the flux is lower. He also states that it will now take less time to complete each orbit. By considering the change of dose between the case $e = 0$ and the case of a small finite e, estimate the fractional change in X-ray dose when $\sin\theta = 0.1$. Hence comment on the statements made by the captain and the crew.

[You may quote the result for T_{12} in part **(i)** without proof if necessary.]

TURN OVER...

Question 8Z – Topics in Astrophysics

(i) The hot gas in the core of a cluster of galaxies forms a static atmosphere with power-law density profile $\rho(r)$ and temperature profile $T(r)$ given by $\rho(r) \propto r^{-0.8}$ and $T(r) \propto r^{0.3}$ respectively. By considering hydrostatic equilibrium deduce the total binding mass (in Solar units) within 100 kpc, if the temperature at that radius is 2×10^7 K. Assume that the gas is all hydrogen for the purpose of your calculation.

If the gas mass fraction is 10% within a radius of 100 kpc, estimate the number density of protons at a radius of 10 kpc.

What is responsible for most of the mass of the cluster?

(ii) A spaceship travelling at $50\,\mathrm{km\,s^{-1}}$ encounters a close binary system of orbital period 600 s, consisting of two white dwarfs each of mass $1\,\mathrm{M_\odot}$ and radius 10^7 m. What are the orbital velocities of the white dwarfs?

What effect causes the two stars slowly to approach each other?

If the spaceship undergoes a close gravitational encounter with one of the white dwarfs (without any impact taking place) so that the spaceship departs back along its original path, show that its velocity can be about $1450\,\mathrm{km\,s^{-1}}$.

Estimate a rough minimum value for the acceleration during the encounter. Would this have destroyed the spaceship and occupants (you should state why you reach your conclusion)?

Approximate the spaceship as two sections of equal mass separated by distance d. If $d = 40$ m show that the maximum tidal acceleration between the two sections is about $1\,g$, where g is the gravitational acceleration on Earth.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Thursday 7 June 2012 13:30pm – 16:30pm

ASTROPHYSICS - PAPER 3

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.**

Answers must be tied up in separate bundles, marked **X, Y, Z**, *according to the letter associated with each question, and a cover sheet must be completed and attached to each bundle. (For example,* **1X and 3X** *should be in one bundle and* **2Y, 5Y and 6Y** *in another bundle.)*

A master cover sheet listing all Parts of all questions attempted must also be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X – Relativity

(i) Let $x^\mu(\lambda)$ be an affinely-parameterised geodesic and $p^\mu \equiv dx^\mu/d\lambda$ be the tangent vector. Show that the geodesic equation, $p^\mu \nabla_\mu p_\nu = 0$, can be written as

$$\frac{dp_\nu}{d\lambda} = \frac{1}{2} p^\mu p^\tau \frac{\partial g_{\mu\tau}}{\partial x^\nu} \, .$$

What does this imply about p_0 for a metric which is independent of the timelike coordinate $t = x^0$?

In the Schwarzschild metric with mass M, show that the redshift between two stationary observers at constant radii $r_{\rm em}$ and $r_{\rm rec}$ is (with $c = 1$)

$$1 + z = \sqrt{\frac{1 - 2GM/r_{\rm rec}}{1 - 2GM/r_{\rm em}}} \, .$$

(ii) A massive particle falls radially from rest at infinity in the Schwarzschild metric. Show that (with $c = 1$)

$$\dot{t}(1 - 2GM/r) = 1 \qquad \text{and} \qquad \dot{r} + \sqrt{2GM/r} = 0 \, ,$$

where dots denote derivatives with respect to proper time.

Show that the redshift of light emitted from the particle when it is at radius r, as measured by a very distant stationary observer (at radius $\gg 2GM$), is $1 + z = \left(1 - \sqrt{2GM/r}\right)^{-1}$.

The distant observer receives the light emitted from the particle at radius r when his clock reads proper time $\tau_{\rm rec}$. By writing $dr/d\tau_{\rm rec} = (dr/d\tau)(d\tau/d\tau_{\rm rec})$, where τ is the particle's proper time, show that

$$\frac{dr}{d\tau_{\rm rec}} = -\sqrt{\frac{2GM}{r}} \left(1 - \sqrt{\frac{2GM}{r}}\right) \, . \qquad (*)$$

By approximating $(*)$ for r close to $2GM$, integrate to find

$$r - 2GM = A \exp[-\tau_{\rm rec}/(4GM)] \, ,$$

where A is a constant, as the particle is observed to approach the horizon.

Find the dependence of the observed redshift on $\tau_{\rm rec}$ as the horizon is approached.

Question 2Y – Astrophysical Fluid Dynamics

(i) Consider a sound wave of frequency ω propagating in the x-direction through non-dispersive media in which the density is ρ_1 and the sound speed is c_1 for $x < 0$, and the density is ρ_2 and the sound speed is c_2 for $x > 0$. Some fraction f_t of the energy carried by the wave is transmitted and the rest is reflected back in the negative x-direction at $x = 0$. What conditions must apply to the wave displacement and excess pressure at the interface?

Show that these conditions require

$$f_\mathrm{t} = \frac{4\rho_1 c_1 \rho_2 c_2}{(\rho_1 c_1 + \rho_2 c_2)^2}\,.$$

A giant molecular cloud at a temperature 30 K is embedded in the interstellar medium at 10^4 K. Estimate the fraction of energy that can be transmitted to the cloud from sound waves propagating through the interstellar medium. You may ignore any differences in the composition or ionization state of the two media.

(ii) Use the Rankine–Hugoniot relations to show that, for an adiabatic shock in which the gas pressure $p \propto \rho^\gamma$, the pre-shock density ρ_1, sound speed c_1 and Mach number $M_1 > 1$ are related to the post-shock quantities ρ_2, c_2 and M_2 by

$$\begin{aligned} \rho_1 M_1 c_1 &= \rho_2 M_2 c_2\,, \\ M_2 c_1 (1 + \gamma M_1^2) &= M_1 c_2 (1 + \gamma M_2^2)\,, \\ c_1^2 (2 + (\gamma - 1) M_1^2) &= c_2^2 (2 + (\gamma - 1) M_2^2)\,. \end{aligned}$$

Show that M_1 and M_2 satisfy the relation

$$\left[2 + (\gamma - 1)(M_1^2 + M_2^2) - 2\gamma M_1^2 M_2^2\right](M_1^2 - M_2^2) = 0\,.$$

Hence show that, if there is a shock, the post-shock flow is subsonic, and explain physically why this must be so.

Sketch how M_2 varies with M_1 when: (a) the fluid is composed of monatomic gas; and (b) the fluid is composed of diatomic gas.

TURN OVER...

Question 3X – Physical Cosmology

(i) Explain what is meant by the term particle horizon.

Show that in a Friedmann–Robertson–Walker universe with scale factor $R(t)$, where $R = 0$ at time $t = 0$, the (proper) particle horizon at time t is

$$d_{\rm H}(t) = R(t) \int_0^t \frac{c dt'}{R(t')} \, .$$

Find $d_{\rm H}(t)$ for a flat matter-dominated universe.

Assume that at last scattering the age of the Universe is $t = 4 \times 10^5$ yr and the redshift $z = 1100$. Estimate the horizon size (in pc) at the epoch of last scattering and calculate the angular size on the sky today subtended by this distance. You may assume a flat matter-dominated universe for all relevant epochs.

Use your result to explain the horizon problem.

(ii) Show that in a Friedmann–Robertson–Walker universe with scale factor $R(t)$, Hubble parameter $H(t)$ and curvature k, the density parameter Ω satisfies

$$|\Omega - 1| = \frac{|k| \, c^2}{H^2 R^2} \, .$$

Find the time dependence of $H^2 R^2$ for the two cases of a universe dominated by either radiation or matter and use your results to explain the flatness problem.

What is meant by inflationary expansion and how does a period of early inflation solve the flatness problem?

Use the Friedmann equations with $\Lambda = 0$ to show that

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) ,$$

where ρ is the (mass) density, p the pressure and overdots denote differentiation with respect to cosmic time t. Hence establish a constraint on the equation of state $p/(\rho c^2)$ for inflation to occcur.

Explain briefly how an early period of inflation can also resolve the horizon problem.

Question 4Z – Structure and Evolution of Stars

(i) Show that the equation of hydrostatic equilibrium may be written as

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln\tau} = \frac{g\tau}{\kappa P}\,,$$

where P is the pressure, τ is the optical depth, κ is the opacity and g is the gravitational acceleration.

Starting from the condition for hydrostatic equilibrium, derive a limit for the minimum central pressure in the Sun and compute its numerical value.

(ii) We know from geological and fossil records that the Sun's luminosity has remained constant to within approximately 10% over the last billion years, and the wavelength of the peak emission has changed by less than 0.1 nm. From these statements, deduce a limit on the acceleration of the stellar surface and comment on the accuracy of the approximation of hydrostatic equilibrium during this period.

[At the surface of the Sun, the gravitational acceleration is $250\,\mathrm{m\,s^{-2}}$.]

TURN OVER...

Question 5Y – Statistical Physics

(i) A ferromagnet has magnetization order parameter m and temperature T. The free energy is given by

$$F(T, m) = F_0(T) + \frac{a}{2}(T - T_{\rm c})\, m^2 + \frac{b}{4}\, m^4 \,,$$

where a, b and $T_{\rm c}$ are positive constants. Sketch F as a function of m at both high and low temperatures and find the equilibrium value of the magnetization in each case.

(ii) Evaluate the free energy of the ground state of the ferromagnet described in part **(i)** as a function of temperature for both $T > T_{\rm c}$ and $T < T_{\rm c}$.

Hence compute the entropy and heat capacity. Determine the jump in the heat capacity and identify the order of the phase transition.

After imposing a background magnetic field B, the free energy becomes

$$F(T, m) = F_0(T) + Bm + \frac{a}{2}(T - T_c)\, m^2 + \frac{b}{4}\, m^4 \,.$$

Explain graphically why the system undergoes a first-order phase transition at low temperatures as B changes sign.

The *spinodial* point occurs when the meta-stable equilibrium ceases to exist. Determine the temperature T of the spinodial point as a function of $T_{\rm c}$, a, b and B.

Question 6Y – Principles of Quantum Mechanics

(i) A Hamiltonian H_0 has non-degenerate energy levels E_n and corresponding non-degenerate normalised eigenstates $|n\rangle$. Show that if a small perturbation δH is added to H_0, the states are perturbed by

$$|\delta n\rangle = \sum_{m \neq n} \frac{\langle m|\delta H|n\rangle}{E_n - E_m} |m\rangle \, ,$$

and derive the change in energy eigenvalue E_n to first order in δH.

(ii) Discuss the consequences of indistinguishability for a quantum mechanical state consisting of two identical, non-interacting particles when the particles have: (a) spin zero; and (b) spin 1/2.

The stationary Schrödinger equation for one electron in the potential of a helium nucleus,

$$V(\boldsymbol{r}) = \frac{-2e^2}{4\pi\epsilon_0 r} \, ,$$

has normalised, spherically-symmetric wave functions $\psi_n(\boldsymbol{r})$ and energy eigenvalues E_n with $E_0 < E_1 < E_2 < \cdots$. What are the consequences of the Pauli exclusion principle for the two-electron ground state of the helium atom?

Write down the two-electron ground state in the form of a spatial wavefunction times a spin state.

Ignoring wavefunctions which are not spherically symmetric, and the interaction between the electrons, what are the states of the first excited energy level of the helium atom?

What combined angular momentum quantum numbers J and M does each state have?

Write down the perturbation to the Hamiltonian of the helium atom arising from the electron–electron interaction. Hence derive a multi-dimensional integral, which you need not evaluate, for the first-order correction to the helium ground-state energy.

TURN OVER...

Question 7Z – Stellar Dynamics and the Structure of Galaxies

(i) A stellar system in which all stars are of mass m is described by the phase-space distribution function

$$f = \frac{n(z)}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} \exp\left[-\left(\frac{v_x^2}{2\sigma_x^2} + \frac{v_y^2}{2\sigma_y^2} + \frac{v_z^2}{2\sigma_z^2}\right)\right] ,$$

where the velocity dispersions σ_x, σ_y and σ_z are independent of position. Show that if the gravitational potential is $\Phi(z)$, this distribution function is a solution of the steady-state collisionless Boltzmann equation if the number density $n(z)$ satisfies

$$\frac{1}{n}\frac{dn}{dz} + \frac{1}{\sigma_z^2}\frac{d\Phi}{dz} = 0 .$$

Hence determine $n(z)$, subject to $n(0) = n_0$, in the case $\Phi = \Phi_0 \ln\left[\cosh(z/z_s)\right]$, where Φ_0 is a constant and z_s is the scale length of the system.

If Φ is produced purely by the self-gravity of the stellar system, use Poisson's equation to demonstrate that the density $\rho(z)$, and the parameters Φ_0 and z_s are given by

$$\rho(z) = mn_0 \,\mathrm{sech}^2\left(\frac{z}{z_s}\right), \quad \Phi_0 = 2\sigma_z^2 , \quad z_s = \frac{\sigma_z}{(2\pi G m n_0)^{1/2}} . \qquad (*)$$

Comment on any similarities and differences between this system and an analogous gaseous system.

(ii) A body of mass M initially travelling with velocity $\boldsymbol{v}_M$ encounters a star of mass m (where $m \ll M$) moving with velocity $\boldsymbol{v}_m$. The mass M experiences a change in velocity parallel to the direction of initial relative motion of magnitude

$$\left|\Delta\boldsymbol{v}_{M||}\right| = \frac{2mV_0}{M}\left[1 + \left(\frac{bV_0^2}{GM}\right)^2\right]^{-1} ,$$

where b is the impact parameter of the encounter and $V_0 = |\boldsymbol{v}_M - \boldsymbol{v}_m|$. Show that the deceleration of the body due to encounters with all stars within a phase-space volume $\mathrm{d}^3\boldsymbol{v}_m$ centred on velocity $\boldsymbol{v}_m$ and out to an impact parameter $b_{\max}$ is given by

$$\frac{d\boldsymbol{v}_M}{dt} = 2\pi \ln(1 + \Lambda^2) G^2 m M f(\boldsymbol{v}_m) d^3\boldsymbol{v}_m \frac{(\boldsymbol{v}_m - \boldsymbol{v}_M)}{|\boldsymbol{v}_m - \boldsymbol{v}_M|^3} ,$$

where $f(\boldsymbol{v}_m)d^3\boldsymbol{v}_m$ is the number of stars per unit volume within this element of velocity phase space and

$$\Lambda = \frac{b_{\max}V_0^2}{GM} .$$

Hence show that if the stellar velocity dispersion is isotropic, the total deceleration of the body due to dynamical drag is

$$\frac{d\boldsymbol{v}_M}{dt} = -8\pi^2 \ln(1+\Lambda^2) G^2 M m \left(\int_0^{v_M} f(v_m) v_m^2 \, dv_m \right) \frac{\boldsymbol{v}_M}{v_M^3} ,$$

where $v_m = |\boldsymbol{v}_m|$ and $v_M = |\boldsymbol{v}_M|$. You should assume that v_M is large compared to the velocity dispersion so that Λ^2 can be evaluated at speed v_M.

Consider the system described in part **(i)** in the case that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \sigma^2$ and where the body passes through the system in the positive z-direction with initial speed $v_M \gg \sigma$. Neglecting the change in the object's velocity as it crosses the system, evaluate the magnitude of the maximum accleration due to gravity $g_{\max}$ and the maximum deceleration due to dynamical drag $f_{\max}$. Hence show that

$$\frac{f_{\max}}{g_{\max}} = \ln(1+\Lambda^2) \frac{GM}{2v_M^2 \, z_s} ,$$

where z_s is defined in $(*)$ above.

Provide a physical interpretation of the quantity $l = GM/v_M^2$.

TURN OVER...

Question 8Z – Topics in Astrophysics

(i) An interstellar cloud has mass $1\,\mathrm{M}_\odot$, and density 10^{10} hydrogen atoms cm^{-3}. It rotates with a period of 10^4 years. What would be the rotation period if the cloud were to condense into a Sun-like star with no loss of angular momentum?

Explain why a Sun-like star cannot sustain this rotation rate and how this problem might be alleviated during the formation process.

(ii) Spherical dust grains, all with the same density ρ (approximately that of water), are in circular orbits about the new sun in part **(i)**. Estimate the minimum radius a_0 dust particles must have to avoid being blown out of their solar system. (You should ignore any effects on the dust grains from solar wind, magnetic fields, and any mass other than the star.)

One of the spherical dust grains with radius a larger than a_0 is initially orbiting with speed v at distance D (with $D \gg \mathrm{R}_\odot$) from the parent star which has luminosity $\mathrm{L}_\odot$. Derive an expression for the drag force acting along the direction of orbital motion of the dust grain (the Poynting–Roberston effect) to leading order in v/c by considering the aberration of the incident light in the grain's rest frame.

The dust grain slowly spirals into the star due to the drag effect. By approximating this process as a sequence of circular orbits, or otherwise, show that the time T_{PR} for orbital decay is given by

$$T_{\mathrm{PR}} = \frac{D^2\, m\, c^2}{a^2\, \mathrm{L}_\odot}\,,$$

where m is the mass of the dust grain.

Calculate the value of T_{PR} for a dust grain initially at $D = 1\,\mathrm{AU}$, with mass a factor two larger than the dust grain blowout limit.

END OF PAPER

NATURAL SCIENCES TRIPOS Part II

Friday 8 June 2012 09:00am – 12:00pm

ASTROPHYSICS - PAPER 4

Before you begin read these instructions carefully.

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

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It is essential that every cover sheet bear the candidate's examination number and desk number.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Formulae Booklet*
Blue Cover Sheets	*Approved Calculators Allowed*
Yellow Master Cover Sheets	
1 Rough Work Pad	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Question 1X – Relativity

(i) The line element for a non-rotating black hole carrying electric charge Q is of the form (with $c = 1$)

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad (*)$$

for some function $f(r)$, and the electromagnetic 4-potential can be taken to have only a single non-zero component $A_t = Q/r$ (with $4\pi\epsilon_0 = 1$). The electromagnetic field-strength tensor $F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Show that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and, hence, that the non-zero components are $F_{tr} = -F_{rt} = Q/r^2$.

Show that the vacuum electromagnetic field equation, $\nabla_\mu F^{\mu\nu} = 0$, is satisfied away from the origin.

(ii) The stress-energy tensor for the electromagnetic field is

$$T_{\mu\nu} = \frac{1}{4\pi}\left(-F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4}g_{\mu\nu}F_{\rho\tau}F^{\rho\tau}\right)\,.$$

Explain why the vacuum Einstein field equation in the presence of electromagnetic fields can be written in the form

$$R_{\mu\nu} = 2G\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\tau}F^{\rho\tau}\right)\,.$$

For the line element $(*)$ and electromagnetic field in part **(i)**, solve the Einstein field equation to find $f(r)$ that correctly reduces to the Schwarzschild solution for $Q = 0$.

[For the line element $(*)$, the non-zero components of the Ricci tensor are

$$\begin{aligned} R_{tt} &= -f\left(\frac{f''}{2} + \frac{f'}{r}\right) = -f^2 R_{rr}\,, \\ R_{\theta\theta} &= rf' + f - 1 = R_{\phi\phi}/\sin^2\theta\,, \end{aligned}$$

where primes denote differentiation with respect to r.]

Question 2Y – Astrophysical Fluid Dynamics

(i) Describe what is meant by a conservative force and show that gravitational acceleration can be written in the form $\boldsymbol{g} = -\nabla\Phi$, where Φ is a scalar potential.

Show that for a mass distribution of density ρ within a volume V enclosed by a surface S,

$$\int_S \boldsymbol{g} \cdot d\boldsymbol{S} = -4\pi G \int_V \rho \, dV \,,$$

and hence derive Poisson's equation for the potential Φ.

For a gas in hydrostatic equilibrium, show that the pressure and density are constant on equipotential surfaces.

(ii) There are many filamentary structures in astrophysics that can be approximated as infinite cylindrically-symmetric mass distributions in which the density ρ depends only on the distance R from the central axis. Show that the gravitational acceleration toward the axis of symmetry has magnitude

$$g(R) = \frac{2G}{R} \int_0^R 2\pi R' \rho(R') dR' \,.$$

When the equation of state is polytropic, so that $p \propto \rho^{(n+1)/n}$, show that the pressure p and gravitational potential Φ are related by

$$p = p_0 \left(\frac{\Phi_\mathrm{s} - \Phi}{\Phi_\mathrm{s} - \Phi_0} \right)^{n+1} \,,$$

where the subscript "0" denotes the values at $R = 0$ and subscript "s" denotes those at the radius where $p = 0$.

Write $\theta = (p/p_0)^{1/(n+1)}$ and $\xi = \sqrt{4\pi G \rho_0 / (\Phi_\mathrm{s} - \Phi_0)} R$, where ρ_0 is the central density, to show that

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\theta}{d\xi} \right) = -\theta^n \,.$$

Show that the surface of a filament for which $n = 0$ is at $\xi = 2$.

Briefly describe how the solution is affected if the filament is threaded by a magnetic field oriented along its axis of symmetry.

TURN OVER...

Question 3X – Physical Cosmology

(i) Consider a flat ($k = 0$) Friedmann–Robertson–Walker universe with pressureless matter of density $\rho_{\rm m}$, negligible radiation energy density and a cosmological constant Λ. Use the Friedmann equations, or otherwise, to show that the deceleration parameter is given by

$$q \equiv -\frac{\ddot{R}R}{\dot{R}^2} = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda\,,$$

where overdots denote derivatives with respect to cosmic time, and the density parameters are

$$\Omega_{\rm m} \equiv \frac{8\pi G\rho_{\rm m}}{3H^2} \qquad \text{and} \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$$

with $H \equiv \dot{R}/R$ the Hubble parameter.

What are the present-day values $\Omega_{\rm m,0}$ and $\Omega_{\Lambda,0}$ if the Universe changed from deceleration to acceleration at redshift $z = 1$?

(ii) Show that in a Friedmann–Robertson–Walker universe, the observed redshift of a distant comoving source changes with proper time t measured by a comoving observer according to

$$\frac{dz}{dt} = H_0(1+z) - H(z)\,,$$

where $H(z)$ is the Hubble parameter with value H_0 today.

The spectrum of a high-redshift quasar shows an absorption line due to gas in a foreground galaxy at $z = 1$. If the quasar is monitored for 10 years, what is the expected fractional change in wavelength $\delta\lambda/\lambda$ for an Einstein–de Sitter universe with $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$?

Estimate how this fractional wavelength change compares to the line width if the gas has a velocity dispersion of $200\,\mathrm{km\,s^{-1}}$ and comment on your result.

How does the predicted wavelength change for our current best model of the Universe differ from the Einstein–de Sitter case?

Question 4Z – Structure and Evolution of Stars

(i) Describe the most important features of the optical spectra of white dwarfs, outlining the physical properties indicated by each spectral characteristic.

Explain in a few sentences why astronomers are interested in discovering white dwarfs with low values of the effective temperature $T_{\rm eff}$.

(ii) Show for a spherically-symmetric star that the equations of mass continuity and hydrostatic equilibrium can be combined into the second-order differential equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho\,, \qquad (*)$$

where $P(r)$ is the pressure and $\rho(r)$ the density at radius r.

For a gas with an equation of state $P = K\rho^\gamma$, where K and γ are constants, use $(*)$ to derive a second-order differential equation involving only density and radius.

Using the dimensionless variables $r' \equiv r/R_*$ and $\rho' \equiv \rho/\rho_0$, where R_* is the surface stellar radius and ρ_0 is the central density, show that the quantity $K\rho_0^{\gamma-2}/(GR_*^2)$ is dimensionless and the same for all stars, and hence that $R_* \propto \rho_0^{(\gamma-2)/2}$.

White dwarfs obey the equation of state $P = K\rho^\gamma$, with $\gamma = 5/3$ for non-relativistic conditions and $\gamma = 4/3$ in the relativistic regime. Show that for non-relativistic white dwarfs $R_* \propto M^{-1/3}$, while for relativistic white dwarfs the mass M is independent of radius.

TURN OVER...

Question 5Y – Statistical Physics

(i) For a three-dimensional distribution of non-interacting electrons show that the number of single-particle states in a phase-space element $d^3\boldsymbol{x}d^3\boldsymbol{p}$ is $(2/h^3)d^3\boldsymbol{x}d^3\boldsymbol{p}$.

In a cold non-relativistic gas electrons fill all available states up to the Fermi energy $E_{\rm F}$. Show that

$$E_{\rm F} = \frac{h^2}{8m_{\rm e}}\left(\frac{3n}{\pi}\right)^{\frac{2}{3}},$$

where n is the number density of electrons and $m_{\rm e}$ is the electron mass.

(ii) Non-relativistic, non-interacting electrons of mass $m_{\rm e}$ are confined to move in a two-dimensional plane of area A. Each electron has two spin states. Compute the density of states $g(E)$ and show that it is constant.

Write down expressions for the number of particles N and the average energy $\langle E\rangle$ of such a two-dimensional gas of fermions in terms of the temperature T and chemical potential μ.

Find an expression for the Fermi energy $E_{\rm F}$ in terms of N.

For $k_{\rm B}T \ll E_{\rm F}$, you may assume that the chemical potential does not change with temperature. Compute the low-temperature heat capacity of a gas of fermions.

[You may use the integral

$$\int_0^\infty \frac{x^n}{z^{-1}e^x+1}\,dx \approx \frac{1}{n+1}(\ln z)^{n+1} + \frac{\pi^2 n}{6}(\ln z)^{n-1}\,. \quad]$$

Question 6Y – Principles of Quantum Mechanics

(i) What are the commutation relations satisfied by the components of an angular momentum vector $\boldsymbol{J}$?

State the possible eigenvalues of the component J_3 when $\boldsymbol{J}^2$ has eigenvalue $j(j+1)\hbar^2$.

Describe how the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are used to construct the components of the spin angular momentum vector $\boldsymbol{S}$ for a spin-1/2 system and show that they obey the required commutation relations.

Show that S_1, S_2 and S_3 each have eigenvalues $\pm\hbar/2$ and verify that $\boldsymbol{S}^2$ has eigenvalue $3\hbar^2/4$.

(ii) Setting $\hbar = 1$, the raising and lowering operators $J_\pm = J_1 \pm iJ_2$ for angular momentum satisfy

$$J_\pm |j\ m\rangle = [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} |j\ m \pm 1\rangle\,,$$

where the state $|j\ m\rangle$ has total angular momentum quantum number j and azimuthal angular momentum quantum number m about the 3-direction. Find a matrix representation for $J_\pm$ acting on a vector of $j = 1$ states expressed in the basis of $\{|1m\rangle\}$ states.

Thus, calculate the matrix representation of $\boldsymbol{J}$.

Suppose the angular momentum of a $j = 1$ state is measured in the direction $\boldsymbol{n} = (0, \sin\theta, \cos\theta)$ to be $+1$. Find the state, expressing your answer as a linear combination of the $\{|1m\rangle\}$ states.

Suppose that two measurements of a system with $j = 1$ are made. The first is made in the 3-direction and returns a value $+1$. The second measurement is subsequently and immediately made in the direction $\boldsymbol{n}$. What is the probability that the second measurement also returns $+1$?

TURN OVER...

Question 7Z – Stellar Dynamics and the Structure of Galaxies

(i) A star cluster of mass m is in a circular orbit about a galaxy of mass M (with $M \gg m$). The centre of mass of the galaxy and the cluster are separated by a distance R. Consider a test particle which is located along the vector separating the cluster and galaxy centre at a distance r from the cluster centre (with $r \ll R$). Write down an expression for the acceleration of the particle in the frame co-rotating with the galaxy–cluster system when it is stationary in this frame.

Hence show that in the limit of small r/R there are two locations

$$r_J = \pm \left(\frac{m}{3M} \right)^{1/3} R$$

at which the direction of the acceleration changes sign.

The radius r_J provides a reasonable estimate of the *tidal radius* $r_{\rm t}$ of a cluster. Explain what is meant by tidal radius. Your answer should explain: how stars arrive at $r_{\rm t}$ even in clusters that are initially much more compact than $r_{\rm t}$; what happens to these stars at $r_{\rm t}$; and how the long-timescale evolution of the cluster density profile is affected by $r_{\rm t}$.

(ii) A spherical star cluster of radius r_0 is described by the radial density profile $\rho(r)$ where

$$\rho(r) = \begin{cases} \rho_0 \dfrac{\sin(\pi r/r_0)}{\pi r/r_0} & \text{for } r < r_0 \,, \\ 0 & \text{for } r \geq r_0 \end{cases}$$

and ρ_0 is the central density. Verify that the gravitational potential satisfies

$$\Phi(r) = \begin{cases} -\dfrac{4G\rho_0 r_0^2}{\pi} \left(\dfrac{\sin(\pi r/r_0)}{\pi r/r_0} + 1 \right) & \text{for } r < r_0 \,, \\ -\dfrac{4G\rho_0 r_0^3}{\pi r} & \text{for } r \geq r_0 \,, \end{cases}$$

and hence obtain an expression for the total gravitational binding energy of the cluster.

The cluster has mass $3 \times 10^5\,\mathrm{M}_\odot$ and orbits at a radius 1 kpc in a galaxy of mass $10^{11}\,\mathrm{M}_\odot$. If the cluster is tidally limited then, using any expressions derived in part **(i)**, or otherwise, provide values for ρ_0 and r_0 and hence find the total gravitational binding energy of the cluster.

Describe what is meant by core collapse in the context of the evolution of star clusters, including a description of the underlying physics and its effect on cluster evolution. Explain the role of binary stars in averting core collapse.

In the cluster described above, the collapsing core contains around 1% of the gravitational binding energy of the entire cluster. Discuss whether it is feasible for a binary star, comprising two $10\,M_\odot$ stars with separation $20\,R_\odot$, to avert core collapse in this system.

TURN OVER...

Question 8Z – Topics in Astrophysics

(i) An isolated gas cloud at temperature T and density ρ collapses under gravity if it is dynamically unstable. Use scaling arguments to deduce the condition for stablity in terms of a critical size $R_{\rm J}$ (the Jeans' length) and a corresponding critical mass $M_{\rm J}$ (the Jeans' mass).

A high redshift damped Lyman-alpha cloud has a line-of-sight neutral hydrogen column density $3 \times 10^{21}\,{\rm cm}^{-2}$. Assuming a cloud temperature of $30\,{\rm K}$ and a line-of-sight depth for the cloud of $1\,{\rm kpc}$, estimate its Jeans' mass in Solar units and comment on the value obtained.

(ii) Consider a universe in which the mass is currently dominated by massive neutrinos, each of mass m_ν. At redshift zero these act as classical particles, forming gravitationally-bound systems with mass density $\rho_\nu(r)$, as a function of radial distance r, given by

$$\rho_\nu(r) = \frac{2N_0 m_\nu^4\,\sigma^3}{(2\pi)^{3/2}\hbar^3}\, e^{-\Phi(r)/\sigma^2}\,,$$

where $N_0 \lesssim 0.5$ is the maximum phase-space density, σ is the one-dimensional velocity dispersion, and $\Phi(r)$ is the gravitational potential defined such that $\Phi(0) = 0$. By a change of variables, show that the Poisson equation for such a system can be written in the Lane–Emden form

$$\frac{d^2u}{dz^2} + \frac{2}{z}\frac{du}{dz} + e^u = 0\,, \qquad (*)$$

where z is a length-scale variable, and u is a potential-scale variable.

Equation $(*)$ has a solution for large z of the form $e^u = 2/z^2$. Show the corresponding mass distribution far from the centre is an isothermal sphere.

Apply the phase-space bound, $N_0 \lesssim 0.5$, to deduce a lower limit on the neutrino mass in terms of the velocity dispersion and the scale length of the potential well for the system.

For a galaxy like the Milky Way, with circular velocity $200\,{\rm km\,s^{-1}}$ and length scale $1\,{\rm kpc}$, deduce the corresponding bound on the neutrino mass.

END OF PAPER